Asset Pricing in Dynamic \((B, S)\)-Markets

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Abstract. We study the model \((B, S, \pi, X^\pi)\) of asset pricing where the probabilities of the dynamic evolution of risky asset \(S\) follow a dynamic random walk model. A thorough theory from equivalence of martingale-measures, the fundamental theorem of dynamic asset pricing, the completeness of the related financial market, the fair price and related hedging strategy are presented. The standard static binomial model is contained as a special case. We are aiming at deriving a dynamic formula which is similar to the Cox-Rubinstein formula for European options in the static binomial model.

Keywords: Dynamic asset pricing; \((B, S, \pi, X^\pi)\)-market model; Stochastic difference equations; Equivalent martingale-measures; Discrete Girsanov transform; Fundamental theorem of asset pricing; Fair price; Hedging; European option; Dynamic Cox-Rubinstein formula; Binary market model; Convergence to Black-Scholes model

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1 Introduction

Many authors have investigated the discrete \((B, S)\)-model for the dynamic pricing of derivative securities in financial markets, e.g. see Föllmer and Schied [9], Melnikov [29] and Shiryaev [40], originating from the analysis of Cox, Ross and Rubinstein [4]. Usually, the dynamics of the current amount hold in the fairly risky asset \(B\) is modelled by the stochastic process \(B = \{B_n : n \in \mathbb{N}\}\) and the current amount of the risky asset \(S\) by the stochastic process \(S = \{S_n : n \in \mathbb{N}\}\). Under some fairly mild assumption of having nonnegative semimartingale structure for \(B\) and \(S\) (cf. multiplicative decompositions of nonnegative semimartingales, e.g. in Melnikov [29]), one can show that both \(B\) and \(S\) must be governed by the linear stochastic

1
difference equations

\[ \Delta B_n = r_n B_{n-1}, \]
\[ \Delta S_n = \rho_n S_{n-1}, \]

(1) (2)

where \( \Delta B_n = B_n - B_{n-1} \) and \( \Delta S_n = S_n - S_{n-1} \) for \( n \geq 1 \), started at the nonrandom initial vector \( (B_0, S_0) \in \mathbb{R}_+^2 \) at \( n = 0 \) and driven by the stochastic processes \( r = \{r_n : n \in \mathbb{N}\} \) and \( \rho = \{\rho_n : n \in \mathbb{N}\} \). Throughout this exposition, we suppose that \( B, S, r, \rho \) are defined over the complete, filtered probability space \( (\Omega, \mathcal{F}, \{\mathcal{F}_n\}_{n \in \mathbb{N}}, \mathbb{P}) \). The stochastic process \( r \) is interpreted as the interest rate of dynamics of the fairly riskless asset \( B \) and \( \rho \) as the interest rate belonging to the risky asset \( S \). For the sake of simplicity, we confine us to the more practice-relevant case of finite atomic sample spaces \( \Omega \) (i.e. \( \#(\Omega) < +\infty \)) with \( \mathbb{P}(\{\omega_i\}) > 0 \) for all \( \omega_i \in \Omega \).

We are aiming at deriving pricing and hedging formulas for the \((B, S)\)-market governed by (1) and (2) with dynamic probabilities for the interest rates \( \rho \) instead of the commonly studied binomial model with static probabilities for the distribution of \( \rho \). For this purpose, we shall study existence and equivalence of martingale measures, the fundamental theorem of dynamic asset pricing with dynamic multinomial distributions, the completeness of the dynamic \((B, S)\)-market, the fair price and optimal hedging strategies. In order to compute a fair price and hedging strategies, conditions on the parameters of \( \rho \) and \( r \) need to be found to guarantee the (unique) existence of equivalent martingale-measure \( \mathbb{P}^* \) to the original measure \( \mathbb{P} \). The approach we pursue should be computationally feasible, efficient and practice-relevant. That is why we prefer the discrete model of \((B, S)\)-market instead of continuous models. The dynamic aspect is obvious since the real data in the market undergo time-varying fluctuations and gives us more flexibility. Most of the commonly used models in mathematical finance suppose that the sums of the interest rates

\[ V = \{V_n : V_n = \sum_{k=0}^{n} \rho_k\} \]

follow the classic symmetric random walk with constant probability transitions such that \( \{\rho_n\}_{n \in \mathbb{N}} \) are independently identically distributed (i.i.d.) random variables. Therefore, an i.i.d. sequence of random variables \( \rho_n \) is generated to model the dynamics of \( \{S_n\}_{n \in \mathbb{N}} \) governed by (2). We rather want to consider the more general case when \( \rho_n \) are just independently, but nonidentically distributed. As the consequence, the sequence of \( \{r_n\}_{n \in \mathbb{N}} \) cannot be modelled by i.i.d. random variables \( r_n \) either. Thus, the process

\[ U = \{U_n : U_n = \sum_{k=0}^{n} r_k\} \]

must be modelled by nonidentically distributed variables \( r_n \) too.

In particular we are interested in computations when \( V \) is modeled by the dynamic random walk and its consequences for the pricing theory. This dynamic concept has been introduced by Guillotin [10] and further studied in Guillotin [11], [12], [13], [14], [15] or Guillotin-Plantard and Schott [16], [17] with respect to several asymptotic aspects and possible applications in computer sciences. Consider the following definition.
Definition 1.1 The random sequence $V = \{V_n : n \in \mathbb{N}\}$ on $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}_{n \in \mathbb{N}}, \mathbb{P})$ is called a \textit{dynamic random walk} over the dynamical system $(E, T)$ where $E$ is a metric space and $T : E \to E$ is a transformation iff

(i) $V_n = \sum_{k=1}^{n} Z_k$ for $n \in \mathbb{N}$, where $Z_k : (\Omega, \mathcal{F}_k, \mathbb{P}) \to (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$

(ii) $V_0 = Z_0 = 0$ (a.s.)

(iii) $\exists m \in \mathbb{N} : m \geq 2 \exists a_k^{(j)} \in \mathbb{R}^d : a_k^{(i)} \neq a_k^{(j)}$ (nonrandom) for $i \neq j$ and $i, j \in \{1, 2, ..., m\}$ such that

\[
p_k := \mathbb{P}(Z_k = z) = \begin{cases} 
\frac{1}{m}(1 + g_1(T^k x)) & \text{if } z = a_k^{(1)} \\
\frac{1}{m}(1 + g_2(T^k x)) & \text{if } z = a_k^{(2)} \\
\vdots & \vdots \\
\frac{1}{m}(1 + g_m(T^k x)) & \text{if } z = a_k^{(m)}
\end{cases}
\]

where $x \in E$ and $g_j : E \to [-1, +1]$ is a Borel-measurable function with identity $\sum_{j=1}^{m} g_j(x) = 0$ for all $x \in E$ are fixed.

(iv) $Z = \{Z_k : k \in \mathbb{N}\}$ are independent random variables.

This allows us to model the probability distributions of $V$ and $\rho_n$ in a time-dependent manner. If the $(B, S)$-market is modelled by $V_n = \sum_{k=0}^{n} \rho_k$ as a dynamic random walk then we call it the model of \textit{dynamic $(B, S)$-market}. We concentrate on the univariate dynamic random walk with two- or three-point distributed increments at first, i.e. $d = 1$ and $m = 2, 3$. In particular, the specific choices of

\[
p_k = \mathbb{P}(Z_k = z) = \begin{cases} 
\frac{1}{2}(1 + g(T^k x)) & \text{if } z = a_k \\
\frac{1}{2}(1 - g(T^k x)) & \text{if } z = b_k
\end{cases}
\]

with $a_k < b_k$ and

\[
p_k = \mathbb{P}(Z_k = z) = \begin{cases} 
\frac{1}{3}(1 + g(T^k x)) & \text{if } z = a_k \\
\frac{1}{3}(1 - g(T^k x)) & \text{if } z = b_k \\
\frac{1}{3} & \text{if } z = c_k
\end{cases}
\]

with $a_k < c_k < b_k$ are in the center of our interest. This can replicate the practically observed scenario of values of assets $B$ and $S$ to go up, down or stay at the same level when compared at consecutive time-steps (effectuated by their interest rates $r$ and $\rho$, respectively). However, the related $(B, S)$-market model is considered to be a small step towards the analysis of financial markets with time-dependent parameters. The more general model using dynamic random walks with multipoint-distributions for $Z_k$ is conceivable, but left to future research due to its complexity.

The following aims of mathematical analysis of $(B, S)$-market are pursued during this paper.

1. Thorough computation for fair price and hedging strategies for one example of $(B, S)$-market which allows dynamic flexibility in the interest rates $r, \rho$ and which is more general than the Cox-Ross-Rubinstein market model based on static binomial probabilities for the random variables $(\rho_n)_{n \in \mathbb{N}}$. 

3
2. Construction of equivalent martingale-measure $\mathbb{P}^*$ to the original probability measure $\mathbb{P}$ which renders $(S_n/B_n)_{n \in \mathbb{N}}$ to be a $\{\mathcal{F}_n\}_{n \in \mathbb{N}}$-martingale under $\mathbb{P}^*$ and Proof of absence of arbitrage by the fundamental theorem of asset pricing for dynamic $(B, S)$-markets.

3. Completeness of the dynamic $(B, S)$-market.

4. Calculation of fair call prices $C = C(N)$ with time $N > 0$ of maturity (i.e. the exercising time).

5. Construction of a.s. minimal hedging strategies $\pi^*$ and its value $X^{\pi^*}$ with probability $\mathbb{P}(X^{\pi^*}_N = f(S_N)) = 1$ for given pay-off function $f$.

6. $\gamma$-Hedging and pricing with positive probability, i.e. when $\mathbb{P}(X^{\pi^*}_N \geq f(S_N)) \geq 1 - \gamma$ and $X^{\pi^*}_0 = (1 - \gamma)C(N)$.

7. Asymptotic behavior of the results as $N \to +\infty$ or parameters in dynamic random walk converge to certain values.

As a simple example of pay-off functions we are motivated by the European call option with $f(S_N) = (S_N - K(N))_+$ where $K(N)$ is the striking price to be agreed with by market participants and $N$ is supposed to be a nonrandom finite exercising time. Due to the well-known Call-Put parity relation, the presented analysis shall be relevant for both call and put options.

2 Absence of Arbitrage of Dynamic $(B, S)$-Markets

Let $\Delta Z_n = Z_n - Z_{n-1}$ denote the $n$-th increment of random sequence $Z = \{Z_n : n \in \mathbb{N}\}$. For the sake of simplicity, we assume that the underlying $\sigma$-algebra $\mathcal{F} = \sigma(\mathcal{F}^S, \mathcal{F}^B)$ is determined by

$$\mathcal{F}^S_n = \sigma\{S_0, S_1, ..., S_n\}, \quad \mathcal{F}^B_n = \sigma\{B_0, B_1, ..., B_n\}$$

as the smallest $\sigma$-algebras generated by the sequences $S = \{S_k : 0 \leq k \leq n\}$ and $B = \{B_k : 0 \leq k \leq n\}$ governed by (2) and (1), respectively. We shall follow a notation which is similar to that of Melnikov [29].

2.1 Key definitions and auxiliary results

Fix $N$ as the time of maturity and let $d = 1$. We recall some standard definitions at first.

**Definition 2.1** An investment strategy or portfolio $\pi = (\pi_n)_{0 \leq n \leq N}$ of $(B, S)$-market is defined by $\pi_n = (\beta_n, \alpha_n)$ where the random variables $\alpha_n, \beta_n : (\Omega, \mathcal{F}_n, \mathbb{P}) \to (\mathbb{R}^1, \mathcal{B}(\mathbb{R}^1))$ are interpreted as the amount $\beta_n$ of riskless asset $B_n$ hold in the portfolio $\pi$ at the time $n$ and $\alpha_n$ of risky asset $S_n$ hold in the portfolio $\pi$ at the time $n$, respectively. An investment strategy $\pi$ is called self-financing for the $(B, S)$-market if

$$\forall n : 1 \leq n \leq N \quad \Delta \beta_n B_{n-1} + \Delta \alpha_n S_{n-1} = 0.$$

The set of all self-financing strategies is defined by

$$SF := \{ \pi : \pi \text{ portfolio satisfying (5)} \}.$$
The quantity $X_n^n = \{X_n^n : 0 \leq n \leq N\}$ is called the value of the investment strategy $\pi = (\beta_n, \alpha_n)_{0 \leq n \leq N}$ if

$$\forall n : 0 \leq n \leq N \quad X_n^n = \beta_n B_n + \alpha_n S_n.$$ (6)

**Lemma 2.1** Any portfolio $\pi = (\beta_n, \alpha_n)_{0 \leq n \leq N} \in SF$ admits the a.s. representation

$$X_n^n = X_0^n + \sum_{k=1}^{n} \left( \beta_k \Delta B_k + \alpha_k \Delta S_k \right)$$ (7)

for all $1 \leq n \leq N$. If additionally $\Delta B_0 = \Delta S_0 := 0$ then we have the equivalence

$$\forall n \in \{0, 1, ..., N\} \quad X_n^n = X_0^n + \sum_{k=1}^{n} \left( \beta_k \Delta B_k + \alpha_k \Delta S_k \right) \iff \pi \in SF \ (i.e. \ (5)).$$

**Proof.** Observe that

$$X_n^n = \alpha_n S_n + \beta_n B_n = \alpha_n (S_n - S_{n-1}) + \beta_n (B_n - B_{n-1}) + \alpha_n S_{n-1} + \beta_n B_{n-1}$$

$$= \alpha_n \Delta S_n + \beta_n \Delta B_n + \alpha_{n-1} S_{n-1} + \beta_{n-1} B_{n-1} + (\alpha_n - \alpha_{n-1})S_{n-1} + (\beta_n - \beta_{n-1})B_{n-1}$$

$$= \alpha_n \Delta S_n + \beta_n \Delta B_n + X_{n-1}^n + \Delta \alpha_n S_{n-1} + \Delta \beta_n B_{n-1}.$$  

This is equivalent to

$$\Delta X_n^n - \alpha_n \Delta S_n - \beta_n \Delta B_n = \Delta \alpha_n S_{n-1} + \Delta \beta_n B_{n-1}.$$  

Consequently, we have the equivalence

$$\Delta X_n^n - \alpha_n \Delta S_n - \beta_n \Delta B_n = 0 \iff \Delta \alpha_n S_{n-1} + \Delta \beta_n B_{n-1} = 0$$

for all $1 \leq n \leq N$. However, by telescoping, the case $\Delta X_n^n - \alpha_n \Delta S_n - \beta_n \Delta B_n = 0$ for all $1 \leq n \leq N$ is equivalent to $X_n^n = X_0^n + \sum_{k=1}^{n} \left( \alpha_n \Delta S_n + \beta_n \Delta B_n \right)$ for all $1 \leq n \leq N$. Therefore, the representation (7) is verified. Now, suppose that $\Delta B_0 = \Delta S_0 = 0$ and $S_{-1} := 0, B_{-1} := 0$. Then the above equivalence is even true for all $0 \leq n \leq N$, and the proof of Lemma 2.1 is complete. $\diamond$

**Definition 2.2** An investment strategy $\pi \in SF$ is said to realize arbitrage iff there exists a number $N > 0$ such that $\mathbb{P}(X_N^n > 0) > 0$ with $X_0^n = 0$ and $X_n^n \geq 0$ for all $1 \leq n \leq N$. The set $SF_{arb}$ of all arbitrage-realizing strategies $\pi \in SF$ is defined by

$$SF_{arb} = \{ \pi \in SF : \pi \text{ realizes arbitrage} \}.$$  

A $(B,S)$-market is called arbitrage-free or no-arbitrage market iff $SF_{arb} = \emptyset$ (the empty set).
Solutions to linear systems of homogeneous stochastic difference equations such as (1) and (2) can be expressed explicitly in terms of stochastic exponentials under some mild regularity conditions. For this purpose, consider the following definition.

**Definition 2.3** Given a sequence \( Z = \{Z_n : n \in \mathbb{N}\} \) of random variables \( Z_n : (\Omega, \mathcal{F}_n, \mathbb{P}) \to (\mathbb{R}^1, \mathcal{B}(\mathbb{R}^1)) \). Then, the random sequence \( \mathcal{E}(Z) = \{\mathcal{E}_n(Z) : n \in \mathbb{N}\} \) defined by

\[
\mathcal{E}_n(Z) := \prod_{k=0}^{n} (1 + \Delta Z_k)
\]

for \( n \in \mathbb{N} \), where \( Z_n = Z_0 + \sum_{k=1}^{n} \Delta Z_k \) and \( \Delta Z_0 = 0 \), is called discrete stochastic exponential of \( Z \).

**Lemma 2.2** Assume that

(i) \( U = \{U_n : n \in \mathbb{N}\} \) is a random sequence with \( U_n : (\Omega, \mathcal{F}_n, \mathbb{P}) \to (\mathbb{R}^1, \mathcal{B}(\mathbb{R}^1)) \),

(ii) \( N = \{N_n : n \in \mathbb{N}\} \) is a random sequence with \( N_n : (\Omega, \mathcal{F}_n, \mathbb{P}) \to (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)) \),

(iii) \( Z = \{Z_n : n \in \mathbb{N}\} \) as a random sequence with \( Z_n : (\Omega, \mathcal{F}_n, \mathbb{P}) \to (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)) \) satisfies the initial value problem (IVP)

\[
\Delta Z_n = \Delta N_n + Z_{n-1} \Delta U_n, \quad n \geq 1,
\]

\[
Z_0 = N_0,
\]

(iv) \( \forall k \geq 1 \quad \mathbb{P}(\Delta U_k = -1) = 0 \).

Then, the sequence \( Z = \{Z_n : n \in \mathbb{N}\} \) possesses the a.s. \( (\mathcal{F}_n, \mathcal{B}(\mathbb{R}^d)) \)-measurable representation

\[
Z_n = \mathcal{E}_n(U) \cdot \left\{ N_0 + \sum_{k=1}^{n} \mathcal{E}_k^{-1}(U) \Delta N_k \right\} \quad (8)
\]

for all \( n \geq 1 \). In particular, for the case \( N_n = X_0 = N_0 \) (\( n \geq 1 \)), we have

\[
Z_n = \mathcal{E}_n(U) \cdot Z_0, \quad n \geq 0.
\]

**Proof.** The proof is a slight modification of Melnikov’s one ([29], p. 14-15).

**Remark.** It is rather obvious that

(i) \( \mathcal{E}(U) = \{\mathcal{E}_n(U) : n \in \mathbb{N}\} \) \( \{\mathcal{F}_n\}_{n \in \mathbb{N}} \)-martingale \( \iff \) \( U = \{U_n : n \in \mathbb{N}\} \) \( \{\mathcal{F}_n\}_{n \in \mathbb{N}} \)-martingale,

(ii) \( \forall n \in \mathbb{N} : \mathcal{E}_n(U) \cdot \mathcal{E}_n(V) = \mathcal{E}_n(U + V + < U, V >) \) where \( < U, V > \) is the quadratic covariation defined by \( < U, V > = \sum_{k=0}^{n} \Delta U_k \cdot \Delta V_k \) where \( \Delta U_0 = \Delta V_0 = 0 \), and
(iii) regularity condition (iv) in Lemma 2.2 guarantees the invertibility of discrete exponentials \( \mathcal{E} \) at all times \( n \).

For proving a version of the fundamental theorem of dynamic asset pricing within the framework of dynamic \((B, S)\)-markets, we recall some results on the relation of \( \{\mathcal{F}_n\}_{n \in \mathbb{N}} \)-martingales and risk-neutral (martingale) probability measures.

**Definition 2.4** A probability measure \( \mathbb{P}^* \sim \mathbb{P} \) over \( (\Omega, \mathcal{F}, \{\mathcal{F}_n\}_{n \in \mathbb{N}}) \) is said to be a martingale-measure or risk-neutral for the \((B, S)\)-market iff the random sequence \( R = \{R_n : 0 \leq n \leq N\} \) defined by \( R_n = S_n/B_n \) is a \( \{\mathcal{F}_n\}_{n \in \mathbb{N}} \)-martingale with respect to \( \mathbb{P}^* \). The set of all equivalent martingale-measures for the \((B, S)\)-market is defined by \( \mathbb{P}^* \).

**Lemma 2.3** Assume that \((B, S)\)-market over \( (\Omega, \mathcal{F}, \{\mathcal{F}_n\}_{n \in \mathbb{N}}, \mathbb{P}) \) satisfies that

(i) \( r = \{r_n : n \in \mathbb{N}\} \) is predictable,

(ii) \( r_n > -1 \) for all \( 0 \leq n \leq N \) (a.s.), and

(iii) \( \mathbb{P}^* \) is a probability measure on \( (\Omega, \{\mathcal{F}_n\}_{n \in \mathbb{N}}) \).

Then, we have that

\[
R = \{R_n : 0 \leq n \leq N\} \text{ with } R_n = \frac{S_n}{B_n} \text{ is a } \mathcal{F}_{0 \leq n \leq N} \text{-martingale w.r.t. } \mathbb{P}^*
\]

\[
\iff \left( \sum_{k=0}^{n} (\rho_k - r_k) \right)_{0 \leq n \leq N} \text{ is a } \mathcal{F}_{0 \leq n \leq N} \text{-martingale w.r.t. } \mathbb{P}^*.
\]

**Proof.** The proof is a slight modification of Melnikov’s one ([29], Theorem 3.1, p. 22-23).

**Remark.** Lemma 2.3 provides us an important criteria for martingale-measures \( \mathbb{P}^* \). Suppose that \( r = \{r_n : n \in \mathbb{N}\} \) is predictable with respect to the filtration \( \{\mathcal{F}_n\}_{n \in \mathbb{N}} \). Then, \( R \) is a martingale iff

\[
\mathbb{E}[\rho_n | \mathcal{F}_{n-1}] = r_{n-1}, \quad n \geq 1
\]

Thus, for martingale-measures, there is a well-determined relation between the dynamics of interest rates \( r \) and \( \rho \), and hence they cannot be chosen arbitrarily independent from each other within linear \((B, S)\)-market models.

### 2.2 Fundamental theorem of asset pricing in \((B, S)\)-markets

A theorem which connects the important concepts of arbitrage (economics) and martingale measures (probability) plays a fundamental role in the analysis of financial markets. This is particularly true for our \((B, S)\)-market model with dynamic probabilities. For a more general
context, see Schachermeyer [34], [35], and Delbaen and Schachermeyer [7] who build up their ideas based on the fundamental works of Merton [30], Kreps [23], Stricker [44] and Dalang, Morton and Willinger [6] who proved the equivalence between the existence of martingale-measures and the absence of arbitrage in discrete time with finite time-horizons. Note that there are counterexamples when the following equivalence break down in the case of infinite time-horizons, e.g. see Back and Pliska [2].

**Theorem 2.1** Let \((B, S, \pi, X^\pi)\) be a financial market over the filtered probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_n\}_{n \in \mathbb{N}}, \mathbb{P})\). Assume that

(i) \(r = \{r_n : n \in \mathbb{N}\}\) is predictable and \(r_n > -1\) for all \(0 \leq n \leq N\) (a.s.),

(ii) \(\sigma(r_n)\) is independent of \(\mathcal{F}_k\) for all \(n > k\),

(iii) \(\alpha = \{\alpha_n : n \in \mathbb{N}\}\) is predictable,

(iv) \(B_0 > 0\) and \(B_0\) is \((\mathcal{F}_0, \mathcal{B}(\mathbb{R}^1))\)-measurable, and

(v) cardinality \(\#(\Omega) < +\infty\) with \(\mathbb{P}((\omega_i)) > 0\) for all \(\omega_i \in \Omega\).

Then, we have that

\[
P^* \neq \emptyset \iff SF_{arb} = \emptyset.
\]

**Proof.** The proof is a slight modification of Melnikov’s one ([29], Theorem 3.2, p.24-27)

3 Completeness of Dynamic \((B, S)\)-Markets

A very natural requirement of "fair" financial markets is that every yield (value) can be reproduced. This property relates to the concept of completeness.

**Definition 3.1** A \((B, S, \pi, X^\pi)\)-market defined on the filtered, discrete time probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_n\}_{0 \leq n \leq N}, \mathbb{P})\) is said to be a (pathwise) **complete** iff

\[
\forall (\mathcal{F}_N, \mathcal{B}(\mathbb{R}^1)) - \text{measurable } f : \Omega \to \mathbb{R}^1 \exists \pi \in SF \forall \omega \in \Omega : X^\pi_N(\omega) = f(\omega).
\]

**Remark.** Completeness also means the absence of any constraints for investing in related assets and assures the accessibility of all assets involved.

**Theorem 3.1** Fix \(N \in \mathbb{N} \setminus \{0\}\) as the time of maturity. Assume that \(P^* \neq \emptyset\) and \(P^* \in P^*\), and \(r = \{r_n\}_{0 \leq n \leq N}\) is predictable. Then the following assertions are equivalent:

(1) Completeness of the \((B, S, \pi, X^\pi)\)-market,

(2) Uniqueness of the equivalent martingale-measure, i.e. \(P^* = \{P^*\}\),

8
(3) All \((\mathcal{F}_n)_{0 \leq n \leq N}\)-martingales \(M = (M_n)_{0 \leq n \leq N}\) possess the representation

\[
M_n = M_0 + \sum_{k=1}^{n} \alpha_k \Delta m_k
\]

for \(1 \leq n \leq N\), where \(\alpha_k\) is predictable and \(\Delta m_k = S_k/B_k - S_{k-1}/B_{k-1}\).

Remark. Note that \(\Delta m_k = S_{k-1}B_k^{-1}(\rho_k - r_k)\). Also, in view of Lemma 2.3, the martingale representation (9) is equivalent to

\[
M_n = M_0 + \sum_{k=1}^{n} \tilde{\alpha}_k (\rho_k - r_k)
\]

where \(\tilde{\alpha}_k = \alpha_k S_{k-1}B_k^{-1}\) which is predictable under the above assumptions.

Proof. The proof is a slight modification of Melnikov's one ([29], Theorem 3.3, p27-29).

4 Fair Pricing and Hedging Strategies in Complete Dynamic Markets

This section aims at the calculation of fair prices and hedging strategies for European options in complete dynamic \((B, \mathcal{S})\)-markets.

4.1 Definition of Contingent Claims, European Options, and Related Notions

Consider our dynamic \((B, \mathcal{S}, \pi, X^\pi)\)-market model over the complete probability space \((\Omega, \mathcal{F}, (\mathcal{F}_n)_{0 \leq n \leq N}, \mathbb{P})\). Recall the following notions from literature, e.g. Shiryaev [40].

**Definition 4.1** The pair \((f, N)\) is called contingent claim with repayment date \(N\) iff \(f : (\Omega, \mathcal{F}_N, \mathbb{P}) \rightarrow (\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))\), i.e. \(f\) is a \((\mathcal{F}_N, \mathcal{B}(\mathbb{R}_+))\)-measurable random variable. The contingent claim \((f, N)\) is said to be (pathwise) attainable for all market participants iff \(\exists\) portfolio \(\pi\) such that \(X^\pi_N \geq f\) on \(\Omega\) (a.s.). The procedure of building up attainable contingent claims with appropriate portfolios \(\pi\) is called hedging and its result \(\pi\) is called hedging portfolio.

Remark. One of the most important problems in financial markets is to calculate the fair premium \(c\) at time 0 which guarantees to purchase an asset with a derivative security (for selling, buying, etc.) at later time \(N > 0\). Such an example is provided with one of the most studied options, the European option. This exhibits the right to exercise a given security at deterministic time \(N > 0\) and the guarantee to receive a payment in the amount of \(f\) which is fixed by contracts with options.
**Definition 4.2** Suppose that a \((B,S,\pi,X^\pi)\)-market is given with initial capital \(x > 0\) and contingent claim \((f,N)\). Then, a portfolio \(\pi \in SF\) is called \((x,f,N)\)-hedge iff

\[
\forall \omega \in \Omega : X^\pi_0(\omega) = x \quad \text{and} \quad X^\pi_N(\omega) \geq f(\omega).
\]

The related \((x,f,N)\)-hedge \(\pi \in SF\) is said to be minimal iff \(X^\pi_N(\omega) = f(\omega)\) for all \(\omega \in \Omega\). \(H(x,f,N)\) is defined to be the set of all \((x,f,N)\)-hedges. The minimal investment cost of \((f,N)\) is identified by

\[
c(N) = \inf\{x > 0 : H(x,f,N) \neq \emptyset\}
\]

and is also called the fair price of the option in case of options.

**Remark.** Note that \(c(N)\) is bounded since \(\#(\Omega) < +\infty\) in this paper. Suppose that \((f,N)\) is a contingent claim on an option. Then, (11) corresponds to a ”fair contract” that both the seller can attain the claim \((f,N)\) acting on the \((B,S)\)-market and the buyer pays a minimal premium to the seller.

**Definition 4.3** Given a dynamic \((B,S,\pi,X^\pi)\)-market. Then, an **European call (put) option** is an option to buy an asset (shares) with the contingent claim \(f = (S_N - K)_+\) \((f = (K - S_N)_+ \text{ in case of put})\) at the strike price \(K\) at the exercise time \(N\).

**Remark.** It is clear that the option is exercised by the buyer if indeed \(S_N > K\), but certainly not if \(S_N < K\).

### 4.2 Computing Prices and Hedging Strategies for European Options

There is an obvious need for general formulas for the fair price, minimal hedge and the related value for European options on nonarbitrage dynamic \((B,S)\)-markets with interest rates \(r = (r_n)_{0 \leq n \leq N}\).

**Theorem 4.1** Let \((B,S,\pi,X^\pi)\) be a financial market over the filtered probability space \((\Omega,\mathcal{F},\{\mathcal{F}_n\}_{n \in \mathbb{N}},\mathbb{P})\). Assume that

(i) \(r = \{r_n : n \in \mathbb{N}\}\) is predictable and \(r_n > -1\) for all \(0 \leq n \leq N\) (a.s.),

(ii) \(\sigma\)-algebra \(\sigma(r_n)\) is independent of \(\mathcal{F}_k\) for all \(n > k\),

(iii) \(\alpha = \{\alpha_n : n \in \mathbb{N}\}\) is predictable,

(iv) \(B_0 > 0\) and \(X^\pi_0 = x\) are nonrandom,

(v) cardinality \(\#(\Omega) < +\infty\) with \(\mathbb{P}(\{\omega_i\}) > 0\) for all \(\omega_i \in \Omega\),

(vi) completeness of \((B,S,\pi,X^\pi)\)-market (i.e. \(\exists \mathbb{P}^* \in \mathcal{P}^* : \mathbb{P}^* \sim \mathbb{P}\)) holds, and
(vii) no arbitrage (i.e. $\exists ! \mathbb{P}^* \in \mathcal{P}^*$) can be realized.

Then, for the European option with contingent claim $(f, N)$,

[1] the fair price is given by

$$c(N) = \mathbb{E}_{\mathbb{P}^*}[\mathcal{E}^{-1}_N(U)f],$$

(12)

[2] the minimal $(c(N), f, N)$-hedge $\pi^* = (\beta^*_n, \alpha^*_n)_{0 \leq n \leq N}$ exists and has the value

$$X^*_n = \mathbb{E}_{\mathbb{P}^*}[\mathcal{E}^{-1}_N(U)\mathcal{E}_n(U)f|\mathcal{F}_n],$$

(13)

[3] the coefficients of the minimal $(c(N), f, N)$-hedge $\pi^* = (\beta^*_n, \alpha^*_n)_{0 \leq n \leq N}$ satisfy

$$\alpha^*_n = \alpha_n, \quad \beta^*_n = \frac{X^*_n - \alpha^*_n S_{n-1}}{B_{n-1}}$$

(14)

for $0 \leq n \leq N$ (i.e. $(\alpha^*_n)_{0 \leq n \leq N}$ and $(\beta^*_n)_{0 \leq n \leq N}$ can be chosen as predictable sequences).

Proof. The proof is a slight modification of Melnikov’s one ([29], Theorem 4.1, p. 33-35)

4.3 Some Properties of the Binary Dynamic $(B, S, \pi, X^*)(\mathbb{M})$-Market

Definition 4.4 A $(B, S, \pi, X^*)$-market over the probability space $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{0 \leq n \leq N}, \mathbb{P})$ is called binary if there are two sequences of distinct real numbers $(a_n)_{0 \leq n \leq N}$ and $(b_n)_{0 \leq n \leq N}$ such that $a_n < b_n$, $0 < q_n = \mathbb{P}(\{\rho_n = a_n\}) < 1$ and $0 < p_n = \mathbb{P}(\{\rho_n = b_n\}) < 1$ with $p_n + q_n = 1$ for all $0 \leq n \leq N$ and $(\rho_n)_{0 \leq n \leq N}$ are independent random variables.

Remark. Note that, in general, binary markets can possess time-dependent probabilities. An appropriate probability space for binary markets can be constructed as follows. Introduce independent random variables $\varepsilon_n : (\Omega, \mathcal{F}_n, \mathbb{P}) \rightarrow \{+1, -1\}$ with the properties that

$$\varepsilon_n = +1 \iff \rho_n = b_n, \quad \varepsilon_n = -1 \iff \rho_n = a_n.$$

Thus, we may set

$$\rho_n = \frac{a_n + b_n}{2} + \frac{b_n - a_n}{2} \varepsilon_n.$$

Define $\mathcal{F}_0^N = \{\emptyset, \Omega\}$ since $B_0$ and $S_0$ are supposed to be nonrandom (as known certain quantities to the market participants). Also it is reasonable to assume that interest rates $r = (r_n)_{0 \leq n \leq N}$ involved in the dynamics of fairly riskless asset $B$ are nonrandom (hence known to the participants) too. Then, the naturally induced choice for the filtration is given by the $\sigma$-algebras $\mathcal{F}_n^N = \sigma(\varepsilon_1, \varepsilon_2, ..., \varepsilon_n)$ for $1 \leq n \leq N$. Consequently, the product probability space $(\Omega^N, \mathcal{F}_N^N, (\mathcal{F}_n^N)_{0 \leq n \leq N}, \mathbb{P}_N^N = \bigotimes_{k=1}^N \mathbb{P}_k)$ with $\mathbb{P}_k(\varepsilon_k = +1) = p_k$ may serve as the underlying stochastic basis. Note that also $\mathcal{F}_n^N = \sigma(\rho_1, ..., \rho_n)$ for all $1 \leq n \leq N$. 

11
Theorem 4.2 The binary \((B, S, \pi, X^n)\)-market with nonrandom parameters \(-1 < a_n < r_n < b_n < +\infty\) is a no-arbitrage and complete financial market over the filtered probability space \((\Omega^N, \mathcal{F}^N, (\mathcal{F}^n_0 \leq n \leq N, \otimes_{k=1}^N \mathbb{P}^*_k)\).

Proof. Note that any binary probability measure \(\mathbb{P}^*_k\) is completely determined by \(p_k = \mathbb{P}(\rho_k = b_k)\). Choose
\[p_k^* = \frac{r_k - a_k}{b_k - a_k}\]
for all \(1 \leq k \leq N\). Trivially, \(0 < p_k^* < 1\) holds due to the assumption \(r_k \in (a_k, b_k)\) (also \(q_k^* = 1 - p_k^* = (b_k - r_k)/(b_k - a_k)\)). Thus, probability measures \(\mathbb{P}_k^*\) and \(\otimes_{k=1}^N \mathbb{P}_k^*\) can be constructed out of the given data. Define \(M_n = S_n/B_n\) for all \(0 \leq n \leq N\). Calculate
\[
\mathbb{E}_{\mathbb{P}^*} \left[ \sum_{k=1}^{n} (\rho_k - r_k) - \sum_{k=1}^{n-1} (\rho_k - r_k) \right] | \mathcal{F}^{n-1}_n = \mathbb{E}_{\mathbb{P}^*} [\rho_n - r_n] = a_n(1 - p_n^*) + b_np_n^* - r_n = (b_n - a_n)p_n^* + a_n - r_n = r_n - a_n - a_n - r_n = 0,
\]
hence \(\sum_{k=1}^{n} (\rho_k - r_k) | \mathcal{F}^{n-1}_n\) forms a \((\mathcal{F}^n_0 \leq n \leq N)\)-martingale with respect to the measure \(\mathbb{P}^* = \otimes_{k=1}^N \mathbb{P}^*_k\). Thanks to Lemma 2.1, \(M = (M_n)_{0 \leq n \leq N}\) is a \((\mathcal{F}^n_0 \leq n \leq N)\)-martingale under \(\mathbb{P}^*\) too, i.e., \(\mathbb{P}^*\) represents a martingale measure and \(\mathbb{P}^* \in \mathbb{P}^*\) (equivalent by construction). Now, Theorem 2.1 implies that \(SF_{arb} = \emptyset\), hence there is not any arbitrage possibility in the framework of given binary \((B, S, \pi, X^n)\)-market model. Moreover, its completeness follows from the martingale-representation Lemma 4.1 below. Thus, the proof of Theorem 4.2 is complete. \(\diamond\)

Lemma 4.1 Let \((\Omega_n, \mathcal{F}_n, \mathbb{P}_n)\) be a sequence of probability spaces and \(\rho = (\rho_n)_{0 \leq n \leq N}\) be a sequence of independent random variables with nonrandom parameters \(-1 < a_n < r_n < b_n < +\infty\), \(p_n + q_n = 1\), \(0 < q_n = \mathbb{P}_n(\{\rho_n = a_n\}) < 1\) and \(0 < p_n = \mathbb{P}_n(\{\rho_n = b_n\}) < 1\) where \(p_n = (r_n - a_n)/(b_n - a_n)\) for \(1 \leq n \leq N\). Define \(\mathcal{F}_n = \sigma\{\rho_1, ..., \rho_n\}\) and \(\mathcal{F}_0 = \{\emptyset, \Omega\}\). Then, any arbitrary \((\mathcal{F}_n)_{0 \leq n \leq N}\)-martingale \(M = (M_n)_{0 \leq n \leq N}\) with \(\mathbb{E}M_0 = 0\) possesses a discrete integral representation of the form
\[
M_n = \sum_{k=1}^{n} \alpha_k \Delta m_k
\]
where \((\alpha_k)_{1 \leq k \leq N}\) is a predictable sequence and \(m = (m_k)_{1 \leq k \leq N}\) defined by \(m_n = \sum_{k=1}^{n} (\rho_k - r_k)\) for \(1 \leq n \leq N\) and \(m_0 = 0\) is the underlying martingale. with respect to \((\mathcal{F}_n)_{0 \leq n \leq N}\).

Proof. First, note that \(M_n\) is \((\mathcal{F}_n, \mathcal{B}(\mathbb{R}^1))\)-measurable. Therefore, there must exist Borel-measurable functions \(f_n : \{a_1, b_1\} \times \{a_2, b_2\} \times ... \times \{a_n, b_n\} \rightarrow \mathbb{R}^1\) such that \(M_n(\omega) = f_n(\rho_1(\omega), \rho_2(\omega), ..., \rho_n(\omega))\) for all \(\omega \in \Omega\) and \(1 \leq n \leq N\). This implies that
\[
\Delta M_n = \alpha_n \Delta m_n
\]
holds if
\[
f_n(\rho_1, \rho_2, ..., \rho_n) - f_{n-1}(\rho_1, \rho_2, ..., \rho_{n-1}) = \alpha_n \Delta m_n = \alpha_n(\rho_n - r_n).
\]

12
This is equivalent to the following system of equations

\[
\begin{align*}
    f_n(p_1, p_2, \ldots, p_{n-1}, b_n) - f_{n-1}(p_1, p_2, \ldots, p_{n-1}) &= \alpha_n (b_n - r_n), \\
    f_n(p_1, p_2, \ldots, p_{n-1}, a_n) - f_{n-1}(p_1, p_2, \ldots, p_{n-1}) &= \alpha_n (a_n - r_n).
\end{align*}
\]

Solving the latter system with respect to \( \alpha_n \) yields the identities

\[
\alpha_n = \frac{f_n(p_1, p_2, \ldots, p_{n-1}, b_n) - f_{n-1}(p_1, p_2, \ldots, p_{n-1})}{b_n - r_n},
\]

\[
= \frac{f_n(p_1, p_2, \ldots, p_{n-1}, a_n) - f_{n-1}(p_1, p_2, \ldots, p_{n-1})}{a_n - r_n},
\]

(17)

which also follow directly from the martingale property of \( M \)

\[
\mathbb{E}[M_n - M_{n-1} | \mathcal{F}_{n-1}] = \mathbb{E}[f_n(p_1, p_2, \ldots, p_n) - f_{n-1}(p_1, p_2, \ldots, p_{n-1}) | \mathcal{F}_{n-1}] = 0
\]

which can be equivalently rewritten as

\[
\begin{align*}
    p_n f_n(p_1, \ldots, p_{n-1}, b_n) + (1 - p_n) f_n(p_1, \ldots, p_{n-1}, a_n) \\
    = \frac{p_n f_n(p_1, \ldots, p_{n-1}) + (1 - p_n) f_{n-1}(p_1, \ldots, p_{n-1})}{1 - p_n}
\end{align*}
\]

or, in another form,

\[
\frac{f_n(p_1, \ldots, p_{n-1}, b_n) - f_{n-1}(p_1, \ldots, p_{n-1})}{1 - p_n} = \frac{f_n(p_1, \ldots, p_{n-1}) - f_{n-1}(p_1, \ldots, p_{n-1}, a_n)}{p_n}
\]

by taking the choice \( p_n = (r_n - a_n)/(b_n - a_n) \). Backwards, we may also conclude the validity of (16) from (17), and hence the discrete integral representation (15) is confirmed. This completes the proof of Lemma 4.1. \( \diamond \)

5 \( \gamma \)-Pricing and \( \gamma \)-Hedging

So far we have considered claims which are attainable with probability one. It is natural to ask what happens for claims which are only attainable with some positive probability. This leads to an essentially weaker concept and requires less initial capital for investments. For simplicity, we suppose that we have a complete \((B, S)\)-market with nonrandom interest rates \( r_n > -1 \), nonrandom initial values \( X_0^S = x \), nonrandom initial riskless assets \( B_0 \) which are normalized to \( B_0 = 1 \) and contingent claims \( (f, N) \) such that \( \mathbb{E}_{\mathbb{P}^*}[f(S_N)] > 0 \). Define

\[
SF(f, N) = \{ \pi \in SF : X_N^\pi > f - \mathbb{E}_{\mathbb{P}^*}[f] \}.
\]

In analogy to statistics, introduce the level \( \gamma \in (0, 1) \) of significance of contingent claims \((f, N)\).

**Definition 5.1** A portfolio \( \pi \in SF(f, N) \) is called an \( \gamma - (x, f, N) \)-hedge for the probability measure \( \mathbb{P}^* \) iff

\[
\mathbb{P}^*(\{X_N^\pi \geq f\}) \geq 1 - \gamma.
\]

(18)

The set \( H(x, f, N, \gamma) = \{ \pi \in SF(f, N) : \pi \gamma - (x, f, N) \}-hedge \} \) is called the set of \( \gamma - (x, f, N) \)-hedges. The \( \gamma \)-price \( C(N, \gamma) \) of the contingent claim \((f, N)\) is defined by

\[
C(N, \gamma) = \inf\{x > 0 : H(x, f, \gamma) \neq \emptyset\}.
\]
Theorem 5.1 Under the forementioned conditions and for the martingale-measure $\mathbb{P}^* \sim \mathbb{P}$, we have
\[ \forall \pi \in SF(f, N) : \mathbb{P}^*(\{X^\pi_N \geq f\}) \leq \frac{X^\pi_0}{C(N)}, \] (19)
where $C(N) = \mathbb{E}_{\mathbb{P}^*}[\mathcal{E}_N^{-1} f]$ is the fair price of the European option with claim $(f, N)$ with $f \geq 0$.

Proof. Using Chebyshev’s inequality, the definition of $SF(f, N)$, the nonrandomness of $r_n$ and the martingale property of $X^\pi/B$ under $\mathbb{P}^*$ leads to
\[
\mathbb{P}^*(\{X^\pi_N \geq f\}) = \mathbb{P}^*(\{X^\pi_N - f + \mathbb{E}_{\mathbb{P}^*}[f] \geq \mathbb{E}_{\mathbb{P}^*}[f]\}) \\
\leq \frac{\mathbb{E}_{\mathbb{P}^*}[X^\pi_N - f + \mathbb{E}_{\mathbb{P}^*}[f]]}{\mathbb{E}_{\mathbb{P}^*}[f]} = \frac{\mathbb{E}_{\mathbb{P}^*}[X^\pi_N]}{\mathbb{E}_{\mathbb{P}^*}[f]} = \frac{\mathbb{E}_{\mathbb{P}^*}[\mathcal{E}_N^{-1} X^\pi_N]}{\mathbb{E}_{\mathbb{P}^*}[f]} = \frac{X^\pi_0}{C(N)},
\]
hence the assertion of Theorem 5.1 is rather obvious. \(\Box\)

Remark. As a consequence, a necessary condition for $\pi \in H(x, f, N, \gamma)$ is
\[ 1 - \gamma \leq \mathbb{P}^*((\{X^\pi_N \geq f\}) \leq \frac{x}{C(N)} \] (20)
as a determining relation between significance-level $\gamma$ and initial capital $x$.

Following ideas from the proof of Theorem 3.1, we are able to construct a $\gamma$-hedge $\pi$. For this purpose, define the density
\[ Z_N = \frac{d\mathbb{P}}{d\mathbb{P}^*} \bigg|_{\mathcal{F}_N} \]
with $Z_N > 0$. Clearly, there exists a quantity $\lambda = \lambda(\gamma)$ such that
\[ \mathbb{P}^*(\{Z_N \geq \lambda(\gamma)\}) = 1 - \gamma \]
or, equivalently, $\mathbb{P}^*(\{Z_N < \lambda(\gamma)\}) = \gamma$. For simplicity, suppose that $\lambda \geq 1$. Consider the following $(\mathcal{F}_n)_{0 \leq n \leq N}$-martingales defined by
\[ M^\gamma_n = \mathbb{E}_{\mathbb{P}^*}[\mathbb{I}_{\{Z_N < \lambda(\gamma)\}}] \big|_{\mathcal{F}_n}, \quad M^c_n = \mathbb{E}_{\mathbb{P}^*}[\mathcal{E}_N^{-1} f] \big|_{\mathcal{F}_n}. \]
Recall that the market $(B, S)$ is complete (as we assumed it). Hence the martingales $M^\gamma = (M^\gamma_n)_{0 \leq n \leq N}$ and $M^c = (M^c_n)_{0 \leq n \leq N}$ admit the representations
\[ M^\gamma_n = \gamma + \sum_{k=1}^{n} \varphi_k \mathcal{E}^{-1}_k(U) S_{k-1}(\rho_k - r_k), \]
\[ M^c_n = C(N) + \sum_{k=1}^{n} \alpha_k \mathcal{E}^{-1}_k(U) S_{k-1}(\rho_k - r_k) \]
with predictable sequences $(\varphi_k)_{0 \leq k \leq N}$ and $(\alpha_k)_{0 \leq k \leq N}$, where $M^\gamma_0 = \gamma$ and $M^c_0 = C(N)$.
Theorem 5.2 Under the above assumptions and for nonrandom initial value $X_0^\pi = x = (1 - \gamma)C(N)$, the portfolio $\pi_\gamma = (\beta_n^\gamma, \alpha_n^\gamma)_{0 \leq n \leq N}$ defined by

$$
\alpha_n^\gamma = \alpha_n^* - \varphi_n C(N), \quad \beta_n^\gamma = \frac{X_n^{\pi_\gamma} - \alpha_n^* S_{n-1}}{\mathcal{E}_{n-1}(U)}
$$

is a $\gamma - ((1 - \gamma)C(N), f, N)$-hedge.

Proof. From section 2, recall the representation

$$
X_n^{\pi_\gamma} = \mathcal{E}_n(U) \left( X_0^{\pi_\gamma} + \sum_{k=1}^n \mathcal{E}_{k-1}(U) \Delta N_k \right).
$$

In view of the construction (22), we conclude that

$$
\mathcal{E}_{n-1}(U) X_n^{\pi_\gamma} = (1 - \gamma)C(N) + \sum_{k=1}^n \mathcal{E}_{k-1}(U) \alpha_k^* S_{k-1} (\rho_k - r_k)
$$

$$
= (1 - \gamma)C(N) + \sum_{k=1}^n \mathcal{E}_{k-1}(U) \alpha_k^* S_{k-1} (\rho_k - r_k) - C(N) \sum_{k=1}^n \mathcal{E}_{k-1}(U) \varphi_k S_{k-1} (\rho_k - r_k)
$$

$$
= C(N) + \sum_{k=1}^n \mathcal{E}_{k-1}(U) \alpha_k^* S_{k-1} (\rho_k - r_k) - \gamma C(N) - C(N) \sum_{k=1}^n \mathcal{E}_{k-1}(U) \varphi_k S_{k-1} (\rho_k - r_k)
$$

$$
= M_n^\gamma - C(N) M_n^\gamma = \mathbb{E}_{p^*}[\mathcal{E}_{n-1}(U)f|\mathcal{F}_n] - C(N) \mathbb{E}_{p^*}[\mathbb{I}_{\{Z_N < \lambda(\gamma)\}}|\mathcal{F}_n].
$$

Now, just set $n = N$ in these computations. Thus, one obtains the identity

$$
\mathcal{E}^{-1}_N(U) X_N^{\pi_\gamma} = \mathcal{E}^{-1}_N(U)f(S_N) - C(N) \mathbb{I}_{\{Z_N < \lambda(\gamma)\}}.
$$

Therefore, one can estimate

$$
\mathcal{E}^{-1}_N(U) X_N^{\pi_\gamma} \geq \mathcal{E}^{-1}_N(U)f(S_N) - C(N),
$$

hence $\pi_\gamma \in SF(f, N)$. Furthermore, from (21), we know that

$$
\mathbb{P}^*\left(\{X_N^{\pi_\gamma} \geq f\}\right) = \mathbb{P}^*\left(\{f - C(N) \mathcal{E}_N(U) \mathbb{I}_{\{Z_N < \lambda(\gamma)\}} \geq f\}\right) = \mathbb{P}^*\left(\{\mathbb{I}_{\{Z_N < \lambda(\gamma)\}} \leq 0\}\right)
$$

$$
= \mathbb{P}^*\left(\{\mathbb{I}_{\{Z_N < \lambda(\gamma)\}} = 0\}\right) = \mathbb{P}^*\left(\{Z_N \geq \lambda(\gamma)\}\right) = 1 - \gamma.
$$

Finally, from (21) and (23), we arrive at

$$
\mathbb{P}\left(\{X_N^{\pi_\gamma} \geq f\}\right) = \mathbb{E}_{p^*}[\mathbb{I}_{\{X_N^{\pi_\gamma} \geq f\}}] = \mathbb{E}_{p^*}[\mathbb{I}_{\{X_N^{\pi_\gamma} \geq f\}} Z_N] = \mathbb{E}_{p^*}[\mathbb{I}_{\{X_N^{\pi_\gamma} \geq f\}} Z_N] \geq \lambda(\gamma) \mathbb{P}^*\left(\{X_N^{\pi_\gamma} \geq f\}\right) = \lambda(\gamma) (1 - \gamma) \geq (1 - \gamma)
$$

if $\lambda(\gamma) \geq 1$. Consequently, $\pi_\gamma$ is a $\gamma - ((1 - \gamma)C(N), f, N)$-hedge, and the proof of Theorem 5.1 is complete. ∗

Remark. So it is possible to hedge contingent claims with a specified probability $1 - \gamma$. Moreover, the initial capital can be reduced to $(1 - \gamma)C(N)$ by the amount of $\gamma C(N)$ with allowing a risk probability $\gamma$ that the accepted claim cannot totally be repaid.
6 Asymptotic Behavior of Binary \((B, S)\)-markets

6.1 A few words on relation to limit-models

Employing the heuristic argument that stock prices are either rising or falling at any moment of time, Cox, Ross, and Rubinstein [4] proposed regarding these changes as discrete and introduced a binomial model of financial markets. They showed that the binomial model has a Brownian motion as a limit, and that the formula obtained for a fair price converges to the classical Black-Scholes formula. In some studies of financial time series [26, 40] it has been demonstrated that the stock market prices exhibit the so-called long-range-dependence property. Therefore, it has been proposed by Sottinen and Valkeila [41, 42] to replace the Brownian motion in the classical Black-Scholes pricing model by the fractional Brownian motion. Sottinen’s construction is based on the path properties of some “disturbed” random walk.

We consider a discrete time approximation of the Black-Scholes model with a dynamic Cox-Ross-Rubinstein’s model based on dynamic random walks. Our asymptotic analysis relies on results proven by the first author in [10, 13, 12, 14]. The results have also been applied recently in computer science [16, 17].

This section aims at proving the convergence of the binary dynamic binomial model to the continuous Black-Scholes model. So consistency of modeling with discrete time dynamic \((B, S)\)-markets and their continuous time counterparts is verified. Especially, we shall prove that, for every \(x \in E\), the sequence \(\left( B_t^{[n]} \right) \) converges weakly in the Skorohod space \(D = D([0, T], \mathbb{R}^2) \) to \((B_t, S_t)_{t \in [0, T]}\) as \(n\) goes to infinity. For this purpose, the binary univariate dynamic \((B, S)\)-market is supposed to possess the probabilities

\[
\mathbb{P}(\rho_k = z) = \begin{cases} 
  g(T^k x) & \text{if } z = b \\
  1 - g(T^k x) & \text{if } z = a \\
  0 & \text{otherwise}
\end{cases}
\]

where \(x \in E, a < b \in \mathbb{R}\).

6.2 Convergence of binary dynamic \((B, S)\)-markets to the continuous Black-Scholes model

In the classical Black-Scholes pricing model two assets are traded continuously over the time interval \([0, T]\). Let us denote by \(B\) the continuous time risk-less asset, or bond, and by \(S\) the continuous time risky asset, or stock. The dynamics of the continuous time assets are given by

\[
\mathrm{d}B_t = rB_t \mathrm{d}t, \quad B_0 > 0
\]

and

\[
\mathrm{d}S_t = S_t \left(-\frac{\sigma^2}{2} \mathrm{d}t + \sigma \mathrm{d}W_t\right), \quad S_0 > 0.
\]

Here \(r\) is a deterministic interest rate, \(\sigma > 0\) is a constant and \((W_t)_{t \geq 0}\) is the classical one-dimensional Brownian motion. We consider a discrete time approximation of the Black-Scholes model with the following sequence of dynamic Cox-Ross-Rubinstein’s models: The assets are traded on time points \(0 = t_0^{(n)} < t_1^{(n)} < \ldots < t_{n}^{(n)} = T\). The dynamics are

\[
B_k^{(n)} = \left(1 + \frac{t_k^{(n)}}{t_{k-1}^{(n)}}\right) B_{k-1}^{(n)}, \quad B_0^{(n)} = B_0
\]
and
\[ S_k^{[n]} = \left(1 + \rho_k^{[n]} \right) S_{k-1}^{[n]}, \quad S_0^{[n]} = S_0.\]

Here, \( \rho_k^{[n]} \) and \( S_k^{[n]} \) are the values of the bond and stock, respectively, on \([t_k^{[n]}, t_{k+1}^{[n]}]\). We assume that for each \( k \),
\[ r_k^{[n]} = \frac{r}{n} \]
where \( r \) is the constant interest rate appearing in (24). The process \( \rho_k^{[n]} = (\rho_k^{[n]})_{k \in \mathbb{N}} \) is defined for \( n \) fixed by
\[ 1 + \rho_k^{[n]} = \exp(X_k^{[n]}), k \in \mathbb{N} \]
where \((X_k^{[n]})_{k \in \mathbb{N}} \) is a sequence of independent random variables with distribution
\[ \mathbb{P}(X_k^{[n]} = \frac{\sigma}{\sqrt{n}}) = g_n(T^k x) = 1 - \mathbb{P}(X_k^{[n]} = -\frac{\sigma}{\sqrt{n}}), \]
where \( D = (E, A, \mu, T) \) is a dynamical system, the functions \( g_n \) are defined on \( E \) with values in \([0, 1]\) and \( x \) is a point of \( E \). We assume that the functions \( g_n \) satisfy the two hypotheses
\[ (H_1) : \sup_{t \in [0,T], x \in E} \left| \frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]} (2g_n(T^k x) - 1) + \frac{\sigma t}{2} \right| = o(1), \]
\[ (H_2) : \sup_{x \in E} \left| \frac{1}{n} \sum_{k=1}^{[nt]} [4g_n(T^k x)(1 - g_n(T^k x))] - t \right| = o(1). \]

6.3 A preliminary result

**Theorem 6.1** For every \( x \in E \), the sequence of processes \( \left( \sum_{k=1}^{[nt]} X_k^{[n]} \right)_{t \in [0,T]} \) weakly converges in the Skorohod space \( \mathcal{D}([0,T]) \) (see [3]) to the real Brownian motion with mean \( -\frac{\sigma^2 t}{2} \) and variance \( \sigma^2 t \).

**Proof.** Introduce the characteristic function of \( \sum_{k=1}^{[nt]} X_k^{[n]} \) by
\[ \phi_n(u) = \mathbb{E}(\exp(iu \sum_{k=1}^{[nt]} X_k^{[n]})). \]

By independence of the random variables \( X_k^{[n]}, k \geq 1 \),
\[ \phi_n(u) = \prod_{k=1}^{[nt]} \mathbb{E}(\exp(iu X_k^{[n]})) = \prod_{k=1}^{[nt]} Q_{k,n}(u) \]
where
\[ Q_{k,n}(u) = \cos\left(\frac{u \sigma}{\sqrt{n}}\right) + i(2g_n(T^k x) - 1) \sin\left(\frac{u \sigma}{\sqrt{n}}\right). \]
A direct calculation gives

\[ |Q_{k,n}(u)|^2 = 1 - 4g_n(T^k x)(1 - g_n(T^k x)) \sin^2 \left( \frac{u\sigma}{\sqrt{n}} \right) \]

\[ = 1 - 4g_n(T^k x)(1 - g_n(T^k x)) \frac{u^2 \sigma^2}{n} + O(n^{-2}) \]

and then

\[ |\phi_n(u)| = \prod_{k=1}^{[nt]} |Q_{k,n}(u)| = \exp \left[ \frac{1}{2} \sum_{k=1}^{[nt]} \log \left( 1 - 4g_n(T^k x)(1 - g_n(T^k x)) \frac{u^2 \sigma^2}{n} + O(n^{-2}) \right) \right] \]

\[ = \exp \left( - \frac{\sigma^2 t}{2} u^2 + o(1) \right) \]

using hypothesis \((H_2)\).

The imaginary part of the characteristic function can be rewritten as

\[ \prod_{k=1}^{[nt]} \exp \left( i \arctan \left( \frac{(2g_n(T^k x) - 1) \sin \left( \frac{u\sigma}{\sqrt{n}} \right)}{\cos \left( \frac{u\sigma}{\sqrt{n}} \right)} \right) \right) \]

\[ = \exp \left( i \sum_{k=1}^{[nt]} (2g_n(T^k x) - 1) \frac{u\sigma}{\sqrt{n}} + o(1) \right) = \exp \left( - \frac{i u\sigma^2 t}{2} \right) + o(1) \]

using hypothesis \((H_1)\). The convergence of the finite dimensional distributions of processes \((\sum_{k=1}^{[nt]} X_k^{(n)})_t\) to the one of the one-dimensional Brownian motion with mean \(-\frac{\sigma^2 t}{2}\) and variance \(\sigma^2 t\) is obtained in a classical way using the independence of the increments.

It remains to prove the tightness in \(D([0,T])\) of the sequence \((\sum_{k=1}^{[nt]} X_k^{(n)})_t\) for \(t \in [0,T]\). The classical criterion: Theorem 15.6 in Billingsley ([3]) is used. Let \(0 \leq t_1 < t < t_2 \leq T\), by independence of the random variables \(X_k^{(n)}\),

\[ \mathbb{E}(\sum_{k=1}^{[nt]} X_k^{(n)} - \sum_{k=1}^{[nt_1]} X_k^{(n)} \mid \sum_{k=1}^{[nt_2]} X_k^{(n)} - \sum_{k=1}^{[nt]} X_k^{(n)}^2) \]

\[ = \mathbb{E}(\sum_{k=1}^{[nt]} X_k^{(n)} - \sum_{k=1}^{[nt_1]} X_k^{(n)} \mid \sum_{k=1}^{[nt_2]} X_k^{(n)} - \sum_{k=1}^{[nt]} X_k^{(n)}^2) \mathbb{E}(\sum_{k=1}^{[nt]} X_k^{(n)} - \sum_{k=1}^{[nt]} X_k^{(n)}^2) \]

Now,

\[ \mathbb{E}(\sum_{k=1}^{[nt]} X_k^{(n)} - \sum_{k=1}^{[nt_1]} X_k^{(n)}^2) = \mathbb{E}(\sum_{k=[nt_1]+1}^{[nt]} X_k^{(n)}^2) \]

\[ = \frac{\sigma^2}{n} \sum_{k=[nt_1]+1}^{[nt]} (4g_n(T^k x)(1 - g_n(T^k x))) + \frac{\sigma^2}{n} \left( \sum_{k=[nt_1]+1}^{[nt]} (2g_n(T^k x) - 1) \right)^2 \]

Using hypothesis \((H_1)\), there exists a constant \(C_1 > 0\) such that for every \(n\),

\[ \left( g_n(T^{[nt_1]+k} x) - \frac{1}{2} \right)^2 \leq C_1 n(t_2 - t_1)^2. \]
And, since for every \( k, n \) and \( x \in E \), \( 0 \leq 4g_n(T^kx)(1 - g_n(T^kx)) \leq 1 \), we have

\[
\sum_{k=\lfloor nt \rfloor + 1}^{\lfloor nt \rfloor} 4g_n(T^kx)(1 - g_n(T^kx)) \leq \lfloor nt \rfloor - \lfloor nt_1 \rfloor \leq \lfloor nt_2 \rfloor - \lfloor nt_1 \rfloor.
\]

If \( t_2 - t_1 \geq \frac{1}{n}, \lfloor nt_2 \rfloor - \lfloor nt_1 \rfloor \leq 2n(t_2 - t_1) \) so there exists \( C > 0 \) such that

\[
\mathbb{E}(|\sum_{k=1}^{\lfloor nt \rfloor} X_k^{(n)} - \sum_{k=1}^{\lfloor nt_1 \rfloor} X_k^{(n)}|^2 - |\sum_{k=1}^{\lfloor nt_2 \rfloor} X_k^{(n)} - \sum_{k=1}^{\lfloor nt \rfloor} X_k^{(n)}|^2 |) \leq C(t_2 - t_1)^2.
\]

If \( t_2 - t_1 < \frac{1}{n} \), either \( t \) and \( t_1 \) belong in the same interval \( \left[ \frac{i-1}{n}, \frac{i}{n} \right] \) or \( t \) and \( t_2 \) do. In both cases, the left hand side of (26) vanishes. The tightness follows from Billingsley’s Theorem 15.6 ([3]).

\[ \diamond \]

### 6.4 The main convergence result.

We now are able to prove that the sequence of dynamic Cox-Ross-Rubinstein’s models converges to the continuous Black-Scholes model.

**Theorem 6.2** For every \( x \in E \), the sequence \( (B_{[nt]}^{(n)}, S_{[nt]}^{(n)})_{t \in [0,T]} \) converges weakly in \( \mathcal{D}([0,T], \mathbb{R}^2) \) to \( (B_t, S_t)_{t \in [0,T]} \) as \( n \) tends to infinity.

**Proof.** By definition,

\[
\log(S_{[nt]}^{(n)}) = \log(S_0^{(n)}) + \sum_{k=1}^{\lfloor nt \rfloor} \log(1 + \rho_k^{(n)}) = \log(S_0^{(n)}) + \sum_{k=1}^{\lfloor nt \rfloor} X_k^{(n)}
\]

From Theorem 6.1, the sequence of processes

\[
\left( \sum_{k=1}^{\lfloor nt \rfloor} X_k^{(n)} \right)_{t \in [0,T]}
\]

weakly converges in the Skorohod space \( \mathcal{D}([0,T]) \) to the one-dimensional Brownian motion with mean \(-\frac{\sigma^2}{2}t\) and variance \(\sigma^2 t\). Since the functional exponential is continuous in the Skorohod space, we deduce that \( (S_{[nt]}^{(n)})_{t \in [0,T]} \) weakly converges to the stock price

\[
S_t = S_0 \exp \left( -\frac{\sigma^2}{2} t + \sigma W_t \right)
\]

which is the unique solution of (25).

The theorem is proved since the deterministic sequence

\[
B_{[nt]}^{(n)} = \left( 1 + \frac{r}{n} \right)^{\lfloor nt \rfloor} B_0
\]

converges to \( B_t = B_0 \exp(rt) \), the solution of equation (24). \( \diamond \)
6.5 Particular examples.

1. Cox-Ross-Rubinstein Theorem:
   Let us fix $n \geq 1$ and for every $k \in \mathbb{N}$, we choose
   \[
   g_n := g_n(T^k x) = \frac{r_n - a_n}{b_n - a_n}
   \]
   with $r_n = r/n$, $a_n = (1 + r_n) \exp(-\sigma/\sqrt{n})$, $b_n = (1 + r_n) \exp(\sigma/\sqrt{n})$ with $-1 < a_n < r_n < b_n$. For each $n$ fixed, this dynamic $(B, S)$-market is the classical Cox-Ross-Rubinstein’s model. For these particular functions $g_n$, hypotheses $(H_1)$ and $(H_2)$ are clearly satisfied and Theorem 6.2 is the well-known Cox-Ross-Rubinstein’s result (see [4]) which asserts that a discrete time approximation of the Black-Scholes model is given by a sequence of Cox-Ross-Rubinstein’s models. It is worth noting that in this particular case, with the notations $\rho_n^{(n)} = \rho_n, n \geq 1$, the sequence of random variables
   \[
   \left( \sum_{k=1}^{n} (\rho_k - r_k) \right)_{k \geq 1}
   \]
   is a martingale with respect to the natural filtration $\mathcal{F}_n = \mathcal{F}_n^S = \sigma(\rho_1, \ldots, \rho_n), n \geq 1$. So, from Lemma 2.3, we deduce that this particular measure is risk-neutral. However, in general, i.e. when the functions $g_n$ are not all constant, the above sequence does not have to be a martingale.

2. A disturbed model:
   We add to the previous model dynamical perturbations as follows. Let $g$ be a strictly positive, bounded function defined on $E$. We denote by $\|g\|_\infty$ the supremum norm of $g$. For every $n \geq 1$, the functions giving the distributions of the dynamic random variables $\rho_k^{(n)}$ (or $X_k^{(n)}$) are chosen as
   \[
   g_n(x) = \frac{r_n(x) - a_n(x)}{b_n(x) - a_n(x)}
   \]
   where for every $x \in E$,
   \[
   a_n(x) = (1 + r_n(x)) \exp\left(-\frac{g(x)}{\sqrt{n}}\right) - 1
   \]
   and
   \[
   b_n(x) = (1 + r_n(x)) \exp\left(\frac{g(x)}{\sqrt{n}}\right) - 1.
   \]
   From this definition, it clearly follows that for every $n \geq 1$, for every $x \in E$, $-1 < a_n(x) < r_n(x) < b_n(x)$ and hence, each function $g_n$ take its values in the interval $[0, 1]$. Let us assume that
   \[
   (H_3) : \sup_{t \in [0, T] \cap E} \left| \frac{1}{n} \sum_{k=1}^{n} g(T^k x) - \sigma t \right| = o(1).
   \]
   Large classes of functions $g$ for which this kind of convergence holds will be given in the next section. For this particular model which is a modification of the so-called Cox-Ross-Rubinstein model, Theorem 6.2 holds. To prove it we have to show that hypotheses $(H_1)$
and \((H_2)\) hold. Let us define for every \(t \in [0, T], x \in E,\)

\[
M_{n,t}(x) = \frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]} (2g_n(T^{k}x) - 1).
\]

By definition of the \(g_n\)s,

\[
M_{n,t}(x) = \frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]} \left( \frac{2r_n(T^{k}x) - a_n(T^{k}x) - b_n(T^{k}x)}{b_n(T^{k}x) - a_n(T^{k}x)} \right)
= \frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]} \left( \frac{(1 + r_n(T^{k}x))(1 - \cosh(\frac{g(T^{k}x)}{\sqrt{n}}))}{(1 + r_n(T^{k}x)) \sinh(\frac{g(T^{k}x)}{\sqrt{n}})} \right)
= -\frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]} \tanh\left( \frac{g(T^{k}x)}{2\sqrt{n}} \right)
= -\frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]} \left[ \tanh\left( \frac{g(T^{k}x)}{2\sqrt{n}} \right) - \frac{g(T^{k}x)}{2\sqrt{n}} \right] - \frac{1}{2n} \sum_{k=1}^{[nt]} g(T^{k}x).
\]

Now, for every \(x \in \mathbb{R}\), we have \(|\tanh(x) - x| \leq \frac{|x|^3}{3}\), thus

\[
\sup_{t \geq 0, x \in E} \left| \frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]} \left[ \tanh\left( \frac{g(T^{k}x)}{2\sqrt{n}} \right) - \frac{g(T^{k}x)}{2\sqrt{n}} \right] \right| \leq C \frac{[nT]|g|_\infty^3}{n^2} = o(1).
\]

Then, hypothesis \((H_1)\) follows from hypothesis \((H_3)\).

Moreover, hypothesis \((H_2)\) can be proved when hypothesis \((H_3)\) is satisfied. Straightforward computations give

\[
\frac{1}{n} \sum_{k=1}^{[nt]} [4g_n(T^{k}x)(1 - g_n(T^{k}x))] \geq \frac{4}{n} \sum_{k=1}^{[nt]} \left[ \frac{(r_n(T^{k}x) - a_n(T^{k}x) - b_n(T^{k}x))}{(b_n(T^{k}x) - a_n(T^{k}x))^2} \right]
= \frac{4}{n} \sum_{k=1}^{[nt]} \frac{(\cosh(\frac{g(T^{k}x)}{\sqrt{n}}) - 1)}{\sinh^2(\frac{g(T^{k}x)}{\sqrt{n}})}
= \frac{[nt]}{n} - \frac{1}{n} \sum_{k=1}^{[nt]} \tanh^2\left( \frac{g(T^{k}x)}{2\sqrt{n}} \right)
\]

Then, from hypothesis \((H_3)\), we get that hypothesis \((H_2)\) is satisfied:

\[
\sup_{x \in E} \left| \frac{1}{n} \sum_{k=1}^{[nt]} [4g_n(T^{k}x)(1 - g_n(T^{k}x))] - t \right| \leq \frac{[nt]}{n} - t + \frac{1}{n} \sup_{x \in E} \sum_{k=1}^{[nt]} \tanh^2\left( \frac{g(T^{k}x)}{2\sqrt{n}} \right)
\leq \left| \frac{[nt]}{n} - t \right| + \frac{T}{4n} \|g\|_\infty^2 = o(1)
\]

since \(|\tanh(x)| \leq |x|\) for every \(x\).
6.6 Example: The rotation on the torus $\mathbb{T}^d$

Explicit calculations are possible in this case. If the angle of rotation is rational, we get a periodic dynamical system.

Given $\alpha \in \mathbb{R}^d$. Consider the map

$$T_\alpha : \mathbb{T}^d \to \mathbb{T}^d$$

$$x \mapsto x + \alpha \mod 1 = x_1 + \alpha_1 \mod 1, \ldots, x_d + \alpha_d \mod 1.$$ 

It is clear that $T_\alpha$ preserves Lebesgue measure $\mu$ on $\mathbb{T}^d$. The angle $\alpha = (\alpha_1, \ldots, \alpha_d)$ is said irrational if $1, \alpha_1, \ldots, \alpha_d$ are linearly independent on $\mathbb{Q}$. It can be proved that $T_\alpha$ is ergodic if and only if $\alpha$ is irrational. Under this condition, for every $g \in L^1(\mu)$, for almost every $x \in \mathbb{T}^d$,

$$M_n(g) = \frac{1}{n} \sum_{i=1}^{n} g(T^i_\alpha x) - \int_{\mathbb{T}^d} g(t) dt \to 0 \quad n \to \infty$$

When $g$ is with bounded variation, this result holds for every $x \in \mathbb{T}^d$ and the convergence of the sequence $M_n$ to 0 is uniform with respect to $x \in \mathbb{T}^d$. But this uniform convergence is not enough to get $(H_3)$ due to the supremum norm on the time $t \in [0, T]$. When $d = 1$, for all irrational badly approximated by rationals, Denjoy-Koksma’s inequality gives us a majorization of $M_n$ uniformly in $x$ for $n$ large enough. But when $d \geq 2$, Denjoy-Koksma’s inequality does not hold (see Yoccoz [46]) and the method of low discrepancy sequences has to be used.

1. $d = 1$

   We denote $a_n$ the $n$-th partial quotient of $\alpha$, i.e.

   $$\alpha = [\alpha] + \frac{1}{a_1 + \frac{1}{a_2 + \cdots}}.$$ 

   From the following proposition, the function $g$ satisfying hypothesis $(H_3)$ can be chosen into the set of functions of bounded variation defined on $\mathbb{T}^1$ with values in $\mathbb{R}^*_+$.

   **Proposition 6.1** Let $g$ be a function with bounded variation $V(g)$. For every irrational $\alpha$ such that the inequality $a_m < m^{1+\epsilon}$, where $\epsilon > 0$, is satisfied eventually for all $m$,

   $$\sup_{x \in \mathbb{T}^1} \left| \sum_{k=1}^{n} (g(T^k_\alpha x) - \int_{\mathbb{T}^1} g(t) dt) \right| = \mathcal{O}(\log^{2+\epsilon} n).$$

   The proof of this result can be found in Appendix.

2. $d \geq 1$

   The following proposition also permits us to choose a large class of functions $g$ for the particular dynamical system $D = (\mathbb{T}^d, \mathcal{B}(\mathbb{T}^d), \mu, T_\alpha)$.
**Proposition 6.2** Let $g$ be a function with bounded variation in the sense of Hardy and Krause, and $\alpha$ an irrational vector of type $\eta$, then

$$\sup_{x \in \mathbb{T}^d} \left| \sum_{k=1}^n (g(T^k\alpha x) - \int_{\mathbb{T}^d} g(t) dt) \right| = \begin{cases} \mathcal{O}(\log^{d+1} n) & \text{if } \eta = 1 \\ \mathcal{O}(n^{1-\frac{1}{(d+1)-1}}) \log n & \text{if } \eta > 1. \end{cases}$$

The proof of this result can be found in Appendix.

**The disturbed model revisited:**

Let us consider the model introduced in Section 6.5 (Item 2). It has been proved that some temporal fluctuations around the parameter $\sigma > 0$ do not affect the approximation of the Black-Scholes model by a sequence of dynamic Cox-Ross-Rubinstein models. However, the fluctuations around $\sigma$ have to be well-controlled, in fact, it is assumed that in the disturbed model, the function $g$ has to be positive, bounded and satisfying the hypothesis:

$$(H_3): \quad \sup_{t \in [0,T], x \in E} \left| \frac{1}{[nT]} \sum_{k=1}^{[nT]} g(T^k\alpha x) - \sigma t \right| = o(1).$$

Let us choose $E = \mathbb{T}$, $T = T_\alpha$ the rotation on the one-dimensional torus with angle $\alpha \in \mathbb{R}$ defined as in Proposition 6.1. From Proposition 6.1, every function $g : [0, 1] \rightarrow \mathbb{R}$ with bounded variation satisfies $(H_3)$ as soon as $\int_{0}^{1} g(t) dt = \sigma$ since there exists a constant $C > 0$ such that

$$\sup_{t \in [0,T], x \in E} \left| \sum_{k=1}^{[nT]} (g(T^k\alpha x) - \sigma) \right| \leq C \log^2(\beta([nT])) = o(n).$$

For instance, the bounded functions

$$g_1(x) = \begin{cases} 2\sigma \cos^2(2\pi x) & \text{if } x \neq \frac{1}{1+\frac{3}{\eta}} \\ 1 & \text{otherwise} \end{cases}$$

and

$$g_2(x) = \begin{cases} 2\sigma(x-[x]) & \text{if } x \neq 0 \\ 1 & \text{otherwise} \end{cases}$$

satisfy $(H_3)$. Explicit numerical calculations can be done for these functions.

**6.7 The Riemann dynamic random variables.**

Let $g_n$ be one-periodic functions defined on $\mathbb{R}$ with values in $[0, 1]$ and $(X_k^{[n]})_k$ be a sequence of independent random variables with values in $\mathbb{Z}$ with distribution

$$\mathbb{P}(X_k^{[n]} = z) = \begin{cases} g_n(x + \frac{k}{n}) & \text{if } z = \sigma/\sqrt{n} \\ 1 - g_n(x + \frac{k}{n}) & \text{if } z = -\sigma/\sqrt{n} \\ 0 & \text{otherwise} \end{cases}$$

for any $x \in \mathbb{R}$ and $\sigma > 0$ is fixed. In the binary dynamic $(B, S)$-market model introduced in Section 6.2 the distribution of the random variables $X_k^{[n]}$ can be replaced by (27). Assume that
the functions $g_n$ satisfy the two hypotheses:

$$(H_4): \sup_{t \in [0,T], x \in [0,1]} \left| \frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]} [2g_n(x + \frac{k}{n}) - 1] + \frac{\sigma t}{2} \right| = o(1),$$

$$(H_5): \sup_{x \in [0,1]} \left| \frac{1}{n} \sum_{k=1}^{[nt]} \left[ 4g_n(x + \frac{k}{n})(1 - g_n(x + \frac{k}{n})) \right] - t \right| = o(1).$$

Then, under hypotheses $(H_4)$ and $(H_5)$, conclusions of Theorems 6.1 and 6.2 hold for this new model. Consider the disturbed model defined in Section 6.5 (Item 2) when the distribution of the random variable $X_k^{(n)}$ is given by (27) and still denote by $g$ the function introduced to define the $g_n$'s. Hypotheses $(H_4)$ and $(H_5)$ are easily deduced from the hypothesis

$$(H_6): \sup_{t \in [0,T], x \in [0,1]} \left| \frac{1}{n} \sum_{k=1}^{[nt]} g \left( x + \frac{k}{n} \right) - \sigma t \right| = o(1).$$

A large class of functions satisfying $(H_6)$ can be chosen from the following proposition.

**Proposition 6.3** Let $g$ be a one-periodic function which can be expanded into a Fourier series $g(x) = \sum_{h \in \mathbb{Z}} c_h e^{2\pi ihx}$.

When there exists $\beta > 1$ such that $|c_n| + |c_{-n}| = O(n^{-\beta})$, then

$$\sup_{x \in [0,1]} \left| \sum_{i=1}^{n} \left( g(x + \frac{i}{n}) - \int_{[0,1]} g(t)dt \right) \right| = O(n^{1-\beta}).$$

The proof of the proposition is straightforward. From this proposition, each function $g$ expandable into a Fourier series

$$g(x) = \sum_{h \in \mathbb{Z}} c_h e^{2\pi ihx}$$

where the coefficients $(c_h)_{h \in \mathbb{Z}}$ verify $c_0 = \sigma > 0$ and there exists $\beta > 1$ such that

$$|c_n| + |c_{-n}| = O(n^{-\beta})$$

satisfies the hypothesis $(H_6)$ and can be chosen to define the $g_n$'s.
7 Appendix

A. Case of one-dimensional torus

Let $\alpha$ be an irrational. We call a rational $p/q$ with $p, q$ relatively prime such that $|\alpha - p/q| < 1/q^2$, a rational approximation of $\alpha$. When $\alpha$ has the continued fraction expansion $\alpha = [a_0] + [a_1, \ldots, a_n, \ldots]$, the $n$-th principal convergent of $\alpha$ is $p_n/q_n$ where, for all $n \geq 2$, we have

$$p_n = a_n p_{n-1} + p_{n-2},$$

$$q_n = a_n q_{n-1} + q_{n-2},$$

and the recurrence is given by defining the values of $p_0, p_1$ and $q_0, q_1$.

**Denjoy-Koksma's inequality** Let $g : \mathbb{R} \to [0, 1]$ be a function with bounded variation $V(g)$ and $p/q$ a rational approximation of $\alpha$. Then, for every $x \in \mathbb{T}^1$,

$$\left| \sum_{l=1}^{q} g(T_{l \alpha} x) - q \int_{\mathbb{T}^1} g(t)dt \right| \leq V(g).$$

**Proof of Proposition 6.1.** The sequence of integers $(q_i)_{i \geq 1}$ being strictly increasing, for a given $n \geq 1$, there exists $m_n \geq 0$ such that

$$q_{m_n} \leq n < q_{m_n+1}.$$ 

By Euclidean division, we have $n = b_{m_n} q_{m_n} + n_{m_n-1}$ with $0 \leq n_{m_n-1} < q_{m_n}$. We can use the usual relations

$$q_0 = 1, q_1 = a_1,$$

$$q_n = a_n q_{n-1} + q_{n-2}, n \geq 2. \tag{28}$$

We obtain that $(a_{m_{n+1}} + 1)q_{m_n} > q_{m_{n+1}} > n$ and so $b_{m_n} \leq a_{m_{n+1}}$. If $m_n > 0$, we may write $n_{m_n-1} = b_{m_n-1} q_{m_n-1} + n_{m_n-2}$ with $0 \leq n_{m_n-2} < q_{m_n-1}$. Again, we find $b_{m_n-1} \leq a_{m_n}$. Continuing in this manner, we arrive at a representation for $n$ of the form

$$n = \sum_{i=0}^{m_n} b_i q_i$$

with $0 \leq b_i \leq a_{i+1}$ for $0 \leq i \leq m_n$ and $b_{m_n} \geq 1$. Using Denjoy-Koksma’s inequality, we get

$$\left| \sum_{l=1}^{n} g(T_{l \alpha} x) - n \int_{\mathbb{T}^1} g(x)dx \right| \leq V(g) \sum_{i=0}^{m_n} b_i \leq V(g) \sum_{i=0}^{m_n} a_{i+1}.$$ 

By hypothesis, there exists $m_0 \geq 1$ such that,

$$a_m \leq m_0^{1+\epsilon}, \forall m \geq m_0.$$ 

Let $n$ be such that $m_n > m_0$. Thus,

$$\left| \sum_{l=1}^{n} g(T_{l \alpha} x) - n \int_{\mathbb{T}^1} g(t)dt \right| \leq V(g)(\sum_{i=0}^{m_0-1} a_{i+1} + (m_n + 1)^{2+\epsilon}).$$

25
We need to know the asymptotic behavior of $m_n$. When $\alpha$ is the golden ratio, $a_n = 1$, $\forall n \geq 1$ and the relation (28) implies that $q_n \sim \frac{1}{\sqrt{n}} \alpha^{n+1}$. Let $\alpha'$ be another irrational; its partial quotients $a'_n$ satisfy necessarily $a'_n \geq 1$. Using the relation (28), we see that $q'_n \geq q_n$, $\forall n \geq 1$. Therefore, $m_n = O(\log n)$ and the proposition is proved. 

B. Generalization to $d$-dimensional torus

We recall some definitions and well known results from the method of low discrepancy sequences in dimension $d \geq 1$.

Suppose we are given a function $g(x) = g(x^{(1)}, \ldots, x^{(d)})$ with $d \geq 1$. By a partition $P$ of $[0,1]^d$, we mean a set of $d$ finite sequences $\eta^{(j)}_0, \eta^{(j)}_1, \ldots,$

$\eta^{(j)}_m (j = 1, \ldots, d), \text{with } 0 = \eta^{(j)}_0 \leq \eta^{(j)}_1 \leq \ldots \leq \eta^{(j)}_m = 1 \text{ for } j = 1, \ldots, d.$

In connection with such a partition, we define, for $j = 1, \ldots, d$ an operator $\Delta_j$ by

$$\Delta_j g(x^{(1)}, \ldots, x^{(j-1)}, \eta^{(j)}_i, x^{(j+1)}, \ldots, x^{(d)}) = \begin{cases} g(x^{(1)}, \ldots, x^{(j-1)}, \eta^{(j)}_{i+1}, x^{(j+1)}, \ldots, x^{(d)}) \\ -g(x^{(1)}, \ldots, x^{(j-1)}, \eta^{(j)}_i, x^{(j+1)}, \ldots, x^{(d)}) \end{cases},$$

for $0 \leq i < m_j$.

**Definition 7.1** For a function $g$ on $[0,1]^d$, we set

$$V^{(d)}(g) = \sup_P \left( \sum_{i_1=0}^{m_1-1} \cdots \sum_{i_d=0}^{m_d-1} |\Delta_{i_1,\ldots,i_d} g(\eta^{(1)}_{i_1}, \ldots, \eta^{(d)}_{i_d})|, \right)$$

where the supremum is extended over all partitions $P$ of $[0,1]^d$. If $V^{(d)}(g)$ is finite, then $g$ is said to be of bounded variation on $[0,1]^d$ in the sense of Vitali.

For $1 \leq p \leq d$ and $1 \leq i_1 < i_2 < \ldots < i_p \leq d$, we denote by $V^{(p)}(g;i_1, \ldots, i_p)$ the $p$-dimensional variation in the sense of Vitali of the restriction of $g$ to

$$E_{i_1 \ldots i_p}^d = \{ (t_1, \ldots, t_d) \in [0,1]^d; t_j = 1 \text{ whenever } j \text{ is none of the } i_d, 1 \leq d \leq p \}.$$ 

If all the variations $V^{(p)}(g;i_1, \ldots, i_p)$ are finite, the function $g$ is said to be of bounded variation on $[0,1]^d$ in the sense of Hardy and Krause.

Let $x_1, \ldots, x_n$ be a finite sequence of points in $[0,1]^d$ with $x_l = (x_{1l}, \ldots, x_{dl})$ for $1 \leq l \leq n$.

We introduce the function

$$R_n(t_1, \ldots, t_d) = \frac{A(t_1, \ldots, t_d; n)}{n} - t_1 \cdots t_d$$

for $(t_1, \ldots, t_d) \in [0,1]^d$, where $A(t_1, \ldots, t_d; n)$ denotes the number of elements $x_l, 1 \leq l \leq n,$ for which $x_{li} < t_i$ for $1 \leq i \leq d$.

**Definition 7.2** The discrepancy $D^*_n$ of the sequence $x_1, \ldots, x_n$ in $[0,1]^d$ is defined to be

$$D^*_n = \sup_{(t_1, \ldots, t_d) \in [0,1]^d} |R_n(t_1, \ldots, t_d)|.$$
For a real number \( t \), let \( \| t \| \) denote its distance to the nearest integer, namely,

\[
\| t \| = \inf_{n \in \mathbb{Z}} | t - n | = \inf(\{ t \}, 1 - \{ t \})
\]

where \( \{ t \} \) is the fractional part of \( t \).

**Definition 7.3** For a real number \( \eta \), a \( d \)-tuple \( \alpha = (\alpha_1, \ldots, \alpha_d) \) of irrationals is said to be of type \( \eta \) if \( \eta \) is the infimum of all numbers \( \sigma \) for which there exists a positive constant \( c = c(\sigma; \alpha_1, \ldots, \alpha_d) \) such that

\[
d^*(h) \| < h, \alpha > \| \geq c
\]

holds for all \( h \neq 0 \) in \( \mathbb{Z}^d \), where \( d(h) = \prod_{i=1}^d \max(1, |h_i|) \) and \( < \cdot, \cdot > \) denotes the Euclidean inner product in \( \mathbb{R}^d \).

The type \( \eta \) of \( \alpha \) is also equal to

\[
\sup\{ \gamma : \inf_{h \in \mathbb{Z}^d} d^*(h) \| < h, \alpha > \| = 0 \}.
\]

We always have \( \eta \geq 1 \) (see [32]). Now we give a result (see [24]) which yields the asymptotic behavior of the discrepancy of the sequence \( w = (x_1 + l\alpha_1, \ldots, x_d + l\alpha_d), l = 1, 2, \ldots \) as a function of the mutual irrationality of the components of \( \alpha \).

**Proposition 7.1** Let \( \alpha = (\alpha_1, \ldots, \alpha_d) \) be an irrational vector. Suppose there exists \( \eta \geq 1 \) and \( c > 0 \) such that

\[
d^*(h) \| < h, \alpha > \| \geq c
\]

for all \( h \neq 0 \) in \( \mathbb{Z}^d \). Then, for every \( x \in [0,1]^d \), the discrepancy of the sequence \( w = (x_1 + l\alpha_1, \ldots, x_d + l\alpha_d), l = 1, 2, \ldots \) satisfies \( D_n^*(w) = O(n^{-1}\log^{d+1} n) \) for \( \eta = 1 \) and \( D_n^*(w) = O(n^{-\frac{1}{(n-1)\log n}}) \) for \( \eta > 1 \).

The proof is based on the Erdős-Turán-Koksma’s theorem: For \( h \in \mathbb{Z}^d \), define \( p(h) = \max_{1 \leq j \leq d} |h_j| \). Let \( x_1, \ldots, x_n \) be a finite sequence of points in \( \mathbb{R}^d \). Then, for any positive integer \( m \), we have

\[
D_n^* \leq C_d \left( \frac{1}{m} + \sum_{0 \leq p(h) \leq m} \frac{1}{d(h)} \left| \frac{1}{n} \sum_{t=1}^n e^{2\pi i \langle h, x_t \rangle} \right| \right)
\]

where \( C_d \) only depends on the dimension \( d \). This theorem combined with the results of [24] (p.131) gives us the result.
Theorem 7.1 (Hlawka, Zaremba) Let $g$ be of bounded variation on $[0,1]^d$ in the sense of Hardy and Krause, and let $\omega$ be a finite sequence of points $x_1, \ldots, x_n$ in $[0,1]^d$. Then, we have
\[
\left| \frac{1}{n} \sum_{i=1}^{n} g(x_i) - \int_{\mathbb{R}^d} g(t) dt \right| \leq \sum_{p=1}^{d} \sum_{1 \leq i_1 < \ldots < i_p \leq d} V^p(g; i_1, \ldots, i_p) D_{n}^*(\omega_{i_1 \ldots i_p}),
\]
where $D_{n}^*(\omega_{i_1 \ldots i_p})$ is the discrepancy in $E_{i_1 \ldots i_p}^d$ of the sequence $\omega_{i_1 \ldots i_p}$ obtained by projecting $\omega$ onto $E_{i_1 \ldots i_p}^d$.

Proof of Proposition 6.2. Let $\eta'$ be such that $\eta \leq \eta' < 1 + \frac{1}{d}$. There exists $c > 0$ such that $dn'(h) < h, \alpha > \| \geq c$ holds for all $h \neq 0$ in $\mathbb{Z}^d$. Suppose we are given a p-tuple $\alpha_p = (\alpha_{i_1}, \ldots, \alpha_{i_p})$ of $\alpha$ for $1 \leq p \leq d$, then $r_{\eta'}(h) < h, \alpha_p > \| \geq c$ holds for all $h \neq 0$ in $\mathbb{Z}^d, 1 \leq p \leq d$. Thus, every p-tuple, $1 \leq p \leq d$, is of type $\delta$ such that $1 < \delta \leq \eta$ and $(\alpha_{i_1}, \ldots, \alpha_{i_p})$ is an irrational vector. For every $p, 1 \leq p \leq d$, we define $w_{i_1 \ldots i_p}$ by the projection of $w$ on $E_{i_1 \ldots i_p}^d$.

From the previous proposition, we have for every $p, 1 \leq p \leq d$,
\[
\begin{cases}
    nD_{n}^*(w_{i_1 \ldots i_p}) = O(\log^{p+1} n) & \text{if } \delta = 1 \\
    nD_{n}^*(w_{i_1 \ldots i_p}) = O(n^{1 - \frac{1}{(\delta - 1)p + 1} \log n}) & \text{if } 1 < \delta \leq \eta.
\end{cases}
\]

Now, for all $p = 1, \ldots, d$, we arrive at
\[
0 \leq 1 - \frac{1}{(\delta - 1)p + 1} \leq 1 - \frac{1}{(\eta - 1)d + 1} \leq 1.
\]

Therefore, using Hlawka-Zaremba’s theorem, we obtain Proposition 6.2. \quad \star

References


30