The missing new moon of A.D. 16399 and other anomalies of the Gregorian calendar

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Abstract

The Gregorian calendar is a luni-solar calendar introduced in 1582 in order to remedy defects of the Julian calendar. It contains some very rare idiosyncrasies and this article studies one of them related to epacts.

Introduction

In 1582, under pope Gregory XIII, the Catholic Church adopted a new calendar now known as the Gregorian calendar. The main reason to modify the calendar then in use — the Julian calendar — was its inadequacy at reckoning the seasons. With a year length of an average of 365.25 days, the calendar year was longer than the tropical year, or year of the seasons, by little more than 11 minutes.

The Gregorian calendar would bring the calendar year much closer to the tropical year by removing 10 minutes and 48 seconds, or three days every 400 years, thereby producing an average year length of 365.2425 days, just about 27 seconds too long. At the same time, ten days were dropped in order to put the vernal equinox around March 21, as it was supposed to have been at the Nicæa Concile in 325.

The church calendar also had a lunar component, and so it was truly a luni-solar calendar. Whereas the periodic beginning of the year corresponded to a solar year, this very year was also used to reckon the phases of the moon. Superimposed to the solar calendar was a lunar calendar with lunar months starting at specific dates. In the Julian calendar the new moons in a year would recur again after 19 years. Indeed, 19 solar years are almost equal to 235 mean lunar months. There were actually two periods of 19 years, and the position in such a cycle was given by a number between 1 and 19. This value was later called the *golden number*. Evidence suggests that this name appeared in the 9th century [24].

Like the Julian year which was slightly too long with respect to the return of the Sun, 19 Julian years were also too long with respect to the Moon: it had actually moved more than expected.

$$\frac{235 \times 29.530589}{365.25} \text{ years} = 18.9998 \text{ years} = 19 \text{ years} - 1\text{h}29\text{m}$$

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The end of the 235th lunar month occurred on average one hour and 29 minutes before the end of the 19th Julian year.

The shortening of the year in 1582 made the 19 year cycle even worse, but in the opposite direction:

$$\frac{235 \times 29.530589}{365.2425} \text{ years} = 19.00022 \text{ years} = 19 \text{ years} + 1\text{h}57\text{m}$$

Only one hour and 57 minutes after the 19th average Gregorian year would we reach the end of the 235th lunar month.

1 Epacts

In order to compensate for this, the 19 year cycle had to be dropped. Instead, more complex rules were devised and the golden number was replaced by the epact which does not always return after 19 years. The calendar which was adopted is based on the proposal of Aloisius Lilius. This proposal, now lost, was summarized by Pedro Chacon in his *Compendium Novæ Rationis Restituendi Kalendarium* in 1577 [6, 19, 27]. It assigned 31 possible numerical values to the year, that is, as many different values as there would be different cases for the layout of the new moons in the year. And it defined rules for finding the situation appropriate for each year.

In most cases, the epact of a year can be defined as follows [18]. The first of January is part of a lunar month. This lunar month has either started in the previous year, or it is starting with the first of January. In the former case, the epact is the number of days of this lunar month elapsed in the previous year, and in the latter it is 0. For instance, if the epact is III, there will already have been three days in the lunar month, namely December 29, 30 and 31. The lunar month therefore started on December 29. If the epact is $\star$, no part of the lunar month was in the previous year. If the epact is XXIX, 29 days (from December 3 to December 31) were in the previous year. The first day of each lunar month is the day of the new moon.

In reality, this definition is not completely correct, because we will see that there can be gaps in the epact. Therefore, a correct definition of the epact is the following:

> The epact of the year is the age of the moon on Jan. 1st, minus 1 (mod. 30).

or [11, p. 319]:

> The epact of the year is the age of the moon on Jan. 30.

This is not the same as saying that the epact is the age of the moon on December 31st, because the ages of the moon on December 31st and January 1st can be either identical (when the moon is slown down), they can either differ by 1 (normal case), or by 2 or 3 (when the moon is sped up) (see [1, p. 122]).

In particular when the year is not a century year, we can say that when the golden number $G > 1$ the epact is the age of the moon on December 31 and when $G = 1$, it is the age of the moon on December 31, plus 1 (mod. 30).

Lilius' proposal was to define all the possible layouts for the new moons, and to label the new moons in the perpetual calendar with the epact of the year.

This proposal was slightly changed by Clavius and it is the amended proposal which is the basis for the Gregorian calendar. In it, there are exactly 32 different cases for the epact, written $\star$ (or XXX), I, II, III, IV, etc., XXIX, 19 and 25. These values are found on the perpetual lunar calendar shown in figure 1.

A brief description of the calendar was published in 1582 in several places [15]. Clavius gave more details in 1588 when answering to Michael Maestlin [7]. But a complete description of the new calendar was only published by Clavius in 1603 [8].

Table 1 is read as follows. For any epact value except 19, the table shows on which day of the year there is a new moon (for the church). For instance, if the epact is XVI, there will be new moons on January 15, February 13, March 15, April 13, etc.

Obviously, since a lunar month has an average length of 29.53 days, certain days can count as new moons for several values of the epact. For instance, there is a new moon on June 3rd, both when the epact is XXV and XXIV. These epact duplications, as well as the introduction of the value 25 are the result of several constraints, among them the need to have the Gregorian perpetual lunar calendar compatible with the former Julian perpetual lunar calendar.

From one year to the next, the epact value increases on average by 11, which is about the average difference of days between a solar year and twelve lunar months. More exactly, we have:

$$365.2425 - 12 \times 29.530589 = 10.87...md$$

where md is the unit for 'mean days.'

This, however, is not the average epact increase. The average epact increase is obtained by scaling the range of a lunar month to the range of the epacts, namely multiplying by 30/29.530589. Therefore, the average epact increase in a 19 year Gregorian cycle should be:

$$(365.2425 - 12 \times 29.530589) \times \frac{30}{29.530589} = \frac{365.2425 \times 30}{29.530589} - 360 = 11.048...ed$$

where ed is the unit for 'epact days.' A similar result can be obtained for a 19 year Julian cycle.

The distinction between mean days and epact days is usually not made, and sometimes wrong conclusions are drawn in the literature.

From this value, it is obvious that the fractional parts will add up. After 19 years, this fractional part will be close to 1 and the epact will be increased by 12 instead of 11. The difference between 19×0.048 and 1 represents one hour and 58 minutes (ed) and corresponds to the above mentioned error of one hour and 57 minutes (md).

However, there are a two more exceptions, introduced in order to have the perpetual lunar calendar as close as possible to the mean astronomical moon. First, whenever a leap day is dropped (for instance in 2100), the epact value is actually decreased by 1. Then, there are also eight corrections on a span of 2500 years in order to compensate for the difference of two hours mentioned above. This compensation therefore involves 43 corrections in 10000 years [1]. We can summarize all these rules as follows:

[1] This value can be computed as follows. Let $a = 19\times 11+1-19\times\left(\frac{365.2425\times 30}{29.530589} - 360\right) ed \approx 0.0822 ed$ be the amount by which the cyclic moon (as defined by the epacts) must be retarded

Days	Jan.		Feb.		Mar.		Apr.		May.		Jun.		Jul.		Aug.		Sep.		Oct.		Nov.		Dec.	
	E	L	E	L	E	L	E	L	E	L	E	L	E	L	E	L	E	L	E	L	E	L	E	L
1	⋆	A	xxix	D	⋆	D	xxix	G	xxviii	B	xxvii	E	xxvi	G	{xxv xxiv	C	xxiii	F	xxii	A	xxi	D	xx	F
2	xxix	B	xxviii	E	xxix	E	xxviii	A	xxvii	C	25.xxvi	F	25.xxv	A	xxiii	D	xxii	G	xxi	B	xx	E	xix	G
3	xxviii	C	xxvii	F	xxviii	F	xxvii	B	xxvi	D	{xxv xxiv	G	xxiv	B	xxii	E	xxi	A	xx	C	xix	F	xviii	A
4	xxvii	D	25.xxvi	G	xxvii	G	25.xxvi	C	25.xxv	E	xxiii	A	xxiii	C	xxi	F	xx	B	xix	D	xviii	G	xvii	B
5	xxvi	E	{xxv xxiv	A	xxvi	A	{xxv xxiv	D	xxiv	F	xxii	B	xxii	D	xx	G	xix	C	xviii	E	xvii	A	xvi	C
6	25.xxv	F	xxiii	B	25.xxv	B	xxiii	E	xxiii	G	xxi	C	xxi	E	xix	A	xviii	D	xvii	F	xvi	B	xv	D
7	xxiv	G	xxii	C	xxiv	C	xxii	F	xxii	A	xx	D	xx	F	xviii	B	xvii	E	xvi	G	xv	C	xiv	E
8	xxiii	A	xxi	D	xxiii	D	xxi	G	xxi	B	xix	E	xix	G	xvii	C	xvi	F	xv	A	xiv	D	xiii	F
9	xxii	B	xx	E	xxii	E	xx	A	xx	C	xviii	F	xviii	A	xvi	D	xv	G	xiv	B	xiii	E	xii	G
10	xxi	C	xix	F	xxi	F	xix	B	xix	D	xvii	G	xvii	B	xv	E	xiv	A	xiii	C	xii	F	xi	A
11	xx	D	xviii	G	xx	G	xviii	C	xviii	E	xvi	A	xvi	C	xiv	F	xiii	B	xii	D	xi	G	x	B
12	xix	E	xvii	A	xix	A	xvii	D	xvii	F	xv	B	xv	D	xiii	G	xii	C	xi	E	x	A	ix	C
13	xviii	F	xvi	B	xviii	B	xvi	E	xvi	G	xiv	C	xiv	E	xii	A	xi	D	x	F	ix	B	viii	D
14	xvii	G	xv	C	xvii	C	xv	F	xv	A	xiii	D	xiii	F	xi	B	x	E	ix	G	viii	C	vii	E
15	xvi	A	xiv	D	xvi	D	xiv	G	xiv	B	xii	E	xii	G	x	C	ix	F	viii	A	vii	D	vi	F
16	xv	B	xiii	E	xv	E	xiii	A	xiii	C	xi	F	xi	A	ix	D	viii	G	vii	B	vi	E	v	G
17	xiv	C	xii	F	xiv	F	xii	B	xii	D	x	G	x	B	viii	E	vii	A	vi	C	v	F	iv	A
18	xiii	D	xi	G	xiii	G	xi	C	xi	E	ix	A	ix	C	vii	F	vi	B	v	D	iv	G	iii	B
19	xii	E	x	A	xii	A	x	D	x	F	viii	B	viii	D	vi	G	v	C	iv	E	iii	A	ii	C
20	xi	F	ix	B	xi	B	ix	E	ix	G	vii	C	vii	E	v	A	iv	D	iii	F	ii	B	i	D
21	x	G	viii	C	x	C	viii	F	viii	A	vi	D	vi	F	iv	B	iii	E	ii	G	i	C	⋆	E
22	ix	A	vii	D	ix	D	vii	G	vii	B	v	E	v	G	iii	C	ii	F	i	A	⋆	D	xxix	F
23	viii	B	vi	E	viii	E	vi	A	vi	C	iv	F	iv	A	ii	D	i	G	⋆	B	xxix	E	xxviii	G
24	vii	C	v	F	vii	F	v	B	v	D	iii	G	iii	B	i	E	⋆	A	xxix	C	xxviii	F	xxvii	A
25	vi	D	iv	G	vi	G	iv	C	iv	E	ii	A	ii	C	⋆	F	xxix	B	xxviii	D	xxvii	G	xxvi	B
26	v	E	iii	A	v	A	iii	D	iii	F	i	B	i	D	xxix	G	xxviii	C	xxvii	E	25.xxvi	A	25.xxv	C
27	iv	F	ii	B	iv	B	ii	E	ii	G	⋆	C	⋆	E	xxviii	A	xxvii	D	xxvi	F	{xxv xxiv	B	xxiv	D
28	iii	G	i	C	iii	C	i	F	i	A	xxix	D	xxix	F	xxvii	B	25.xxvi	E	25.xxv	G	xxiii	C	xxiii	E
29	ii	A			ii	D	⋆	G	⋆	B	xxviii	E	xxviii	G	xxvi	C	{xxv xxiv	F	xxiv	A	xxii	D	xxii	F
30	i	B			i	E	xxix	A	xxix	C	xxvii	F	xxvii	A	25.xxv	D	xxiii	G	xxiii	B	xxi	E	xxi	G
31	⋆	C			⋆	F			xxviii	D			25.xxvi	B	xxiv	E			xxii	C			19.xx	A

Table 1: Distribution of new moons for the full Gregorian year, according to Clavius. Adapted from the *Encyclopædia Britannica* [13]. Traditionnal tables write XXV.XXIV where we wrote {xxv xxiv to save space. After December 31, we have the note *Haec Epacta 19 nigra numquam est in usu, nisi quando eodem anno concurrit cum Aureo numero xix.*

1. In the latter part of the year 1582, the epact was set to 26;
2. the epact of a new year is obtained by adding 11 to the epact of the preceding year,
3. by adding 1 if the golden number of the new year is 1 (this is the case in 1596 and then every 19 years),
4. by removing 1 if the new year is a century year but not a leap year (such as 1700, 1800, 1900, 2100, etc.),
5. and by adding 1 if the new year (modulo 2500) is 1800, 2100, 2400, 2700, 3000, 3300, 3600 or 3900;
6. finally, the remainder of the value obtained divided by 30 is written in roman digits (except for 0 which is written $\star$ and for 25 which is kept 25 when the golden number is greater than 11).

As a result, the numerical value of the epact (from 0 to 29) can increase (modulo 30) by 10, 11, 12 or 13 [2].

It is not our purpose to discuss these rules here, but we will concentrate on a feature of the lunar perpetual calendar which has been almost completely neglected by all those who wrote on the Gregorian calendar. This feature is the value "19.XX" which appears on December 31 in table 1.

2 The mysterious epact 19

The canons of the Gregorian calendar give as only explanation the following rule [15]:

over a period of 19 Gregorian years. (The Gregorian calendar doesn't have the 19 year cycle anymore, but it is still convenient to do the error calculation on this period.) Let $b = \frac{a\times 10000}{19} \approx 43.26ed$ be the number of epact-days of correction in 10000 years. We have $b = 75\times\frac{30}{29.530589} + \left(19\times 11+1-19\times\left(\frac{365.25\times 30}{29.530589}-360\right)\right)\times\frac{10000}{19} \approx (76.19-32.93)ed$. The second part is the number of epact-days of correction in 10000 years in the Julian calendar, assuming the Julian year to be correct. What this formula readily shows, is that the epact correction in the Gregorian calendar is not the difference of the number of days removed from the Julian calendar (75 days in 10000 years) and the number of units that should be added to the epact over the same period in the Julian calendar (≈ 33). However, $b \approx (75-31.74)ed$, and therefore it is possible to subtract 1 from the epact value every time a leap day is dropped (we subtract 75, but we should have subtracted 76.19) and we add one 32 times (hence, 8 times in 2500 years). The result is then the same. We can see immediately that the lunar corrections should occur every $\frac{10000}{31.74} \approx 315$ years, assuming our values for the synodic month and assuming the correctness of the Gregorian calendar. If instead we compute the error using only the Julian perspective, we obtain that the epact should be incremented every $\frac{10000}{32.93} \approx 304$ years. This would be the right number of corrections for a perpetual lunar Julian calendar. And if we ignore the difference between mean days and epact days, we obtain 309 years. Because of roundings, some authors obtain even other values, for instance 310 years [5, p. 29]. The Gregorian calendar approximates this correction by a correction every 312.5 years on average. Some authors, like Swerdlow, believe that this correction period follows from the *Prutenic tables*, using a computation based only on Julian years. But Swerdlow does not explain why the Gregorian year was not taken into account, nor does he try to obtain the real correction period [22].

[2] We were able to find only one source giving the first year where the leap could be 13 (table 3), namely a manuscript by Heller [14]. Epact leaps of 13 were known since Clavius [8, p. 194] and are mentioned by a few authors, especially those who have implemented the epacts mechanically. See for instance the books by Ungerer [23, p. 104], or Bach [4, p. 229].

Haec Epacta 19 nigra numquam est in usu, nisi quando eodem anno concurrit cum Aureo numero xix

The *Compendium* states the following [6]:

Cum aureus numerus fuerit xix et epacta item xix, nova luna Decembris conficitur ultimo die ipsius mensis. Cum autem alius fuerit aureus numerus, epacta vero eadem xix, nova luna in primum Ianuarii diem incurret.

Hence, we are supposed to use the value 19 when the golden number and the epact are XIX. Though often omitted, an explanation has sometimes been given, but never addressing the concerns of our study [3].

A cursory analysis of this table shows that a problem can occur at the end of a year with the epact XIX. Normally, such a year ends with a new moon on December 2nd. The next year, the epact value is usually $\star$, and therefore there is a new moon on January 1st. However, it can happen that the next epact value is 1, in which case the next new moon is on January 30th. In the first case, we have two lunar months of 30 days, and in the second case there is no new moon between December 2nd and January 30th. In that case, an

[3]Several authors have mentioned or described the epact 19, and we will not attempt to give a complete list. We will content ourselves to point out a few noteworthy authors. First, Clavius has described the epact 19 in 1588 in his answer to Maestlin [7, p. 112]. He again described it in his *Explicatio* [8].

Viète might have been the first to raise some doubts and mentioned uncertainties related to the double epact of December 31 [25, p. 275–276].

Later, *L'art de vérifier les dates* described the epact 19 in details [3, p. 91]. It mentioned the first exceptions of table 4, but stated incorrectly that the problem last happened in 1695 and will not occur again before 8500. Curiously, the years given are almost correct (1690 and 8511). The *Dictionnaire de droit canonique* [16, p. 1241] on the other hand gives all exceptions between 1595 and 8568.

Lalande has also written on it, based on his reading of Clavius [10, p. 316–317]. He gives the exceptions from 1595 until 1690 and then writes that it will not occur again until after the year 8200. The epact 19 is briefly mentioned by Delambre in his *Histoire de l'astronomie moderne* [12, p. 42], where he says only that this epact (which he writes $19'$) is of no need. It is explained by Galloway in the *Encyclopædia Britannica* [13] which borrows from Delambre. But no provision is made for century years. Also a similar explanation appears in [16, p. 1241].

Schmöger gives the correct exceptions of the years 1595 to 1690, but then adds that afterwards it will occur from 8607 to 8807 (meaning every 19 years), which is incorrect [26, p. 61].

Piscé describes the epact 19 problem, saying that it occurred in 1600 and will not recur before several thousand years [20, p. 58].

Schram writes that the exception occurs when the epact I follows the epact XIX [21]. Probably without intending it, Schram gives a definition different from that of the Gregorian calendar, in that it actually catches the year 16399 problem. However, the other problems remain. Schram gives the exceptions of the years 1595 to 8511 correctly, but doesn't expand further, as it is not related to the computation of Easter.

There are also authors like Lefort, who seems to borrow from previous ones, probably the *Annuaires* from the *bureau des longitudes*, who write only that a new moon is added on December 31 when the epact is 19 and there is an epact gap [17, p. 140–141] . He doesn't mention explicitely the golden number being 19.

The various authors often write the epact 19 differently, when it coincides with $G = 19$, and not always in arabic (for instance in bold type in 1916's *Annuaire* from the *bureau des longitudes*[2, p. 126]).

There are also authors who give a table similar to table 1, but without mentioning the epact 19. This is for instance the case of Bach in *Die Osterfest-Berechnung in alter und neuer Zeit* [5, p. 24].

We were not able to find any author who went beyond table 4 and the first entry of table 3.

G (from 1900 until 2199)	Epact	Jan	Dec	
			$G \neq 19$	$G = 19$
19	17	14	4	4
	18	13	3	3
11	19	12	2	2,31
	20	11	1,31	1,31
3	21	10	30	30
1	29	2	22	22
12	$\star$	1,31	21	21
	1	30	20	20

Table 2: Excerpt of the Gregorian perpetual lunar calendar. This table shows the new moons of January and December given several critical values of the epact. This table can easily be obtained from table 1. G is the golden number for 1900–2199.

additional new moon can be introduced on December 31, giving then a month of 29 followed by a month of 30 days.

The rule given by the canons, and which is borrowed from the *Compendium*, states that this change should be made when at the same time the epact value is 19 and the golden number is 19. The question then is: is this rule correct?

A more detailed analysis shows that it is only an approximation, albeit a good one, to what is needed. An example will suffice to illustrate the problem with the *Compendium* rule. There are for instance years where the epact is 19, and the epact gap to the next year is 12, but without the golden number being 19. Such a year is the year 16399, which has a golden number of 3, and which leads to the year 16400 with an epact of 1, because 16400 is a leap year (+0) and there is a lunar correction ($16400 \equiv 3900 \pmod{2500}$) (+1). Hence, the *Compendium* rule forgets to add a new moon between December 2, 16399 and January 30, 16400. This is actually only the first problem of this kind.

A thorough examination (table 7) shows that there are 144 such cases on the period after which epacts and golden numbers recur in the same order, which is also the Easter period, namely 5700000 years.

But that is not the whole story! It is actually also possible to have a year with epact 19, golden number 19, but only an epact leap of 11. In that case, the *Compendium* rule adds an extra new moon on December 31st, when there is actually already a new moon on the next day. There are 51 such cases over the Easter period, the first occurring in 43699 (table 5).

3 The border cases

It is easy to see that there are still problems, namely for the epact values which surround epact 19: epact 18 and epact 20.

None of these problems are taken into account by the Gregorian rule.

3.1 Epact 18

In this first case, a problem also occurs when the epact goes from 18 to 1. In these cases, we normally go from a new moon on December 3 to a new moon on January 30. A new moon should therefore be added on January 1st of the second year, creating two months of 29 days.

This happens eight times in 5700000 years: in 106399, 1466799, 1892399, 2317999, 2743599, 4529599, 4955199 and 5380799 (see table 3).

3.2 Epact 20

A symmetrical problem occurs when the epact goes from 20 to $\star$ (0). In these cases, there are normally new moons on December 1st and 31st of the first year, and January 1st and 31st of the second year. Either the new moon on December 31st or the new moon of January 1st must be removed. In either case, this will create a lunar month of 31 days. This happens 918 times over 5700000 years (table 6).

The following table summarizes the various epact 18, 19 and 20 exceptions and how they are taken into account or ignored by the Gregorian rule.

conditions	number of cases	Gregorian rule	New moons with the Gregorian rule
leap > 11, $e = 19$, $G = 19$	9949	yes	Dec. 2, Dec. 31, Jan. 30 (or 29)
leap > 11, $e = 19$, $G \neq 19$	144	forgotten	Dec. 2, Jan. 30
leap $= 11$, $e = 19$, $G = 19$	51	extra	Dec. 2, Dec. 31, Jan. 1, Jan. 31
leap $= 13$, $e = 18$	8	forgotten	Dec. 3, Jan. 30
leap $= 10$, $e = 20$	918	forgotten	Dec. 1, Dec. 31, Jan. 1, Jan. 31

To our knowledge, these problems with the epact 19 exception have never been published before.

Conclusion

Although the computation of all the exceptions is tedious, the first problems could have easily been found by hand, as these problems only occur at century years. Finding the missing new moon of 16399 involves the computation of the epact and golden number for about 150 years. Moreover, the first epact 20 exception already occurs in 4199 (see table 6). It is therefore surprising that Clavius didn't report these problems, as he went on to compute epact configurations for years much greater than the Easter cycle. But equally surprising is the fact that no past commentator seems to have explored this problem further.

We have described what naïvely seem to be fairly trivial oversights, but our analysis showed that they were not. The exceptions to the epact 19 rule must have been non obvious, and it was certainly not clear to Clavius when they would first occur. Furthermore, the problems we have described being completely irrelevant to Easter, it may explain why nobody, not even Clavius, gave them the full attention they deserved.

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year	e	year	e	year	e	year	e	year	e	year	e
15199	20	965199	15	1915199	10	2865199	5	3815199	0	4765199	25
37999	12	987999	7	1937999	2	2887999	27	3837999	22	4787999	17
60799	4	1010799	29	1960799	24	◇2910799	19	3860799	14	4810799	9
83599	26	1033599	21	1983599	16	2933599	11	3883599	6	4833599	1
•106399	18	1056399	13	2006399	8	2956399	3	3906399	28	4856399	23
136799	7	1086799	2	2036799	27	2986799	22	3936799	17	4886799	12
159599	29	1109599	24	◇2059599	19	3009599	14	3959599	9	4909599	4
182399	21	1132399	16	2082399	11	3032399	6	3982399	1	4932399	26
205199	13	1155199	8	2105199	3	3055199	28	4005199	23	•4955199	18
227999	5	1177999	0	2127999	25	3077999	20	4027999	15	4977999	10
250799	27	1200799	22	2150799	17	3100799	12	4050799	7	5000799	2
◇273599	19	1223599	14	2173599	9	3123599	4	4073599	29	5023599	24
296399	11	1246399	6	2196399	1	3146399	26	4096399	21	5046399	16
326799	0	1276799	25	2226799	20	3176799	15	4126799	10	5076799	5
349599	22	1299599	17	2249599	12	3199599	7	4149599	2	5099599	27
372399	14	1322399	9	2272399	4	3222399	29	4172399	24	◇5122399	19
395199	6	1345199	1	2295199	26	3245199	21	4195199	16	5145199	11
417999	28	1367999	23	•2317999	18	3267999	13	4217999	8	5167999	3
440799	20	1390799	15	2340799	10	3290799	5	4240799	0	5190799	25
463599	12	1413599	7	2363599	2	3313599	27	4263599	22	5213599	17
486399	4	1436399	29	2386399	24	◇3336399	19	4286399	14	5236399	9
516799	23	•1466799	18	2416799	13	3366799	8	4316799	3	5266799	28
539599	15	1489599	10	2439599	5	3389599	0	4339599	25	5289599	20
562399	7	1512399	2	2462399	27	3412399	22	4362399	17	5312399	12
585199	29	1535199	24	◇2485199	19	3435199	14	4385199	9	5335199	4
607999	21	1557999	16	2507999	11	3457999	6	4407999	1	5357999	26
630799	13	1580799	8	2530799	3	3480799	28	4430799	23	•5380799	18
653599	5	1603599	0	2553599	25	3503599	20	4453599	15	5403599	10
676399	27	1626399	22	2576399	17	3526399	12	4476399	7	5426399	2
706799	16	1656799	11	2606799	6	3556799	1	4506799	26	5456799	21
729599	8	1679599	3	2629599	28	3579599	23	•4529599	18	5479599	13
752399	0	1702399	25	2652399	20	3602399	15	4552399	10	5502399	5
775199	22	1725199	17	2675199	12	3625199	7	4575199	2	5525199	27
797999	14	1747999	9	2697999	4	3647999	29	4597999	24	◇5547999	19
820799	6	1770799	1	2720799	26	3670799	21	4620799	16	5570799	11
843599	28	1793599	23	•2743599	18	3693599	13	4643599	8	5593599	3
866399	20	1816399	15	2766399	10	3716399	5	4666399	0	5616399	25
896799	9	1846799	4	2796799	29	3746799	24	◇4696799	19	5646799	14
919599	1	1869599	26	2819599	21	3769599	16	4719599	11	5669599	6
942399	23	•1892399	18	2842399	13	3792399	8	4742399	3	5692399	28

Table 3: Years for which the Gregorian epact changes by 13 to the next year. There are 240 such cases. The first such leap will occur from 15199 to 15200. The second column shows the epact for the year given in the first column. The golden number is 19 for all these years. The years with a '◇' are taken into account by the Gregorian rule, because they correspond to an epact of 19. For the years marked with '•' (these are all the years corresponding to an epact of 18, forgotten by the Gregorian rule), it is necessary to add a new moon on January 1st the following year, otherwise one goes from a new moon on December 3 to a new moon on January 30.

1595	15484	22571	36555	50444
1614	15503	22590	36574	50463
1633	15522	29411	36593	50482
1652	15541	29430	43414	50501
1671	15560	29449	43433	50520
1690	15579	29468	43452	50539
8511	15598	29487	43471	50558
8530	15617	29601	43490	50577
8549	15636	29620	43604	50596
8568	15655	29639	43623	57303
8587	15674	29658	43642	57322
8606	15693	29677	43661	57341
8625	22305	29696	43680	57360
8644	22324	36308	50311	57379
8663	22343	36327	50330	57398
8682	22362	36346	50349	57417
15408	22381	36365	50368	57436
15427	22514	36384	50387	57455
15446	22533	36517	50406	57474
15465	22552	36536	50425	57493

Table 4: The years from 1582 to 60000 for which there is an epact leap of 12 or 13 (from this year to the next), and for which the epact and the golden number are 19. There are 9949 such cases. These are years for which a new moon must be added on December 31st. When the epact is 19, there is normally only one new moon in December, on the 2nd (see table 2). The next year, the epact is usually 0, and there is a new moon on January 1st (see table 2). When the epact leap is 12 or 13, the new moon around January 1st vanishes, because the year that has it (epact 0) has been skipped. This is why the Gregorian rule adds a new moon on December 31st. However, the condition that is proposed is not sufficient. Until 43698, it appears equivalent to having the golden number and the epact equal to 19. However, in 43699, the golden number and the epact are each 19, but there will only be an 11 gap in 43700, which means that no new moon should be added on December 31st, 43699. See table 5. This list shows that approximately every 7000 years, there are a few close exceptions. These exceptions occur every 19 years in a group and there can't be more than 16 of them, because the longest period without an epact leap (both solar and lunar correction, or none) is 299 years. They occur on average every 6977 years, as there are 43 epact corrections in 10000 years, as all combinaisons (epact, golden number) occur and none can be skipped, and as 30 corrections occur in average after $10000 \times 30/43$ years.

43699	1934199	4041299
127299	1955099	4062199
148199	2255299	4083099
169099	2276199	4166699
552899	2359799	4466899
573799	2380699	4487799
594699	2680899	4508699
978499	2764499	4592299
999399	2785399	4892499
1020299	2806299	4913399
1404099	3106499	4996999
1424999	3190099	5017899
1445899	3210999	5318099
1529499	3231899	5338999
1829699	3615699	5401699
1850599	3636599	5422599
1871499	3657499	5443499

Table 5: The 51 cases where the Gregorian rule adds a new moon, but shouldn't. These are all years where the epact jump is 11, the epact is 19 and the golden number is 19. The period of these exceptions is 5700000 years. There is no subperiod. In this case, there is a new moon on December 2 and the next year on January 1st. The Gregorian rule mistakenly adds a new moon on December 31st.

year	G	year	G	year	G	year	G
4199	1	102899	15	213799	12	324699	9
12899	18	112199	5	216699	5	325299	1
22199	8	123799	15	223099	2	336899	11
25099	1	126699	8	234699	12	339799	4
33799	18	133099	5	235299	4	346199	1
36699	11	144699	15	246899	14	357799	11
37299	3	145299	7	249799	7	360699	4
43099	8	156899	17	256199	4	372299	14
54699	18	159799	10	267799	14	372899	6
55299	10	166199	7	270699	7	378699	11
58199	3	177799	17	282299	17	384499	16
69799	13	180699	10	282899	9	390899	13
76199	10	181299	2	288699	14	393799	6
79099	3	192899	12	300899	16	405399	16
90699	13	198699	17	303799	9	414699	6
91299	5	202199	2	306699	2	426299	16

Table 6: The first 64 epact 20 exceptions. These are cases where the epact is 20 and the epact leap is 10. There are 918 such cases in the 5700000 years period. The first such case occurs from 4199 to 4200. For these years, the Gregorian rule forgets to remove an extraneous new moon.

year	G	year	G	year	G	year	G
16399	3	1442399	15	2946799	14	4313599	11
27999	13	1496399	17	3000799	16	4349599	6
117999	10	1520799	2	3025199	1	4403599	8
207999	7	1532399	12	3036799	11	4415199	18
219599	17	1586399	14	3090799	13	4439599	3
297999	4	1622399	9	3126799	8	4493599	5
309599	14	1676399	11	3180799	10	4505199	15
363599	16	1712399	6	3216799	5	4583599	2
387999	1	1766399	8	3270799	7	4595199	12
399599	11	1777999	18	3282399	17	4685199	9
453599	13	1802399	3	3306799	2	4775199	6
489599	8	1856399	5	3360799	4	4786799	16
543599	10	1867999	15	3372399	14	4840799	18
579599	5	1946399	2	3426399	16	4865199	3
633599	7	1957999	12	3450799	1	4876799	13
645199	17	2047999	9	3462399	11	4930799	15
669599	2	2137999	6	3516399	13	4966799	10
723599	4	2149599	16	3552399	8	5020799	12
735199	14	2203599	18	3606399	10	5056799	7
813599	1	2227999	3	3642399	5	5110799	9
825199	11	2239599	13	3696399	7	5146799	4
915199	8	2293599	15	3707999	17	5200799	6
926799	18	2329599	10	3732399	2	5212399	16
1005199	5	2383599	12	3786399	4	5236799	1
1016799	15	2419599	7	3797999	14	5266399	18
1070799	17	2473599	9	3876399	1	5290799	3
1095199	2	2509599	4	3887999	11	5302399	13
1106799	12	2563599	6	3977999	8	5356399	15
1160799	14	2575199	16	3989599	18	5392399	10
1196799	9	2599599	1	4067999	5	5446399	12
1250799	11	2653599	3	4079599	15	5482399	7
1286799	6	2665199	13	4133599	17	5536399	9
1340799	8	2755199	10	4157999	2	5572399	4
1352399	18	2845199	7	4169599	12	5626399	6
1376799	3	2856799	17	4223599	14	5637999	16
1430799	5	2935199	4	4259599	9	5662399	1

Table 7: The 144 cases forgotten by the Gregorian rule. These are cases where the epact is 19, the epact leap is greater than 11 (and actually always equal to 12), but the golden number is not 19. In this case, there is a new moon on December 2, followed by a new moon on January 30. The first such case occurs in 16399. Hence, given the other 51 exceptions, the Gregorian rule doesn't miss a new moon until 16398, inclusive.