Algebra and Coalgebra in the Light Affine Lambda Calculus

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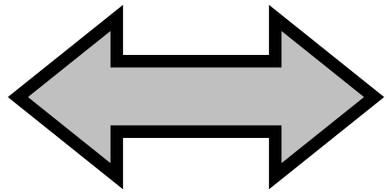
Computability

Complexity

Turing Machines



Lambda Calculus

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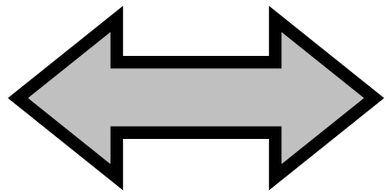
Complexity

Turing Machines



Lambda Calculus

Computability

Complexity

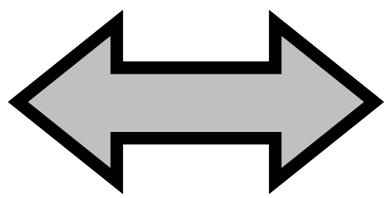
PTIME

NPTIME

LOGSPACE

PSPACE

BPP



LALC

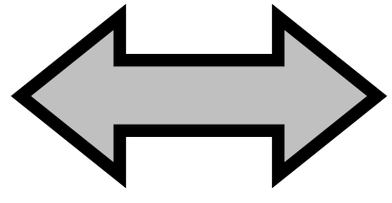
RSLR

INTML

STA

DLAL

Turing Machines



Lambda Calculus

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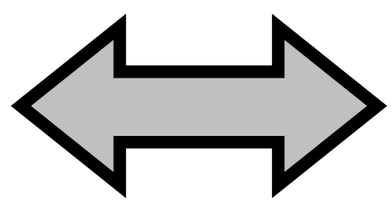

LALC

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Implicit Complexity

Turing Machines

Lambda Calculus

Computability

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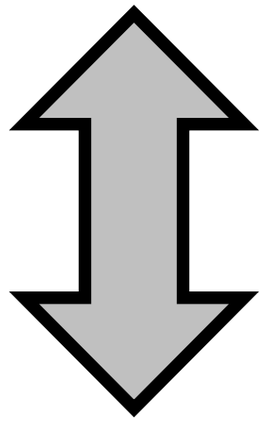
DLAL

(Light Affine Lambda Calculus)

LALC



PTIME

Where these languages could help?

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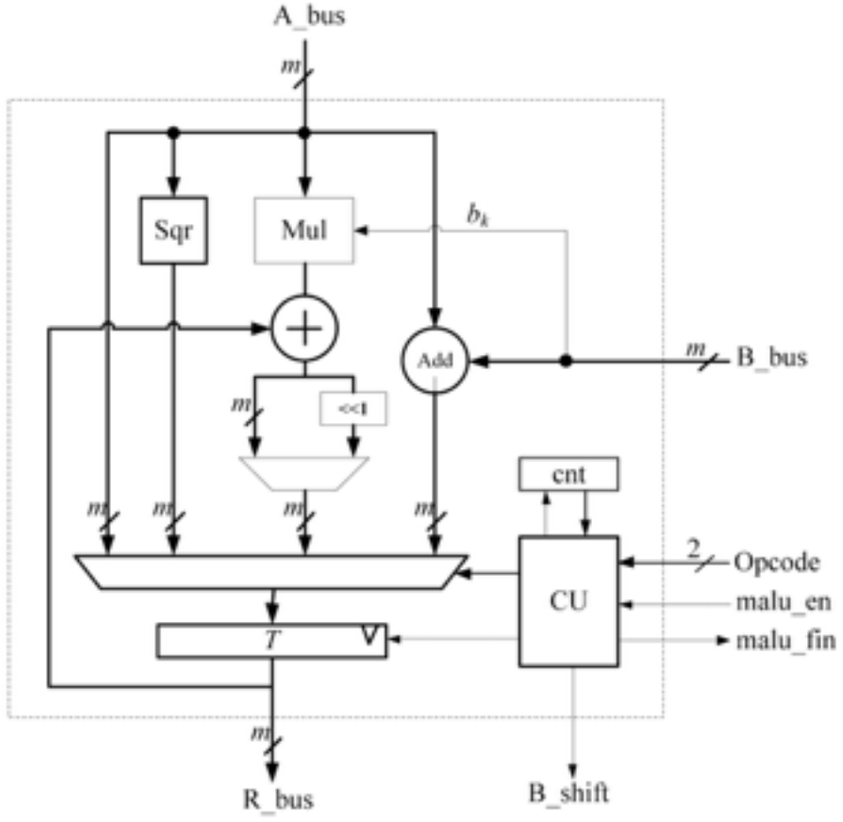
Resource
Analysis

Where these languages could help?

Resource Analysis

Efficient arithmetic implementation



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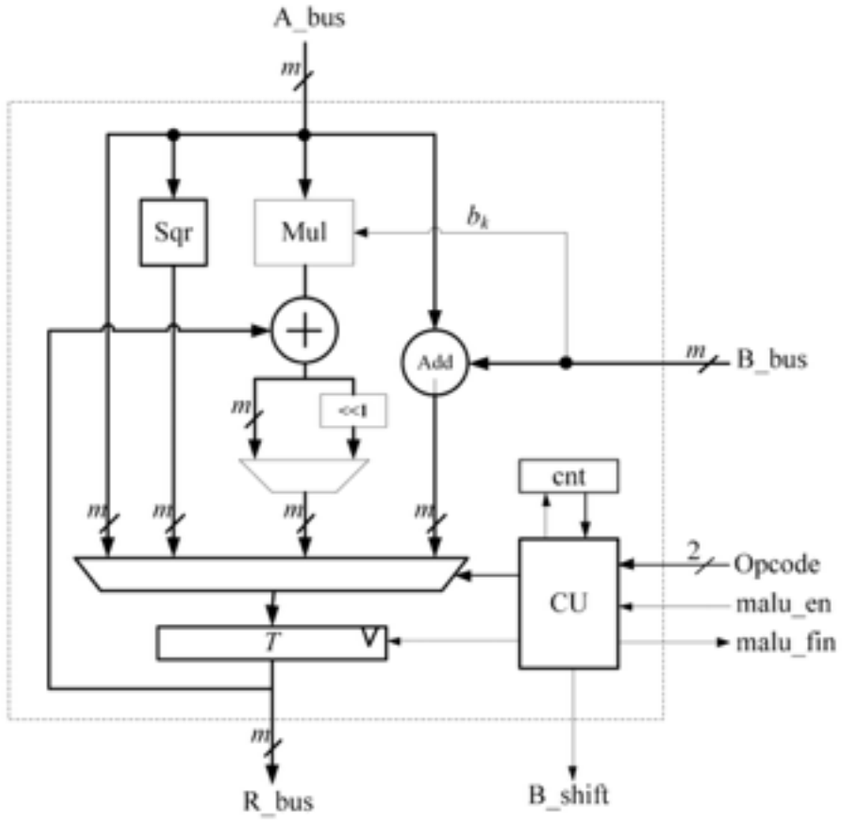
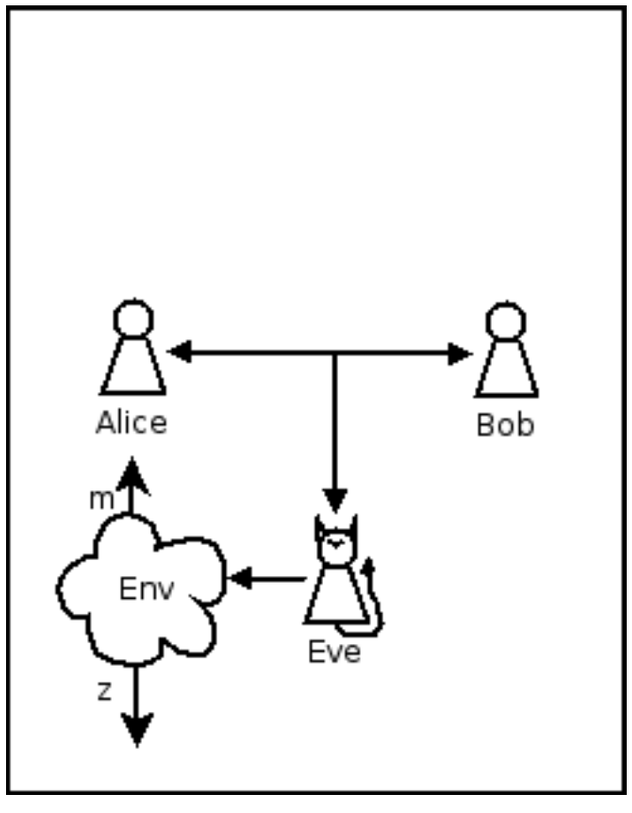


Computational Indistinguishability



PTIME $\Longleftrightarrow$ LALC

Soundness: every LALC program can be run in polynomial time.

Completeness every PTIME Turing Machine can be expressed in LALC.

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Soundness: every LALC program can be run in polynomial time.

Completeness every PTIME Turing Machine can be expressed in LALC.

Expressivity?

Our research question:

Can we express Algebra and Coalgebra in LALC?

Our contribution

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1 - Weak notions of Algebras and Coalgebras can be encoded in LALC.

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2 - Data types:

✔ Inductive types

✗ Coinductive types

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1 - Weak notions of Algebras and Coalgebras can be encoded in LALC.

2 - Data types:

- ✔ Inductive types
- ✗ Coinductive types

3 - LALC restrictions can be slightly relaxed to achieve more expressivity for coinductive types.

Light Affine Lambda Calculus - LALC

LALC $\subset$ Linear (Affine) System F

Light Affine Lambda Calculus - LALC

LALC ⊂ Linear (Affine) System F

Main Idea

$$A \multimap B$$

$A \to B \quad =$

$$!A \multimap \S B$$

Light Affine Lambda Calculus - LALC

LALC ⊂ Linear (Affine) System F

Main Idea

$A \multimap B$

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A non iterative function

-

A function using its argument only once

Light Affine Lambda Calculus - LALC

LALC ⊂ Linear (Affine) System F

Main Idea

$A \multimap B$

$A \to B \quad =$

$!A \multimap \S B$

an iterative function

-

! needed for duplication

§ placeholder witnessing duplication

Iterators in LALC

$$IT : \forall a.!(a \multimap a) \multimap \S(a \multimap a)$$

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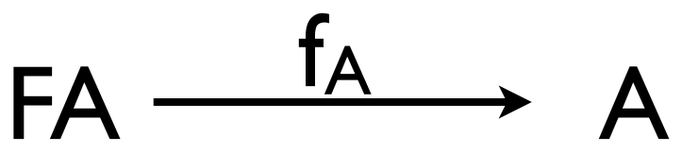
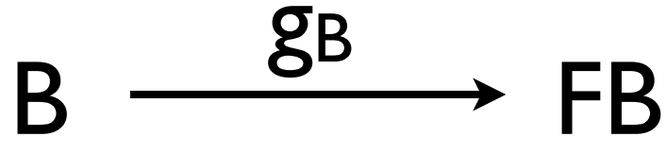
✗ IT A (step: !A ⊸ §A)

Algebras and Coalgebras

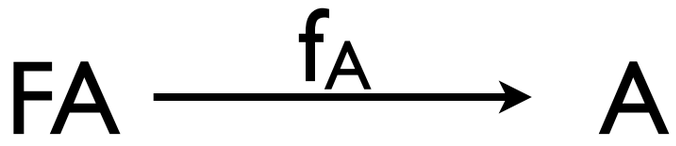


Algebra

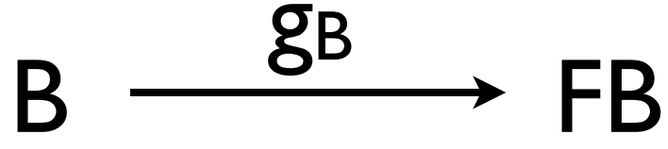


Coalgebra

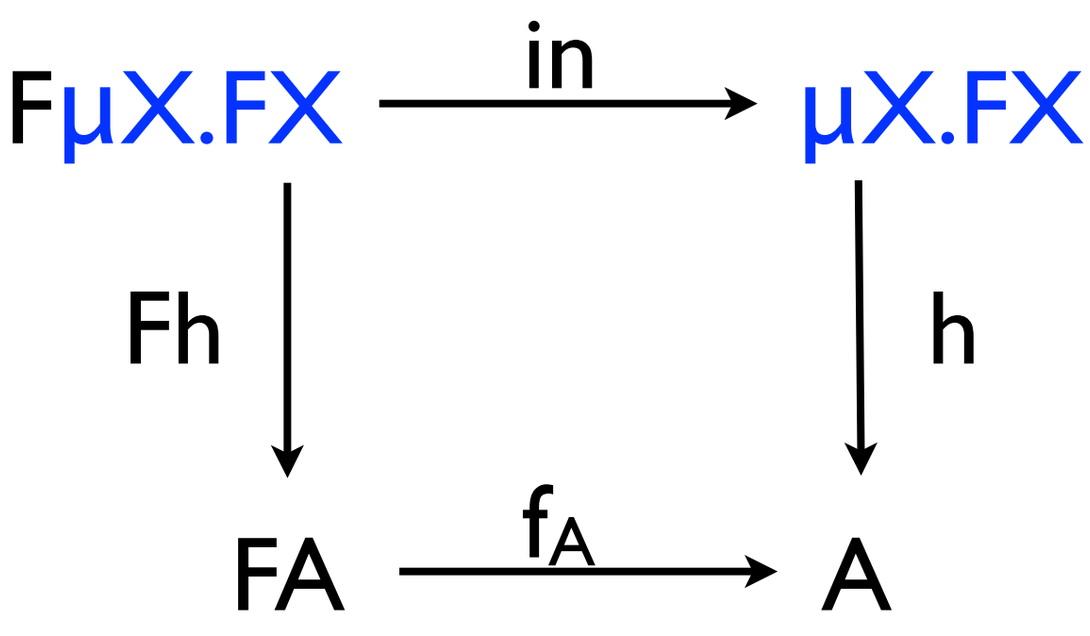
Algebras and Coalgebras



Algebra



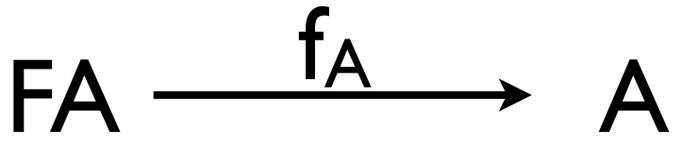
Coalgebra



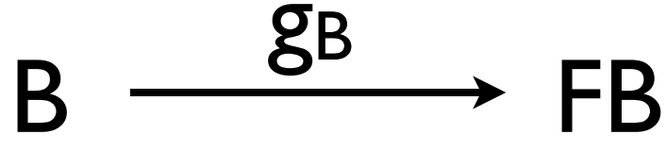
Weakly Initial Algebra

∀A, ∃ h

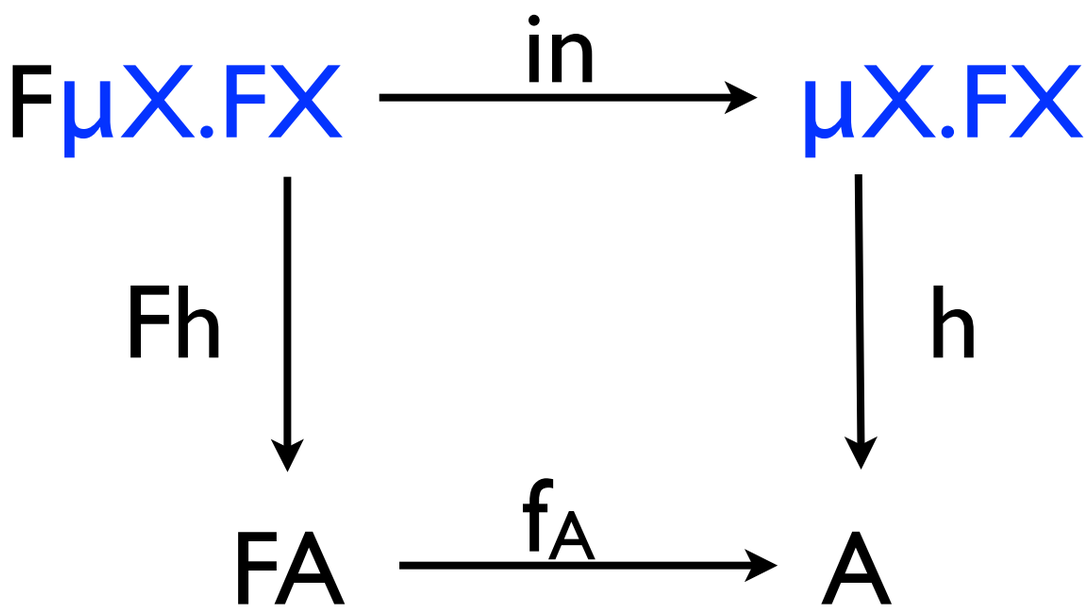
Algebras and Coalgebras



Algebra



Coalgebra



Weakly Initial Algebra

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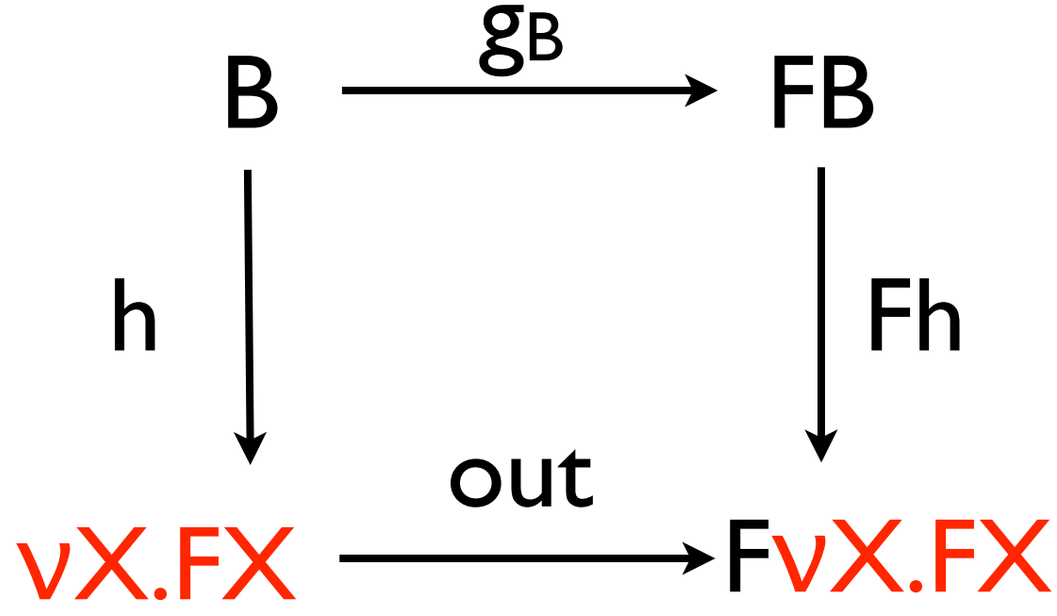

Weakly Final Coalgebra

∀B, ∃h

Examples

	Initial Algebra	Final Coalgebra
$F(-) = 1 + (-)$	$\mathbb{N}$	$\mathbb{N} \cup \{\infty\}$
$F(-) = 1 + A \times (-)$	A^*	A^∞
$F(-) = 1 + A \times (-) \times (-)$	$T^*(A)$	$T^\infty(A)$

(co)algebras in System F

Theorem [Reynolds, Plotkin, Geuvers, …]:
Given F expressible in the polymorphic lambda calculus:

- there exists a weakly initial F-Algebra
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Wraith-Wadler encoding:

$$\mu a.Fa = \forall a.(Fa \to a) \to a$$

$$\nu a.Fa = \exists a.(a \to Fa) * a$$

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F-algebra!

F-coalgebra!

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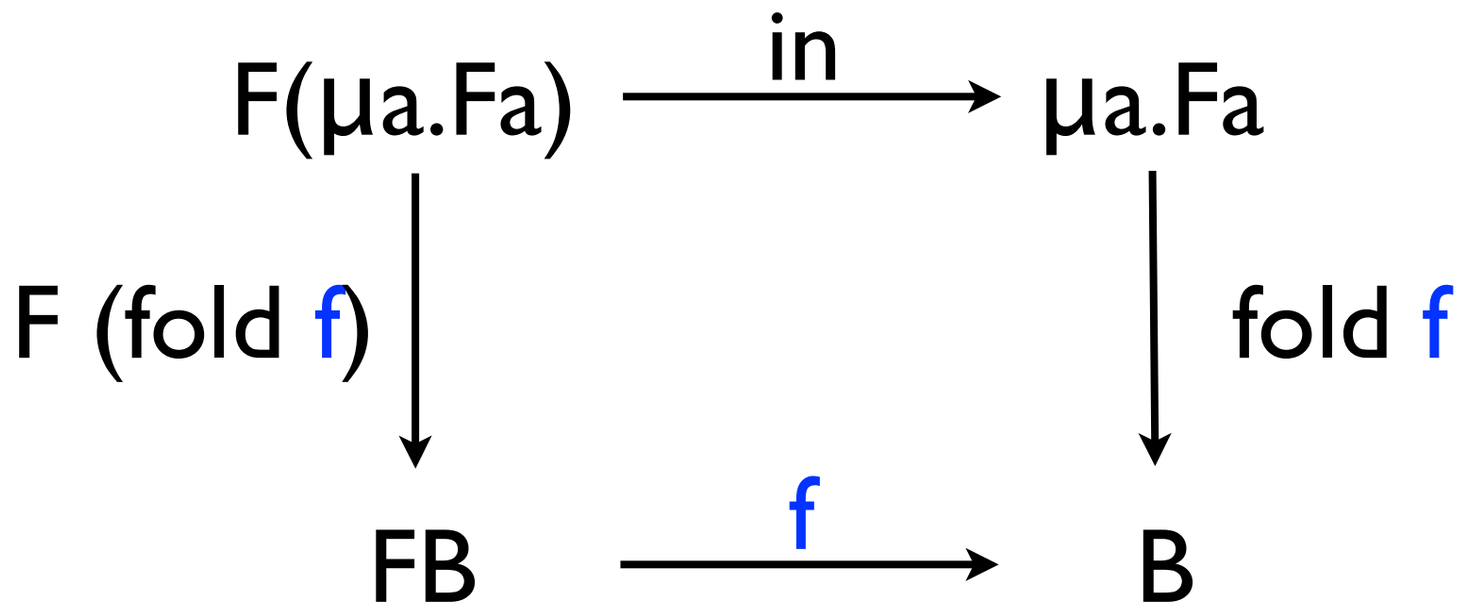

in : F(μa.Fa) → μa.Fa
in = λs.λk.k (F (fold k) s)

fold : ∀b. (Fb → b) → μa.Fa → b
fold = λf.λt. t f

μa.Fa = ∀a.(Fa→a)→a

Wraith-Wadler encoding in LALC

$$\mu a.Fa = \forall a.\ (Fa \to a) \to a$$

$$\begin{aligned}
&in : \ F(\mu a.Fa) \to \mu a.Fa \\
&in = \lambda s.\lambda k.\ k\ (F\ (fold\ \ k)\ s) \\
\\
&fold : \forall b.\ (Fb \to b) \to \mu a.Fa \to b \\
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\end{aligned}$$

Wraith-Wadler encoding in LALC

$$\mu a.Fa = \forall a.\ (Fa \to a) \to a$$

k need to be duplicated!

$$in : F(\mu a.Fa) \to \mu a.Fa$$

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$$fold : \forall b.\ (Fb \to b) \to \mu a.Fa \to b$$

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Wraith-Wadler encoding in LALC

$$\mu a.Fa = \forall a.!(Fa \multimap a) \multimap \S a$$

Let's change the type!

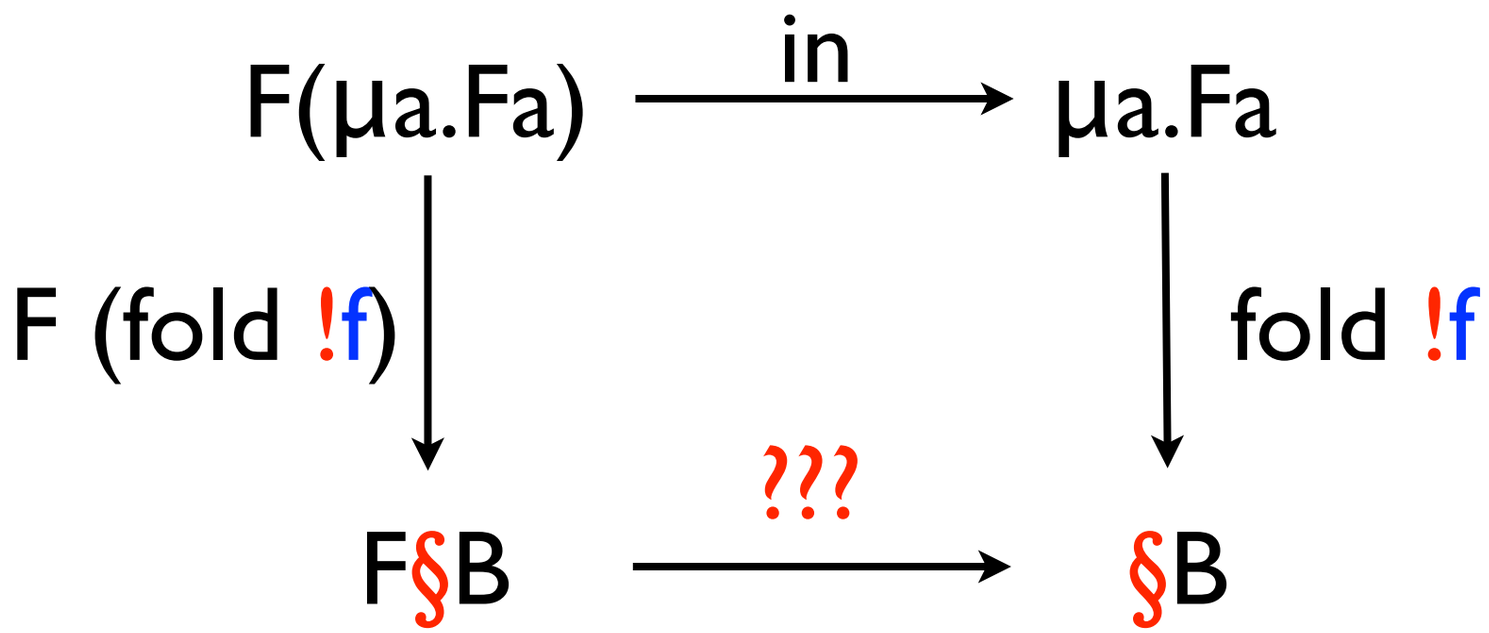
$in : F(\mu a.Fa) \multimap \mu a.Fa$

$in = \lambda s.\lambda k.\S k\ (F\ (fold\ !k)\ s)$

$fold : \forall b.!(Fb \multimap b) \multimap \mu a.Fa \multimap \S b$

$fold = \lambda f.\lambda t.\ t\ !f$

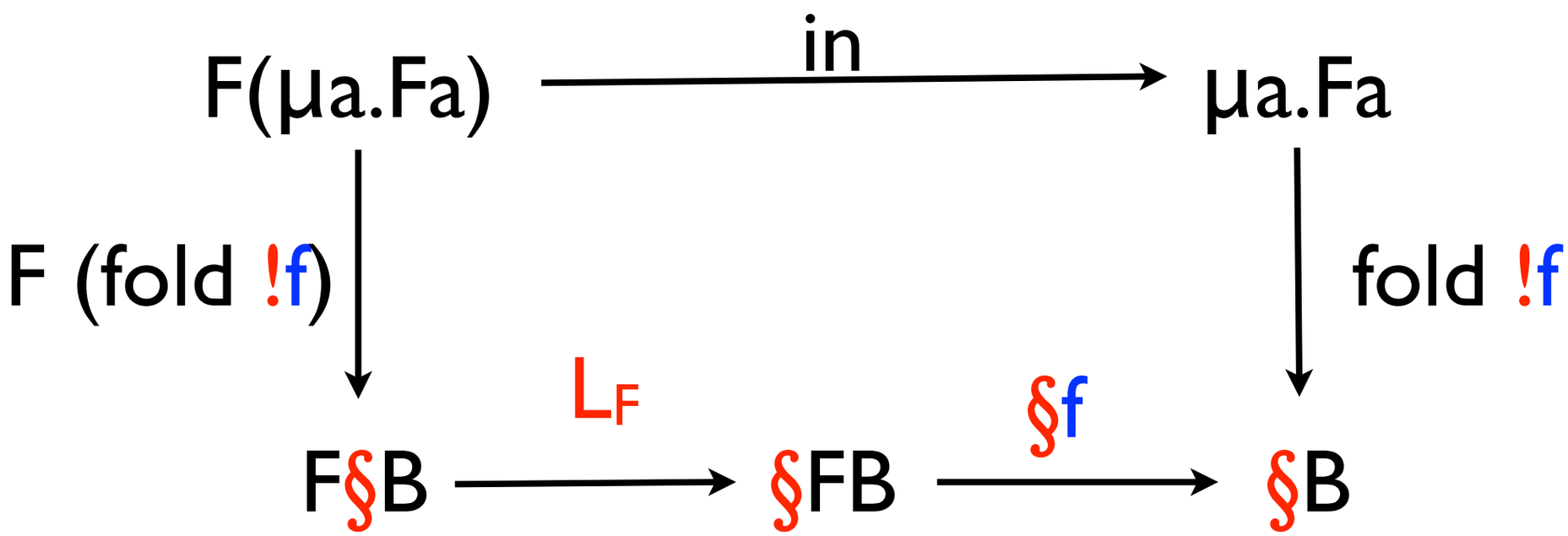
Wraith-Wadler encoding in LALC



in : $F(\mu a.Fa) \multimap \mu a.Fa$

fold : $\forall b.!(Fb \multimap b) \multimap \mu a.Fa \multimap \S b$

Weakly-Initial algebra under §

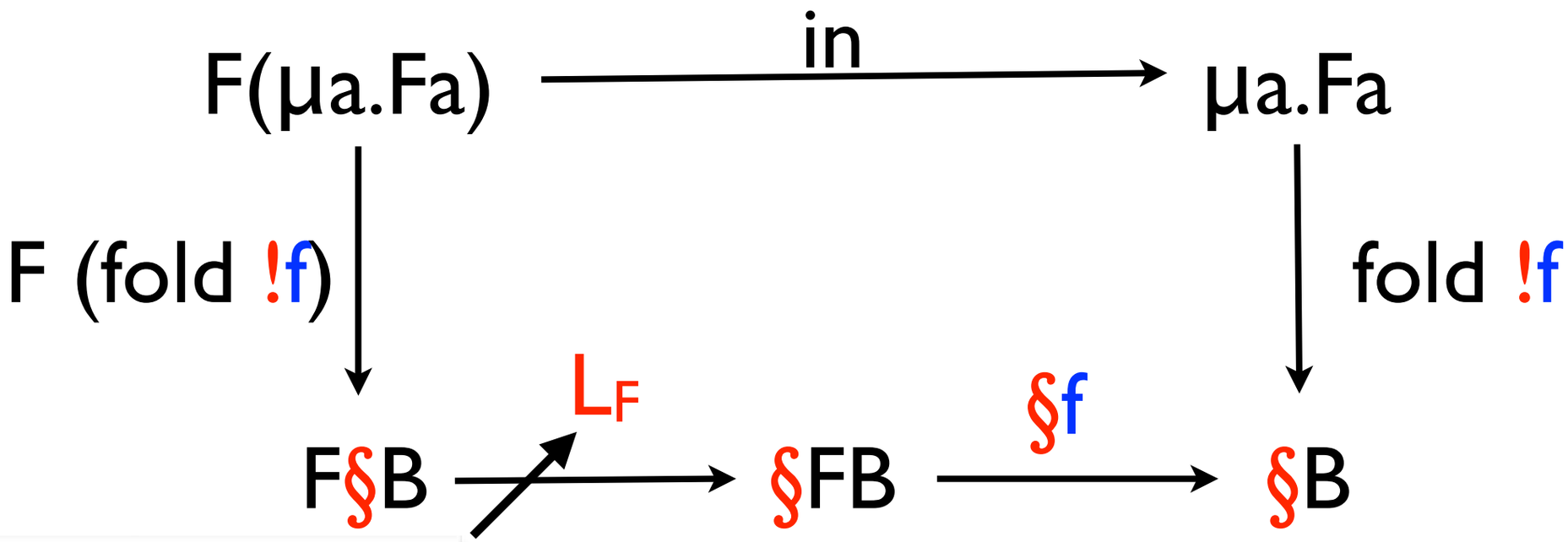


in : F(μa.Fa) ⊸ μa.Fa

L_F : ∀b. F§b ⊸ §Fb

fold : ∀b.!(Fb ⊸ b) ⊸ μa.Fa ⊸ §b

Weakly-Initial algebra under §



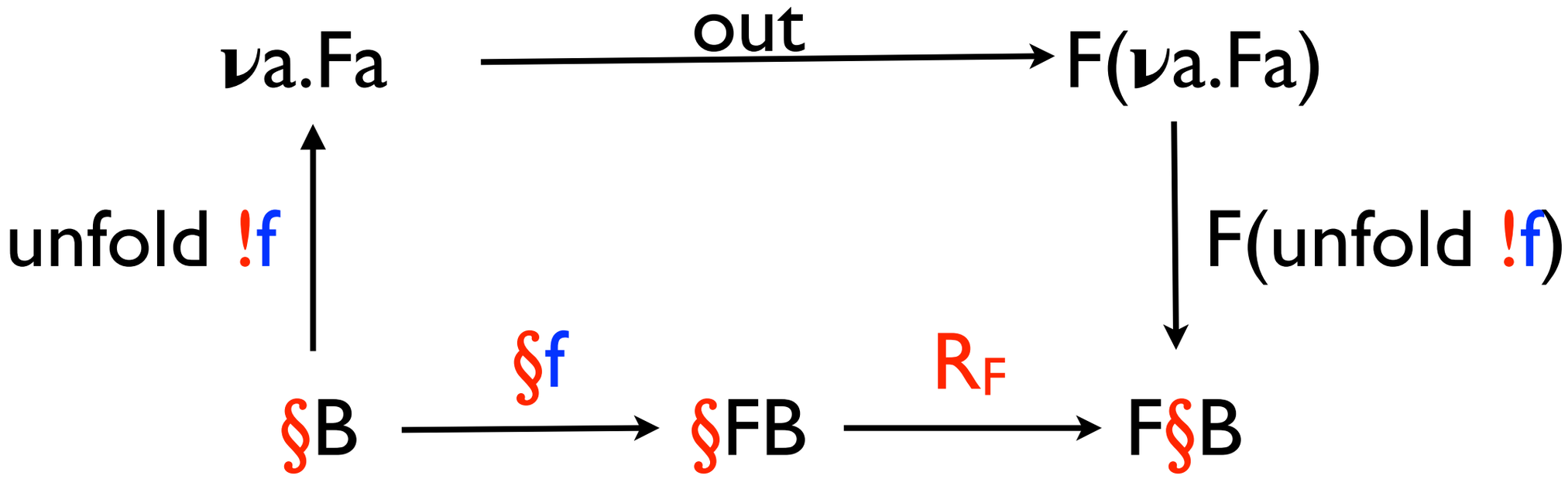
Left distributivity

in : $F(\mu a.Fa) \multimap \mu a.Fa$

$L_F : \forall b.\ F\S b \multimap \S Fb$

fold : $\forall b.!(Fb \multimap b) \multimap \mu a.Fa \multimap \S b$

Weakly-Final coalgebra under §



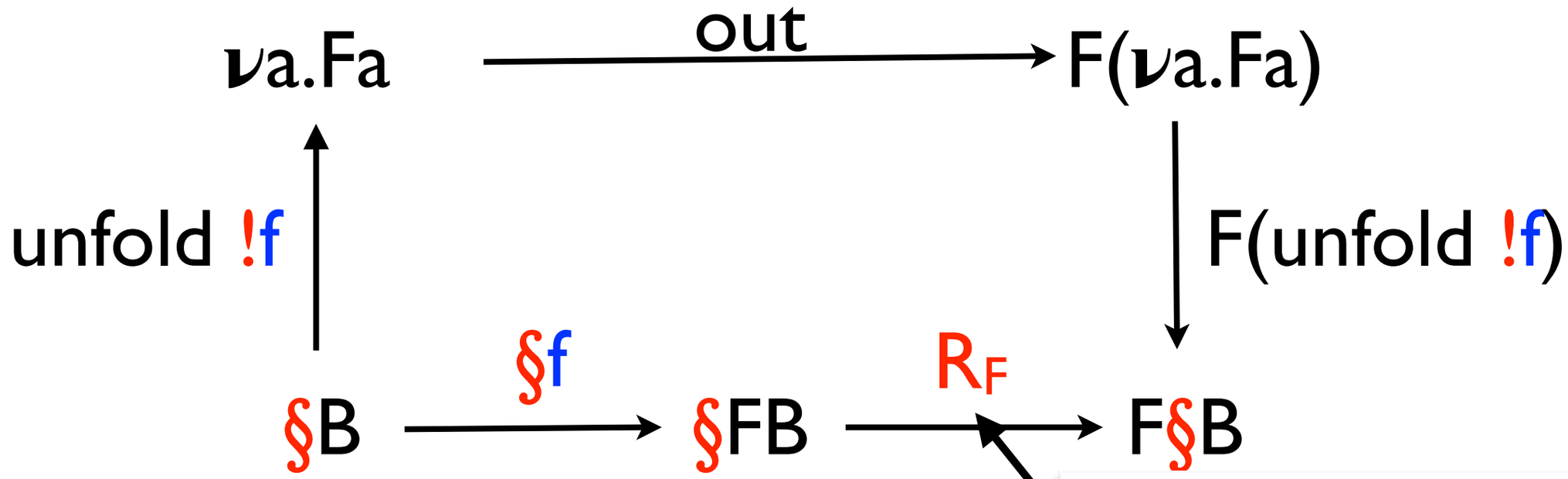
out : $\nu a.Fa \multimap F(\nu a.Fa)$

$R_F : \forall b.\ \S Fb \multimap F\S b$

unfold: $\forall b.!(b \multimap Fb) \multimap \S b \multimap \nu a.Fa$

$\nu a.Fa = \exists a.!(a \multimap Fa) \otimes \S a$

Weakly-Final coalgebra under §



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Expressivity?

- Which Functor satisfies L_F: $\forall b.\ F\S b \multimap \S Fb$?

$$F(-) = I \oplus -$$

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provided $\vdash A \multimap \S A$

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Where is the problem?

The modality § does not commute with the other type constructions!

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The modality § does not commute with the other type constructions!

Solution: make § to commute

Adding terms for distributivity

We can add to LALC the following terms:

$\mathrm{dist}_{\oplus} : \S(A \oplus B) \multimap \S A \oplus \S B$

$\mathrm{dist}_{\otimes} : \S(A \otimes B) \multimap \S A \otimes \S B$

$\mathrm{dist}_{\oplus}\ \S(\mathrm{inj}\ t) \rightarrow \mathrm{inj}\ \S t$

$\mathrm{dist}_{\otimes}\ \S(\langle t_1, t_2\rangle) \rightarrow \langle \S t_1, \S t_2\rangle$

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They require the evaluation of terms inside a §

Problem: this breaks polynomial time soundness.

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Problem: this breaks polynomial time soundness.

New (quite complex) proof in the paper!

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provided $\vdash \S A \multimap A$

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We can have streams and trees at every finite type

Take out?

- Algebras and Coalgebras encodings make sense also for polynomial time languages,
- Due to the restrictive nature of languages for implicit complexity their definitions can be a bit more tricky,
- The expressivity may still depend on the restrictions of the language.