Logical Formalisms

I. Lambda-Calculus

(a crash review)

What is lambda-calculus ?

- An intentional theory of functions.
- A simple functional programming language.
- A theory of free- and bound-variables, of scope and substitution.
- The keystone of higher-order syntax and higher-order logic.
- The algebra of natural-deduction proofs.

Syntax:

$$T \ ::= \ x \mid \lambda x.\, T \mid (T\, T)$$

λ is a binder: thre free occurrences of x in t are bound in $\lambda x.\, t$.

Warning: You should solve, once and for all, any problem you could have with the notions of free and bound occurrences of variables.

Reduction rule: $$(\lambda x.\, t)\, u \ \rightarrow_\beta \ t[x{:=}u]$$

Church-Rosser Theorem: For all λ-terms t, u, and v such that:

$$t \twoheadrightarrow_\beta u \quad \text{and} \quad t \twoheadrightarrow_\beta v$$

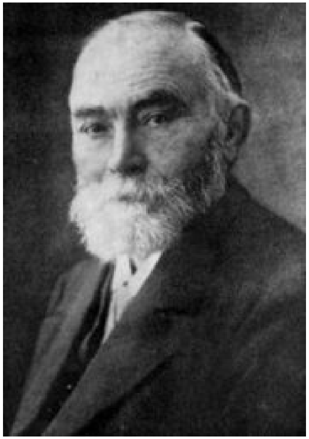
there exists a λ-term w such that:

$$u \twoheadrightarrow_\beta w \quad \text{and} \quad v \twoheadrightarrow_\beta w$$

Corollary: Uniqueness of normal forms.

Turing Completeness: Every recursive function is λ-definable.

Sense and Denotation



F.L.G. Frege
(1848–1925)

Sense/Denotation (Frege)
Intension/Extension (Carnap)

According to Frege, the sense of an expression is its "mode of presentation", while the denotation of an expression is the object it refers to.

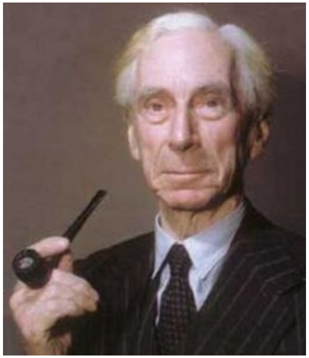
For instance, both expressions "$1+1$" and "2" have the same denotation but not the same sense.

An intensional proposition is a proposition whose validity is not invariant under extensional substitution.

Frege gives the example of the "morning star" and the "evening star" which both refer to the planet Venus.

Compare "the morning star is the evening star" with "John does not know that the morning star is the evening star".

Paradoxes and Type-Theory



B. Russell
(1872–1970)

Compare:

$$\Omega = \delta\,\delta \quad \textit{where} \quad \delta = \lambda x.\, x\, x$$

with:

$$X = \{x \mid x \notin x\}$$

Simply Typed Lambda-Calculus

$$\Gamma, x : A \vdash x : A$$

$$\frac{x : A, \Gamma \vdash t : B}{\Gamma \vdash \lambda x.\, t : A \to B} \qquad \frac{\Gamma \vdash t : A \to B \quad \Gamma \vdash u : A}{\Gamma \vdash (t\, u) : B}$$

Strong-Normalisation Theorem: There is no infinite reduction sequence.

Curry-Howard Isomorphism

Natural deduction	λ-calculus
propositions	types
connectives	type constructors
proofs	terms
introduction rules	term constructors
elimination rules	term destructors
active hypotheses	free variables
discarded hypotheses	bound variables
detour	redex
detour elimination	reduction step
proof normalization	term evaluation

II. Higher-Order Logic

Church's Simple Theory of Types

A. Church
(1903–1995)

Two atomic types:

$$\iota, o$$

Logical constants:

$$\begin{array}{lcl} \bot & : & o \\ \supset & : & o \rightarrow o \rightarrow o \\ \forall_\alpha & : & (\alpha \rightarrow o) \rightarrow o \quad \text{(at each type } \alpha\text{)} \end{array}$$

ι is the type of individuals and o is the type of propositions.

Formulas are defined to be well-typed λ-terms of type o. We write $P \supset Q$ and $\forall x. P$ for $\supset P\, Q$ and $\forall_\alpha\, (\lambda x. P)$, respectively. Similarly for the other connectives $(\neg, \wedge, \vee, \equiv, \exists)$, which are defined in the usual way — the system is classical!

$t = u$ is defined as $\forall P. P\, t \supset P\, u$.

Logical rules:

$$\Gamma, A \vdash A$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \supset B} \qquad \frac{\Gamma \vdash A \supset B \quad \Gamma \vdash A}{\Gamma \vdash B}$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash \forall_\alpha\,(\lambda x_\alpha.\, A)}\ x \text{ of type } \alpha,\ x \notin FV(\Gamma) \qquad \frac{\Gamma \vdash \forall_\alpha\, A}{\Gamma \vdash A\,B}\ B \text{ of type } \alpha$$

$$\frac{\Gamma, \neg A \vdash \bot}{\Gamma \vdash A}$$

Conversion rule:

$$\frac{\Gamma \vdash A}{\Gamma \vdash B}\ \text{where } A =_\beta B$$

Extensionality axioms:

$$\Gamma \vdash (\forall_\alpha x. A\,x = B\,x) \supset (A = B)$$

$$\Gamma \vdash (A \equiv B) \supset (A = B)$$

Higher-order logic as a set theory

$\{x \mid P\}$ as $\lambda x.\, P$

$t \in A$ as $A\, t$

Expressive Power

$$\mathsf{S} \stackrel{\triangle}{=} (\forall x.\, s\, x \neq 0) \;\wedge\; (\forall xy.\, s\, x = s\, y \;\supset\; x = y)$$

$$\mathsf{N} \stackrel{\triangle}{=} \lambda x.\, (\forall R.\, R\, 0 \;\wedge\; (\forall y.\, R\, y \;\supset\; R\, (s\, y)) \;\supset\; R\, x)$$

The only model of $\mathsf{S} \;\wedge\; \forall x.\, \mathsf{N}\, x$ is the set of natural numbers.

Let ϕ be a formula of Peano's Arithmetic, and define ϕ^{N} as follows:

- $\phi^{\mathsf{N}} = \phi$, for ϕ an atomic formula,
- $(\neg\phi)^{\mathsf{N}} = \neg\,\phi^{\mathsf{N}}$,
- $(\phi * \psi)^{\mathsf{N}} = \phi^{\mathsf{N}} * \psi^{\mathsf{N}}$, for $* \in \{\wedge, \vee, \supset, \equiv\}$,
- $(\forall x.\,\phi)^{\mathsf{N}} = \forall x.(\mathsf{N}\,x \supset \phi^{\mathsf{N}})$,
- $(\exists x.\,\phi)^{\mathsf{N}} = \exists x.(\mathsf{N}\,x \wedge \phi^{\mathsf{N}})$.

Let D be the conjunction of the universal closures of the defining equations for addition and multiplication, and let PA be $\mathsf{S} \;\wedge\; \forall x.\,\mathsf{N}\,x \;\wedge\; \mathsf{D}$.

Then, the formula $\mathsf{PA} \supset \phi$ is valid if and only if ϕ is true in the standard model of Peano's arithmetic.

Corollary: incompleteness of higher-order logic.

III. Modal Logic

Necessity and Possibility

G.W. von Leibniz
(1646–1716)

A proposition is necessarily true if it is true in all possible worlds.

A proposition is possibly true if it is true in at least one possible world.

Dr. Pangloss in Voltaire's Candide.

Syntax:

$$F \ ::= \ a \mid \neg F \mid F \vee F \mid \Box F$$

Define the other connectives in the usual way. Define $\Diamond A$ as $\neg\Box\neg A$.

$\Box A$ stands for "necessarily A". $\Diamond A$ stands for "possibly A".

Validity:

let $\mathcal{M} = \langle W, P \rangle$, where W is a set of "possible worlds", and P is a function that asigns to each atomic proposition a subset of W.

$\mathcal{M}, s \models a$ iff $s \in P(a)$.

$\mathcal{M}, s \models \neg A$ iff not $\mathcal{M}, s \models A$.

$\mathcal{M}, s \models A \vee B$ iff either $\mathcal{M}, s \models A$ or $\mathcal{M}, s \models B$, or both.

$\mathcal{M}, s \models \Box A$ iff for every $t \in W$, $\mathcal{M}, t \models A$.

It is easy to establish that:

$\mathcal{M}, s \models \Diamond A$ iff for some $t \in W$, $\mathcal{M}, t \models A$.

System S5:

(P) all propositional tautologies

(K) $\Box(A \supset B) \supset (\Box A \supset \Box B)$

(T) $\Box A \supset A$

(5) $\Diamond A \supset \Box\Diamond A$

Modus ponens:

$$\frac{A \supset B \quad A}{B}$$

Rule of necessitation:

$$\frac{A}{\Box A}$$

Kripke Semantics:

let $\mathcal{M} = \langle W, R, P \rangle$, where W is a set of "possible worlds", R is a binary relation over W, and P is a function that asigns to each atomic proposition a subset of W.

$\mathcal{M}, s \models \Box A$ iff for every $t \in W$ such that sRt, $\mathcal{M}, t \models A$.

$\mathcal{M}, s \models \Diamond A$ iff for some $t \in W$ such that sRt, $\mathcal{M}, t \models A$.

System K:

(P) all propositional tautologies

(K) $\Box(A \supset B) \supset (\Box A \supset \Box B)$

Modus ponens:

$$\frac{A \supset B \quad A}{B}$$

Rule of necessitation:

$$\frac{A}{\Box A}$$

The following theorems of S5 are not valid in the class of all Kripke models:

(D) $\Box A \supset \Diamond A$

(T) $\Box A \supset A$

(B) $A \supset \Box\Diamond A$

(4) $\Box A \supset \Box\Box A$

(5) $\Diamond A \supset \Box\Diamond A$

A binary relation $R \in W \times W$ is serial if and only if for every $s \in W$ there exists $t \in W$ such that sRt.

Some well-known systems

KD	basic deontic logic	serial
KT	basic alethic logic	reflexive
KTB	Brouwersche system	reflexive, symmetric
KT4	Lewis' S4	reflexive, transitive
KT5	Lewis' S5	reflexive, symmetric, transitive

IV. Hybrid Logic

Key idea: provide the formula language with explicit means of speaking about worlds!

Syntax:

Two sorts of atoms: usual atomic propositions ($a, b, c, \ldots$), and nominals ($i, j, k, \ldots$). Nominals will be used for naming worlds.

$$F \ ::= \ a \mid i \mid \neg F \mid F \vee F \mid \Box F \mid {\downarrow} i.\, F \mid @_i F$$

$\downarrow$ is a binder: the free occurrences of i in A are bound in ${\downarrow} i.\, F$. On the, other hand, $@$ is simply a binary connectives whose first term must be a nominal.

Intuition: $\downarrow$ is used for naming the “here-and-now”. It allows a nominal to be bound to the current world. $@_i A$ asserts that proposition A holds at world i.

Semantics:

Let $\mathcal{M} = \langle W, R, P \rangle$ be a Kripke model, and let η be a valuation that assigns to each nominal an element of W.

$\mathcal{M}, \eta, s \models a$ iff $s \in P(a)$.

$\mathcal{M}, \eta, s \models i$ iff $s = \eta(i)$.

$\mathcal{M}, \eta, s \models \neg A$ iff not $\mathcal{M}, \eta, s \models A$.

$\mathcal{M}, \eta, s \models A \vee B$ iff either $\mathcal{M}, \eta, s \models A$ or $\mathcal{M}, \eta, s \models B$, or both.

$\mathcal{M}, \eta, s \models \Box A$ iff for every $t \in W$ such that sRt, $\mathcal{M}, \eta, t \models A$.

$\mathcal{M}, \eta, s \models \downarrow i.\, A$ iff $\mathcal{M}, \eta[i{:=}s], s \models A$.

$\mathcal{M}, \eta, s \models @_i A$ iff $\mathcal{M}, \eta, \eta(i) \models A$.

Axiomatization:

1. $\downarrow i.\, (A \supset B) \supset (A \supset \downarrow i.\, B)$, where i does not occur free in A
2. $\downarrow i.\, A \supset (j \supset A[i{:=}j])$
3. $\downarrow i.\, (i \supset A) \supset \downarrow i.\, A$
4. $\downarrow i.\, A \equiv \neg \downarrow i.\, \neg A$

5. $@_i(A \supset B) \supset (@_i A \supset @_i B)$
6. $@_i A \equiv \neg @_i \neg A$
7. $i \wedge A \supset @_i A$

8. $@_i i$

9. $@_i j \supset (@_j A \supset @_i A)$

10. $@_i j \equiv @_j i$

11. $@_i @_j A \equiv @_j A$

12. $\Diamond @_i A \supset @_i A$

13. $\Diamond i \wedge @_i A \supset \Diamond A$

$$\frac{A \supset B \quad A}{B} \qquad \frac{A}{\Box A} \qquad \frac{A}{\downarrow i.\, A} \qquad \frac{A}{@_i A} \qquad \frac{@_i (j \wedge A) \supset B}{@_i A \supset B}\ (*) \qquad \frac{@_i \Diamond (j \wedge A) \supset B}{@_i \Diamond A \supset B}\ (*)$$

(*) j is distinct from i and does not occur free in A or B.

An example:

The binary operator of temporal logic:

$$A \text{ until } B$$

may be defined as:

$$\downarrow i.\, \Diamond \downarrow j. @_i (\Diamond (j \wedge B) \wedge \Box (\Diamond j \supset A))$$

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