Algebra and Coalgebra in the Light Affine Lambda Calculus

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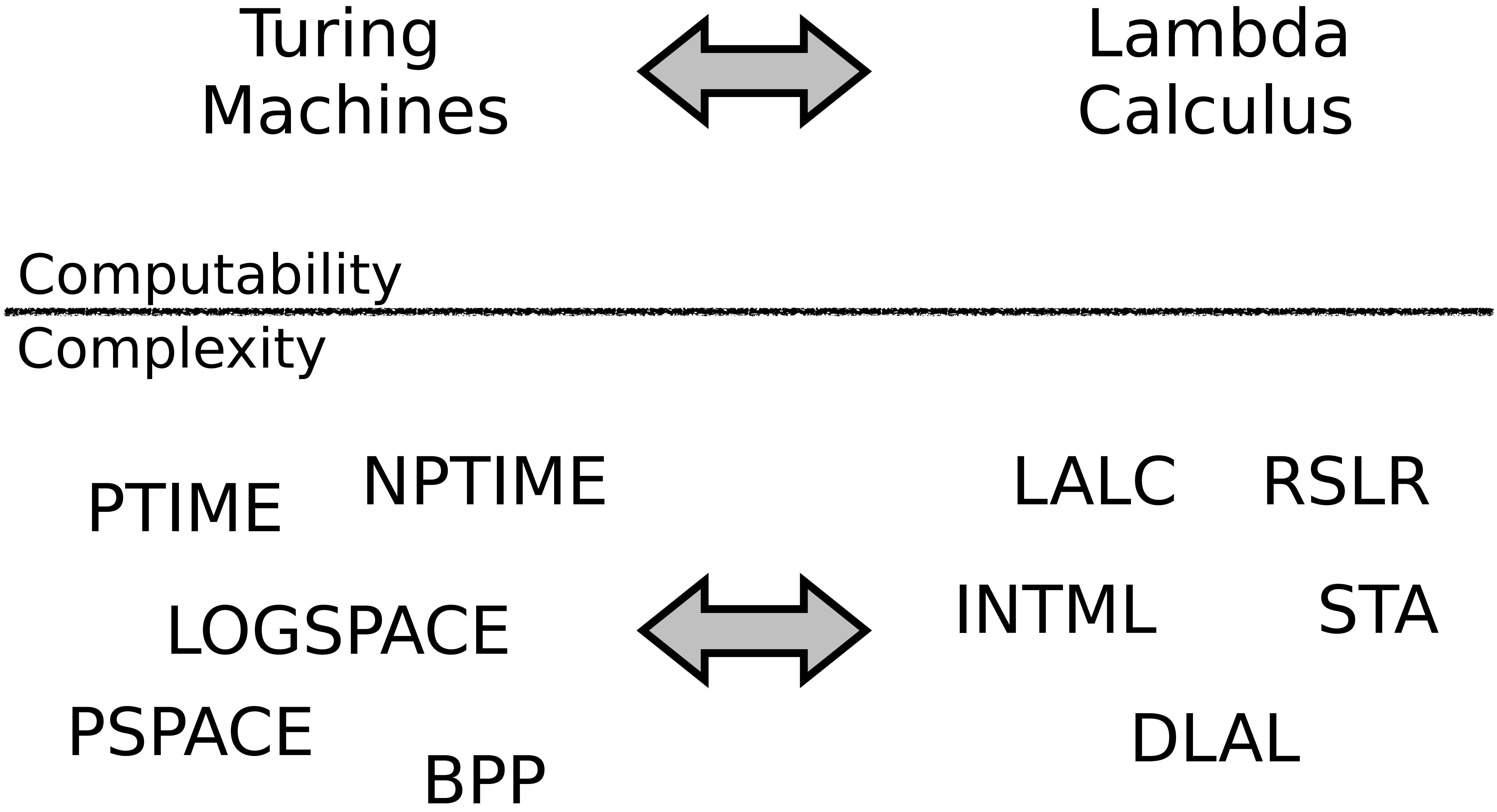
Computability

Complexity

Turing Machines ⟺ Lambda Calculus

Computability

Complexity



Turing Machines

Lambda Calculus

Computability

Complexity

PTIME

NPTIME

LOGSPACE

PSPACE

BPP

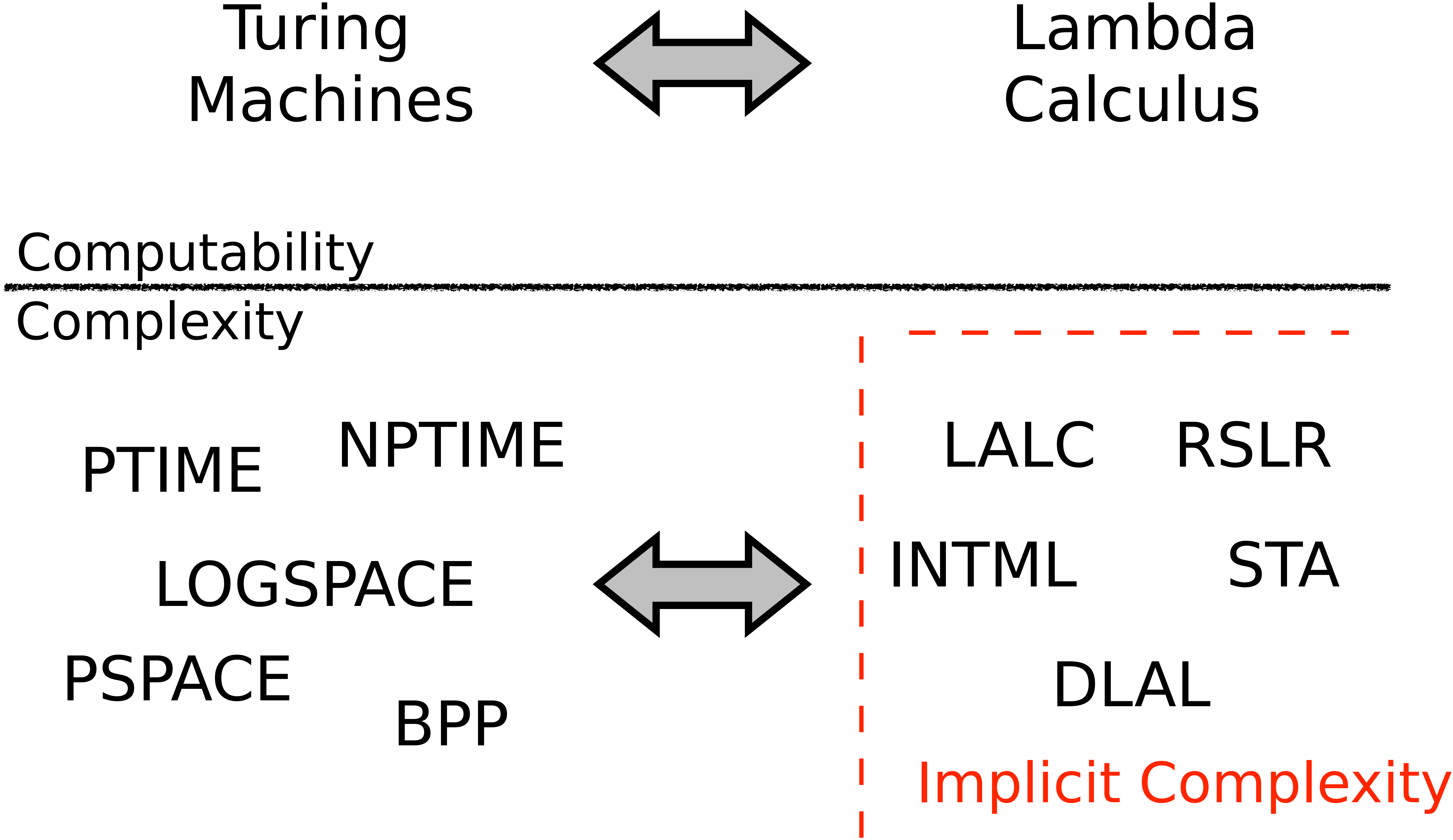
LALC

RSLR

INTML

STA

DLAL



Turing Machines

Lambda Calculus

Computability

Complexity

PTIME

NPTIME

LOGSPACE

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Implicit Complexity

Turing Machines

Lambda Calculus

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NPTIME
PTIME
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LALC
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STA
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(Light Affine Lambda Calculus)

LALC

⇕

PTIME

Where these languages could help?

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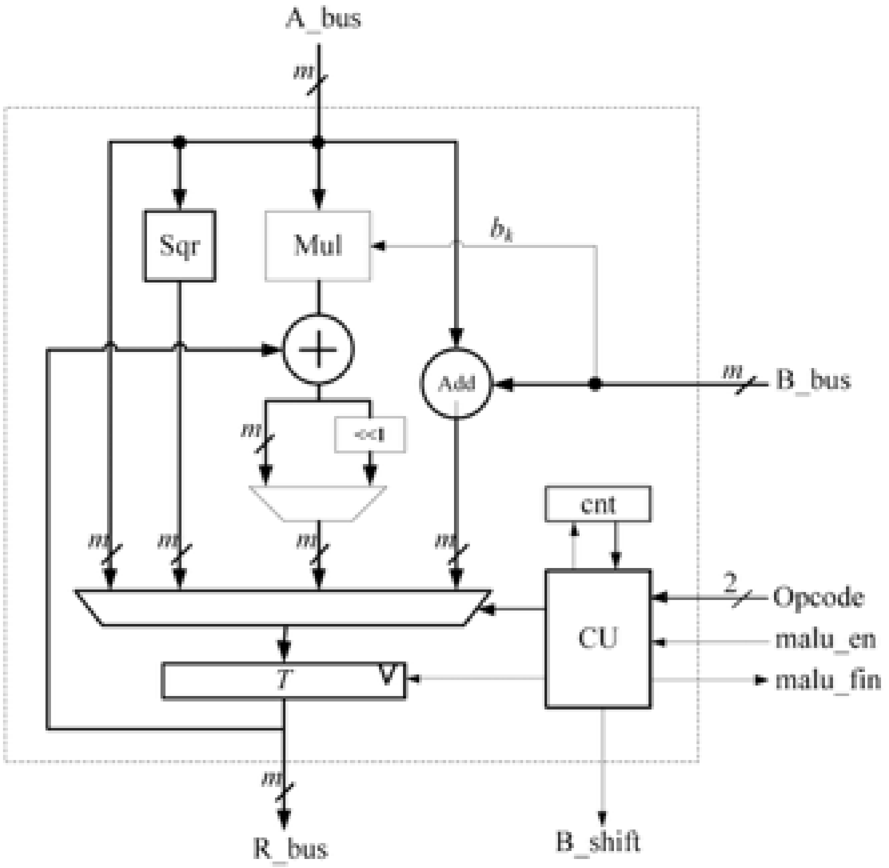
Resource
Analysis

Where these languages could help?

Resource Analysis

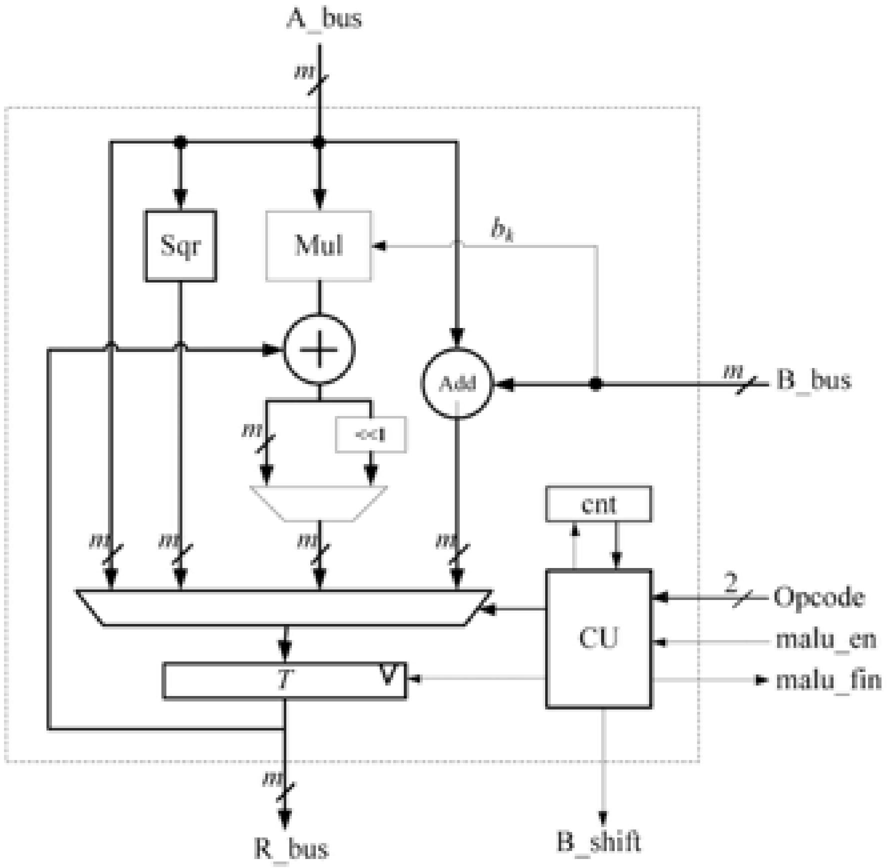
Efficient arithmetic implementation



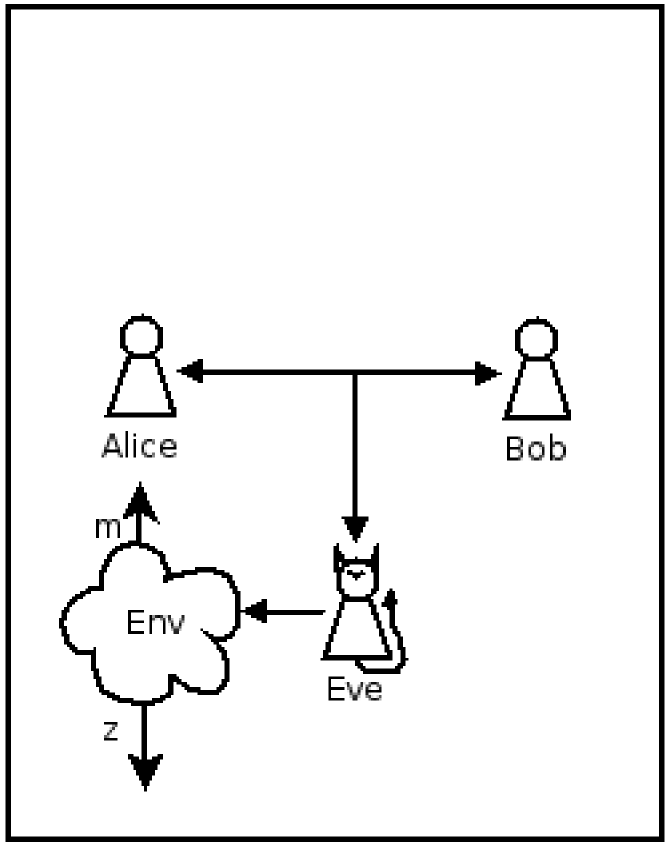
Where these languages could help?

Resource Analysis

Efficient arithmetic implementation



Computational Indistinguishability



PTIME ⟺ LALC

Soundness: every LALC program can be run in polynomial time.

Completeness every PTIME Turing Machine can be expressed in LALC.

PTIME ⇔ LALC

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Completeness every PTIME Turing Machine can be expressed in LALC.

Expressivity?

Our research question:

Can we express Algebra and Coalgebra in LALC ?

Our contribution

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1 - Weak notions of Algebras and Coalgebras can be encoded in LALC.

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2 - Data types:

- ✓ Inductive types
- ✗ Coinductive types

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1 - Weak notions of Algebras and Coalgebras can be encoded in LALC.

2 - Data types:

✓ Inductive types
✗ Coinductive types

3 - LALC restrictions can be relaxed to achieve more expressivity for coinductive types.

Light Affine Lambda Calculus - LALC

LALC ⊂ Linear (Affine) System F

Light Affine Lambda Calculus - LALC

LALC ⊂ Linear (Affine) System F

Main Idea

$$A \multimap B$$

$$A \to B \quad =$$

$$!A \multimap \S B$$

Light Affine Lambda Calculus - LALC

LALC ⊂ Linear (Affine) System F

Main Idea

$$A \to B \quad = \quad \begin{array}{c} A \multimap B \\ \\ !A \multimap \S B \end{array}$$

A non iterative function

-

A function using its argument only once

Light Affine Lambda Calculus - LALC

LALC ⊂ Linear (Affine) System F

Main Idea

$$A \multimap B$$

$$A \to B \quad =$$

$$!A \multimap \S B$$

an iterative function

-

! needed for duplication

§ placeholder witnessing duplication

Iterators in LALC

$$IT : \forall a. !(a \multimap a) \multimap \S(a \multimap a)$$

Iterators in LALC

$$\mathrm{IT} : \forall a.\,!(a \multimap a) \multimap \S(a \multimap a)$$

✓ IT A (step: $A \multimap A$)

Iterators in LALC

$\mathrm{IT} : \forall a.\,!(a \multimap a) \multimap \S(a \multimap a)$

✓ $\mathrm{IT}\;\; A\;(\text{step: } A \multimap A)$

✗ $\mathrm{IT}\;\; A\;(\text{step: } !A \multimap \S A)$

Algebras and Coalgeb

$$\begin{tikzcd} FA \arrow[r, "f_A"] & A \end{tikzcd}$$

Algebra

$$\begin{tikzcd} B \arrow[r, "g_B"] & FB \end{tikzcd}$$

Coalgebra

Algebras and Coalgeb

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Algebra

$$\begin{tikzcd} B \arrow[r, "g_B"] & FB \end{tikzcd}$$

Coalgebra

$$\begin{tikzcd} F\mu X.FX \arrow[r, "in"] \arrow[d, "Fh"'] & \mu X.FX \arrow[d, "h"] \\ FA \arrow[r, "f_A"] & A \end{tikzcd}$$

Weak Initial Algebra

$\forall A, \exists\ h$

Algebras and Coalgeb

$$\begin{tikzcd}
FA \arrow[r, "f_A"] & A
\end{tikzcd}$$

Algebra

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FA \arrow[r, "f_A"] & A
\end{tikzcd}$$

Weak Initial Algebra

$\forall A, \exists h$

$$\begin{tikzcd}
B \arrow[r, "g_B"] \arrow[d, "h"'] & FB \arrow[d, "Fh"] \\
\nu X.FX \arrow[r, "out"] & F\nu X.FX
\end{tikzcd}$$

Weak Final Coalgebra

$\forall B, \exists h$

Examples

	Initial Algebra	Final Coalgebra
$F(-) = 1 + (-)$	N	$N \cup \{\infty\}$
$F(-) = 1 + A \times (-)$	A^*	A^∞
$F(-) = 1 + A \times (-) \times (-)$	$T^*(A)$	$T^\infty(A)$

(co)algebras in System *F*

Theorem [Reynolds, Plotkin, Geuvers, ...]:

Given F expressible in the polymorphic LC:

- there exists a weakly initial F-Algebra
- there exists a weakly final F-Coalgebra

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Wraith-Wadler encoding:

$$\mu a.Fa = \forall a.(Fa \to a) \to a$$

$$\nu a.Fa = \exists a.(a \to Fa) * a$$

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F-coalgebra!

Wraith-Wadler encoding

$$\begin{tikzcd}
F(\mu a.Fa) \arrow[r, "in"] \arrow[d, "F\ (fold\ f)"'] & \mu a.Fa \arrow[d, "fold\ f"] \\
FB \arrow[r, "f"] & B
\end{tikzcd}$$

in : F(μa.Fa) → μa.Fa

in = λs.λk.k (F (fold k) s)

fold : ∀b. (Fb → b) → μa.Fa → b

fold = λf.λt. t f

μa.Fa = ∀a.(Fa→a)→a

Wraith-Wadler encoding in LALC

$\mu a.Fa = \forall a.\ (Fa \to a) \to a$

$\text{in} : F(\mu a.Fa) \to \mu a.Fa$

$\text{in} = \lambda s.\lambda k.\ k\ (F\ (\text{fold}\ k)\ s)$

$\text{fold} : \forall b.\ (Fb \to b) \to \mu a.Fa \to b$

$\text{fold} = \lambda f.\lambda t.\ t\ f$

Wraith-Wadler encoding in LALC

$$\mu a.Fa = \forall a.\ (Fa \to a) \to a$$

k need to be duplicated!

$$\mathrm{in} : F(\mu a.Fa) \to \mu a.Fa$$

$$\mathrm{in} = \lambda s.\lambda k.\ k\ (F\ (\mathrm{fold}\ k)\ s)$$

$$\mathrm{fold} : \forall b.\ (Fb \to b) \to \mu a.Fa \to b$$

$$\mathrm{fold} = \lambda f.\lambda t.\ t\ f$$

Wraith-Wadler encoding in LALC

$$\mu a.Fa = \forall a.!(Fa \multimap a) \multimap \S a$$

Let’s change the type!

$$\mathrm{in} : \ F(\mu a.Fa) \multimap \mu a.Fa$$
$$\mathrm{in} = \lambda s.\lambda k.\S k \ (F \ (\mathrm{fold} \ !k) \ s)$$

$$\mathrm{fold} : \forall b.!(Fb \multimap b) \multimap \mu a.Fa \multimap \S b$$
$$\mathrm{fold} = \lambda f.\lambda t. \ t \ !f$$

Wraith-Wadler encoding in LALC

$$\begin{tikzcd} F(\mu a.Fa) \arrow[r, "\mathrm{in}"] \arrow[d, "F\ (\mathrm{fold}\ !f)"'] & \mu a.Fa \arrow[d, "\mathrm{fold}\ !f"] \\ F\S B \arrow[r, "???"] & \S B \end{tikzcd}$$

$\mathrm{in} : \ F(\mu a.Fa) \multimap \mu a.Fa$

$\mathrm{fold} : \forall b.!(Fb \multimap b) \multimap \mu a.Fa \multimap \S b$

Weakly-Initial algebra under §

$$\begin{tikzcd}
F(\mu a.Fa) \arrow[rr, "in"] \arrow[d, "F\ (fold\ !f)"'] & & \mu a.Fa \arrow[d, "fold\ !f"] \\
F\S B \arrow[r, "L_F"] & \S FB \arrow[r, "\S f"] & \S B
\end{tikzcd}$$

$in :\ F(\mu a.Fa) \multimap \mu a.Fa$

$L_F : \forall b.\ F\S b \multimap \S Fb$

$fold : \forall b.!(Fb \multimap b) \multimap \mu a.Fa \multimap \S b$

Weakly-Initial algebra under §

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Left distributivity

$in :\ F(\mu a.Fa) \multimap \mu a.Fa$

$L_F : \forall b.\ F\S b \multimap \S Fb$

$fold : \forall b.!(Fb \multimap b) \multimap \mu a.Fa \multimap \S b$

Weakly-Final coalgebra under §

$$\begin{tikzcd}
\nu a.Fa \arrow[rr, "\mathrm{out}"] & & F(\nu a.Fa) \arrow[d, "F(\mathrm{unfold}\ !f)"] \\
\S B \arrow[u, "\mathrm{unfold}\ !f"] \arrow[r, "\S f"] & \S FB \arrow[r, "R_F"] & F\S B
\end{tikzcd}$$

$\mathrm{out} : \nu a.Fa \multimap F(\nu a.Fa)$

$R_F : \forall b.\ \S Fb \multimap F\S b$

$\mathrm{unfold} : \forall b.!(b \multimap Fb) \multimap \S b \multimap \nu a.Fa$

$\nu a.Fa = \exists a.!(a \multimap Fa) \otimes \S a$

Weakly-Final coalgebra under §

$$\begin{tikzcd}
\nu a.Fa \arrow[rr, "\mathrm{out}"] & & F(\nu a.Fa) \arrow[d, "F(\mathrm{unfold}\ !f)"] \\
\S B \arrow[u, "\mathrm{unfold}\ !f"] \arrow[r, "\S f"] & \S FB \arrow[r, "R_F"] & F\S B
\end{tikzcd}$$

Right distributivity

$\mathrm{out} : \nu a.Fa \multimap F(\nu a.Fa)$

$R_F : \forall b.\ \S Fb \multimap F\S b$

$\mathrm{unfold} : \forall b.!(b \multimap Fb) \multimap \S b \multimap \nu a.Fa$

$\nu a.Fa = \exists a.!(a \multimap Fa) \otimes \S a$

Expressivity?

- Which Functor satisfies $L_F : \forall b.\ F\S b \multimap \S F b$?

$$F(-) = 1 \oplus -$$

$$F(-) = 1 \oplus A \otimes -$$

$$F(-) = 1 \oplus A \otimes - \otimes -$$

Expressivity?

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provided $\vdash A \multimap \S A$

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Where is the problem?

The modality § does not commute with the other type constructions!

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Solution: make § to commute

Adding terms for distributivity

We can add to LALC the following terms:

$\mathrm{dist}_{\oplus} : \S(A \oplus B) \multimap \S A \oplus \S B$

$\mathrm{dist}_{\otimes} : \S(A \otimes B) \multimap \S A \otimes \S B$

$\mathrm{dist}_{\oplus}\ \S(\mathrm{inj}\ t) \rightarrow \mathrm{inj}\ \S t$

$\mathrm{dist}_{\otimes}\ \S(<t_1,t_2>) \rightarrow <\S t_1,\S t_2>$

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They require the evaluation of terms inside a §

Problem: this breaks polynomial time soundness.

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They require the evaluation of terms inside a §

Problem: this breaks polynomial time soundness.

New (quite technical) proof in the paper!

Expressivity?

- Which Functor satisfies R_F: $\forall b.\ \S F b \multimap F \S b$

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$$F(-) = 1 \oplus A \otimes -$$

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Expressivity?

- Which Functor satisfies R_F: $\forall b.\ \S F b \multimap F \S b$

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provided $\vdash \S A \multimap A$

We can encode streams and trees at every finite type

Take out?

- Algebras and Coalgebras encodings make sense also for polynomial time languages,
- Due to the restrictive nature of languages for implicit complexity their definitions can be a bit more tricky,
- The expressivity may still depend on the restrictions of the language.