A note on Gullstrand's formula

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Abstract

In this note, we derive Gullstrand's formula for a thick lens.

Contents

1 Introduction

We consider two spherical diopters $\mathcal{C}_1$ (center C_1 and radius R_1) and $\mathcal{C}_2$ (center C_2 and radius R_2) enclosing a matter of index n_2, bounded by a matter of index n_1. The system is considered centered and we want to find the focal distance of the pair $(\mathcal{C}_1, \mathcal{C}_2)$ in the Gaussian approximation of small angles.

A well-known formula in optics, named after Allvar Gullstrand (1862–1930, recipient of the 1911 Nobel prize in medicine), gives the focal distance of the system. In this note, we will try to prove the formula *ex nihilo*.

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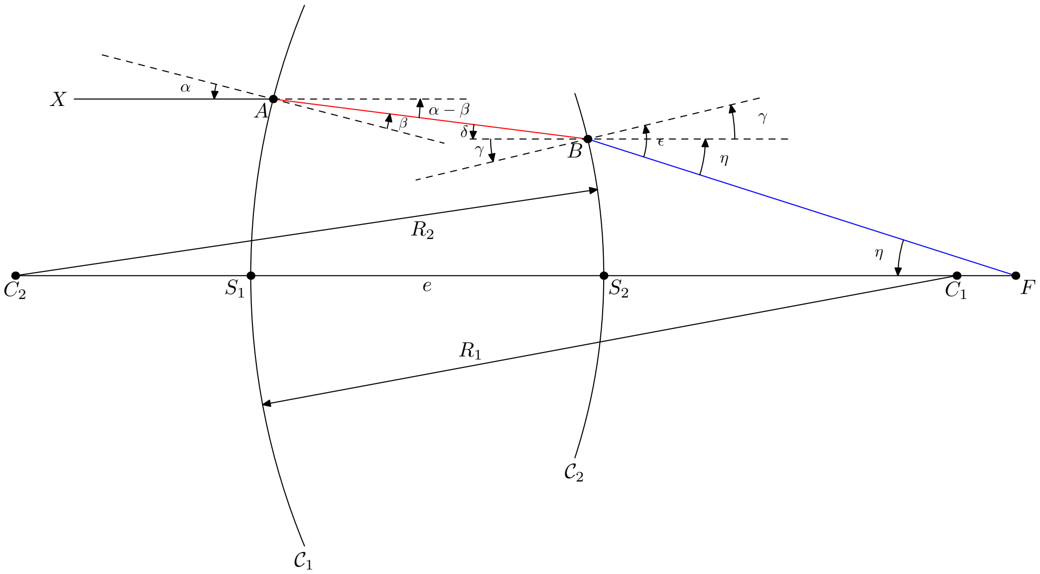

Figure 1: Notations for Gullstrand's formula (drawing not to scale). The angles are measured positively counterclockwise.

2 Finding the focus F

Figure 1 shows the various angles involved in our computation of F. The figure is not to scale and all of the angles α, β, etc., are assumed to be very small. In that case, we can consider $\sin\alpha \approx \alpha$, and likewise for the other angles.

Assuming $n = \frac{n_2}{n_1}$, $\overline{R}_1 = \overline{S_1C_1}$, $\overline{R}_2 = \overline{S_2C_2}$, $e = \overline{S_1S_2}$, an horizontal incident ray XA and the distance from A and B to the axis of the system to be y_A and y_B, we have the following approximations:

$$\alpha \approx \frac{y_A}{\overline{R}_1} \qquad \gamma \approx -\frac{y_B}{\overline{R}_2} \tag{1}$$

$$\beta \approx \frac{\alpha}{n} \qquad \delta = \alpha - \beta \tag{2}$$

$$\delta + \gamma \approx \frac{\epsilon}{n} \qquad \eta = \epsilon - \gamma \tag{3}$$

$$y_A - y_B \approx e(\alpha - \beta) \qquad \frac{y_B}{\overline{S_2F}} \approx \eta \tag{4}$$

The distance $\overline{S_2F}$ can be computed as follows:

$$\overline{S_2F} \approx \frac{y_B}{\eta} = \frac{y_A - e(\alpha - \frac{\alpha}{n})}{n(\delta+\gamma) - \gamma} \approx \frac{\alpha(\overline{R}_1 - e + \frac{e}{n})}{\gamma(n-1) + n(\alpha - \frac{\alpha}{n})} \tag{5}$$

$$\approx \frac{\alpha(\overline{R}_1 - e + \frac{e}{n})}{\frac{e(\alpha-\beta)-y_A}{\overline{R}_2}(n-1) + n(\alpha - \frac{\alpha}{n})} \approx \frac{\alpha(\overline{R}_1 - e + \frac{e}{n})}{\left(\frac{e}{\overline{R}_2}(\alpha-\beta) - \alpha\frac{\overline{R}_1}{\overline{R}_2}\right)(n-1) + n(\alpha - \frac{\alpha}{n})} \tag{6}$$

$$\approx \frac{\alpha(\overline{R}_1 - e + \frac{e}{n})}{\alpha\left(\frac{e}{\overline{R}_2}\frac{n-1}{n} + 1 - \frac{\overline{R}_1}{\overline{R}_2}\right)(n-1)} \tag{7}$$

$$\approx \frac{\overline{R}_1 - e\left(\frac{n-1}{n}\right)}{\left(\frac{e}{\overline{R}_2}\frac{n-1}{n} + 1 - \frac{\overline{R}_1}{\overline{R}_2}\right)(n-1)} \tag{8}$$

We can also express $\frac{1}{\overline{S_2F}}$:

$$\frac{1}{\overline{S_2F}} \approx \frac{\left(\frac{e}{\overline{R}_2}\frac{n-1}{n} + 1 - \frac{\overline{R}_1}{\overline{R}_2}\right)(n-1)}{\overline{R}_1 - e\left(\frac{n-1}{n}\right)} \approx \frac{\left(1 - \frac{\overline{R}_1}{\overline{R}_2}\right)(n-1) + \frac{e}{\overline{R}_2}\frac{(n-1)^2}{n}}{\overline{R}_1 - e\left(\frac{n-1}{n}\right)} \tag{9}$$

If we assume $|\overline{R}_1| \gg e$, we have:

$$\frac{1}{\overline{S_2F}} \approx \left[\left(1 - \frac{\overline{R}_1}{\overline{R}_2}\right)(n-1) + \frac{e}{\overline{R}_2}\frac{(n-1)^2}{n}\right]\frac{1}{\overline{R}_1}\left(1 + \frac{e}{\overline{R}_1}\frac{n-1}{n}\right) \tag{10}$$

$$\approx \frac{n-1}{\overline{R}_1} + \frac{1-n}{\overline{R}_2} + \frac{e}{\overline{R}_1\overline{R}_2}\frac{(n-1)^2}{n} - e\left(\frac{\overline{R}_1 - \overline{R}_2}{\overline{R}_1^2\overline{R}_2}\right)\frac{(n-1)^2}{n} \tag{11}$$

Setting $V' = \frac{1}{\overline{S_2F}}$, $V_1 = \frac{n-1}{\overline{R}_1}$ and $V_2 = \frac{1-n}{\overline{R}_2}$, we have:

$$V' \approx V_1 + V_2 - e\frac{V_1V_2}{n} - e\left(\frac{\overline{R}_1 - \overline{R}_2}{\overline{R}_1^2\overline{R}_2}\right)\frac{(n-1)^2}{n} \tag{12}$$

This expression appears very familiar. In fact, it looks like Gullstrand's formula, except for the last term. One might be tempted to conclude that Gullstrand's formula is incorrect. However, the focal length is not measured from S_2, but from the principal image plane. We will now locate the principal planes.

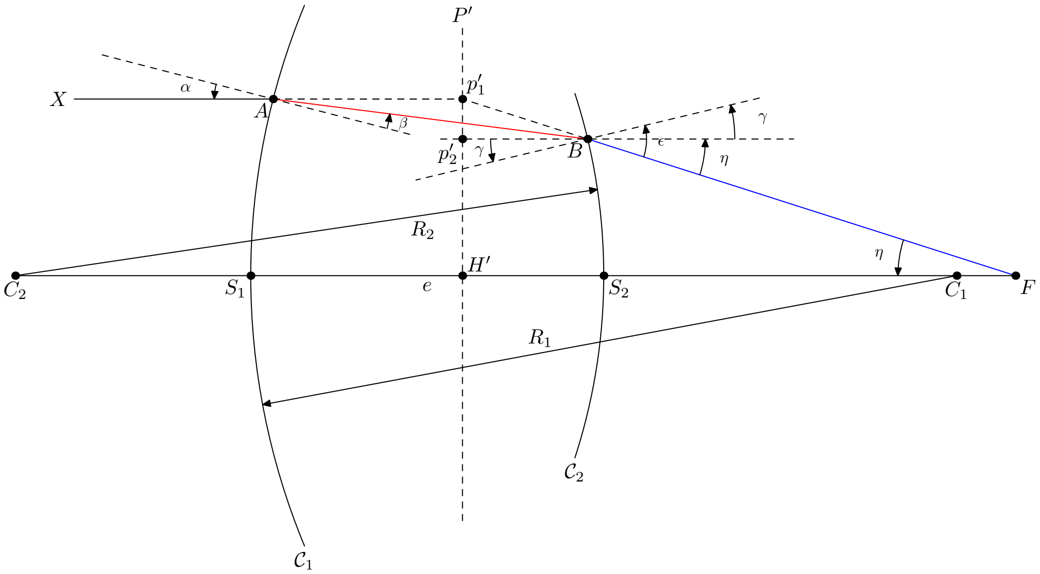

Figure 2: The principal image plane P'. There is also a similar object plane P. (drawing not to scale)

3 Principal planes

The principal image plane P' is defined as the set of the intersections of the incident parallel rays with the outgoing ones. On figure 2, one intersection is shown, namely point p_1'. Although this is *a priori* not obvious, the computation shows that the coordinate of p_1' along the horizontal axis does not depend on the angle α, and this warrants the assertion that p_1' lies in a plane. The distance $\overline{H'S_2}$ can be computed as follows. First, our angles being small, we have

$$\frac{y_A - y_B}{\overline{H'S_2}} \approx \eta \tag{13}$$

Then:

$$\overline{H'S_2} \approx \frac{y_A - y_B}{\eta} \approx \frac{e\left(\alpha - \frac{\alpha}{n}\right)}{n(\delta+\gamma)-\gamma} \approx \frac{e\alpha\frac{n-1}{n}}{n\left(\alpha - \frac{\alpha}{n} + \frac{y_B}{\overline{R}_2}\right) - \frac{y_B}{\overline{R}_2}} \tag{14}$$

$$\approx \frac{\alpha e\frac{n-1}{n}}{n\left(\alpha - \frac{\alpha}{n}\right) + \frac{n-1}{\overline{R}_2}\left(\alpha\overline{R}_1 - e\left(\alpha - \frac{\alpha}{n}\right)\right)} \tag{15}$$

$$\approx \frac{e\frac{n-1}{n}}{n - 1 + (n-1)\frac{\overline{R}_1}{\overline{R}_2} - e\frac{(n-1)^2}{n\overline{R}_2}} \tag{16}$$

4 Gullstrand's formula

We now have:

$$
\begin{aligned}
f = \overline{H'F} &= \overline{H'S_2} + \overline{S_2F} && (17)\\
&\approx \frac{e\frac{n-1}{n}}{n-1+(n-1)\frac{\overline{R}_1}{\overline{R}_2} - e\frac{(n-1)^2}{n\overline{R}_2}} + \frac{\overline{R}_1 - e\left(\frac{n-1}{n}\right)}{\left(\frac{e}{\overline{R}_2}\frac{n-1}{n} + 1 - \frac{\overline{R}_1}{\overline{R}_2}\right)(n-1)} && (18)\\
&\approx \frac{\overline{R}_1}{\left(\frac{e}{\overline{R}_2}\frac{n-1}{n} + 1 - \frac{\overline{R}_1}{\overline{R}_2}\right)(n-1)} && (19)
\end{aligned}
$$

and therefore the vergence V:

$$
\begin{aligned}
V = \frac{1}{f} = \frac{1}{\overline{H'F}} &\approx \frac{\left(\frac{e}{\overline{R}_2}\frac{n-1}{n} + 1 - \frac{\overline{R}_1}{\overline{R}_2}\right)(n-1)}{\overline{R}_1} && (20)\\
&\approx \frac{n-1}{\overline{R}_1} + \frac{1-n}{\overline{R}_2} + \frac{e}{\overline{R}_1\overline{R}_2}\frac{(n-1)^2}{n} && (21)\\
&\approx V_1 + V_2 - e\frac{V_1V_2}{n} && (22)
\end{aligned}
$$

Equation (22) is Gullstrand's formula.

5 Special cases

When $\overline{R}_1 = -\overline{R}_2 = R$, the previous formulæ can be simplified:

$$
\begin{aligned}
f &\approx \frac{R}{\left(2 - \frac{e}{R}\frac{n-1}{n}\right)(n-1)} && (23)\\
\frac{1}{f} &\approx \frac{2(n-1)}{R} - \frac{e}{R^2}\frac{(n-1)^2}{n} && (24)
\end{aligned}
$$

When $\overline{R}_1 \neq -\overline{R}_2 = R$ but e is neglectible, the previous formulæ become:

$$
V = \frac{1}{f} \approx (n-1)\left(\frac{1}{\overline{R}_1} - \frac{1}{\overline{R}_2}\right) \qquad (25)
$$

which is also known as the *spectacles makers' formula.*

6 Association of centered systems

Gullstrand's formula for thick lenses can also easily be obtained by compounding two centered systems, but we will not describe it here. The detailed construction is given in any good textbook. Each diopter can actually be viewed as a centered system, with its focal distances and principal planes. For instance, the first (left) diopter of the thick lens has its two principal planes located at the surface of the diopter, and the focal distances are

$$f_1 = \overline{R}_1 \frac{1}{n-1} \tag{26}$$

and

$$f_1' = \overline{R}_1 \frac{n}{n-1} \tag{27}$$

When associating the systems, the new principal planes are found, as well as the focal distances, using elementary geometry.

More general formulæ can also be obtained if there are three different refraction indices.