

# A computational method for left adjointness

Simon Forest

I2M, Aix-Marseille Université

November 26, 2021

## Usual questions in category theory

Given a category  $\mathcal{C}$ , examples of things that we want to know:

- ▶ is  $\mathcal{C}$  complete or cocomplete?
- ▶ is  $\mathcal{C}$  closed?

Given a functor  $F: \mathcal{C} \rightarrow \mathcal{D}$ , examples of things that we want to know:

- ▶ does  $F$  preserve limits or colimits?
- ▶ is  $F$  part of an adjunction?

# Towards automation

Goal: automate or assist with some reasonings for solving these questions.

This requires:

- ▶ good computational representations
- ▶ efficient algorithms
- ▶ interaction with the user in case of partial decidability

Tools exist for higher categories (Globular, Homotopy.io, Opetopy, *etc.*) but not as many for simple categories.

# Computational representations

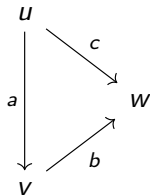
Presentations as computational representations of algebraic structures.

# Computational representations

Presentations as computational representations of algebraic structures.

Example: one can consider a category  $\mathcal{C}$  with

- ▶ objects  $u, v, w$
- ▶ generating arrows  $a: u \rightarrow v$ ,  $b: v \rightarrow w$  and  $c: u \rightarrow w$

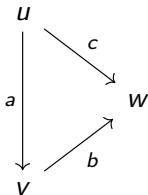


# Computational representations

Presentations as computational representations of algebraic structures.

Also a category  $D$  with

- ▶ objects  $x, y, z$
- ▶ generating arrows  $d: x \rightarrow y$  and  $e: y \rightarrow z$



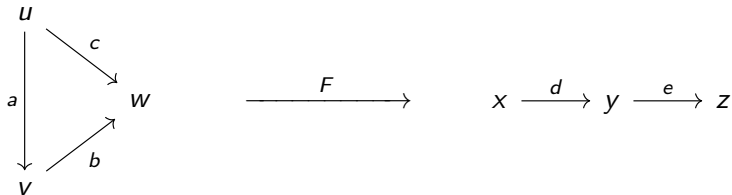
$$x \xrightarrow{d} y \xrightarrow{e} z$$

# Computational representations

Presentations as computational representations of algebraic structures.

Then one can consider the functor  $F$  such that

$$\begin{array}{lll} F(u) = x & F(v) = y & F(w) = z \\ F(a) = d & F(b) = e & F(c) = d * e \end{array}$$



# Computational representations

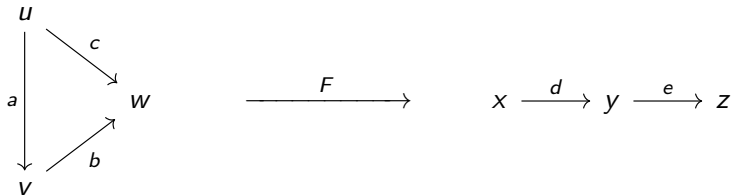
Presentations as computational representations of algebraic structures.

Such data can be given to a computer.

```
A := category {  
  obj := {u,v,w},  
  arr := {a : u => v, b : v => w, c : u => w}  
}  
B := category {  
  obj := {x,y,z},  
  arr := {d : x => y, e : y => z}  
}  
F := functor A => B {  
  u -> x, v -> y, w -> z,  
  a -> d, b -> e, c -> d * e  
}
```

# Computational representations

Presentations as computational representations of algebraic structures.

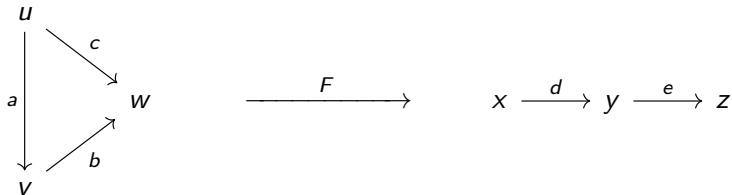


One can ask questions like

- ▶ is  $C$  complete?
- ▶ is  $F$  limit-preserving?
- ▶ *etc.*

# Computational representations

Presentations as computational representations of algebraic structures.



But  $C$ ,  $D$  and  $F$  are very artificial objects that might not be of interest.

What about “real” categories: categories of sets, groups, *etc.* and functors between them?

## Another notion of presentation

Idea: large categories can also be presented in another sense.

$\rightsquigarrow$  notion of *locally presentable categories*

- ▶ category of sets
- ▶ category of groups, rings, monoids
- ▶ category of sheaves and presheaves
- ▶ *etc.*

# Outline

Locally presentable categories

Computational descriptions of functors

Method for left adjointness

Applications

Playing a game

Proof of the criterion

# Outline

Locally presentable categories

Computational descriptions of functors

Method for left adjointness

Applications

Playing a game

Proof of the criterion

## Locally presentable categories: idea

Often, we deal with categories whose object can be **presented**.

## Locally presentable categories: idea

Take **Gph**, the category of graphs.

Every graph can be presented as  $\langle S_V, S_A \mid E \rangle$  where

- ▶  $S_V$  is a set of generating vertices
- ▶  $S_A$  is a set of generating arrows
- ▶  $E$  is a set of equations between sources and targets of arrows, and objects

$$x \xrightarrow{a} y \xrightarrow{b} z$$

$$\langle \emptyset, \{a, b\} \mid \partial^+(a) = \partial^-(b) \rangle$$

## Locally presentable categories: idea

Take **Grp**, the category of groups.

Every group  $G$  can be presented as  $\langle S \mid E \rangle$  where

- ▶  $S$  is a set of generators
- ▶  $E$  is a set of equations

Free commutative group on two elements

$$\langle \{a, b\} \mid ab = ba \rangle$$

## Locally presentable categories: idea

Take  $\mathbf{Cat}$ , the category of small categories.

Every group  $C$  can be presented as  $\langle S_O, S_M \mid E \rangle$  where

- ▶  $S_O$  is a set of generating objects
- ▶  $S_M$  is a set of generating morphisms
- ▶  $E$  is a set of equations on objects and morphisms.

$\mathbb{N}$  seen as a category with one object

$$\langle \emptyset, \{1\} \mid \partial^+(1) = \partial^-(1) \rangle$$

## Locally presentable categories: idea

The notion of **locally finitely presentable categories** describes such theories.

It encompasses a lot of very common categories.

## Locally finitely presentable categories

The abstract definition: a category is locally finitely presentable when

1. it is locally small
2. it has all colimits
3. its class of objects which can be finitely presented is essentially small
4. every objects is a directed colimits of finitely presentable objects

## A more concrete definition

### Proposition (Adámek, Rosický)

*A locally presentable category is the category of models of an **essentially algebraic theory**.*

Essentially algebraic theory  $\mathbb{T}$ : data of

- ▶ sorts
- ▶ operations between sorts
- ▶ equations that should be satisfied

## A more concrete definition

Example: the ess. alg. theory of monoids.

1 sort:

$\mathbf{M}$

2 generating operations:

$$e: 1 \rightarrow \mathbf{M} \quad c: \mathbf{M} \times \mathbf{M} \rightarrow \mathbf{M}$$

## A more concrete definition

Example: the ess. alg. theory of monoids.

1 sort:

$\mathbf{M}$

2 generating operations:

$$e: 1 \rightarrow \mathbf{M} \quad c: \mathbf{M} \times \mathbf{M} \rightarrow \mathbf{M}$$

Note: the domains of the operations are limit cones over the only sort.

## A more concrete definition

Example: the ess. alg. theory of monoids.

1 sort:

$\mathbf{M}$

2 generating operations:

$$e: 1 \rightarrow \mathbf{M} \quad c: \mathbf{M} \times \mathbf{M} \rightarrow \mathbf{M}$$

satisfying the equations

$$c(e(x), y) = y \quad c(x, e(y)) = x \quad c(c(x, y), z) = c(x, c(y, z))$$

## A more concrete definition

Example: the ess. alg. theory of monoids.

1 sort:

$\mathbf{M}$

2 generating operations:

$$e: 1 \rightarrow \mathbf{M} \quad c: \mathbf{M} \times \mathbf{M} \rightarrow \mathbf{M}$$

A model  $\mathcal{M}$  is then the data of

- ▶ a set  $\mathcal{M}(\mathbf{M})$ ,
- ▶ functions  $\mathcal{M}(e): 1 \rightarrow \mathcal{M}(\mathbf{M})$  and  $\mathcal{M}(c): \mathcal{M}(\mathbf{M}) \times \mathcal{M}(\mathbf{M}) \rightarrow \mathcal{M}(\mathbf{M})$  satisfying the equations.

### Proposition

*The category of models (i.e., monoids) is a locally finitely presentable category.*

## A more concrete definition

Example: the ess. alg. theory of small categories.

2 sorts:

$$\mathbf{C}_0 \quad \text{and} \quad \mathbf{C}_1$$

4 operations:

$$\text{id} : \mathbf{C}_0 \rightarrow \mathbf{C}_1 \quad \partial^- : \mathbf{C}_1 \rightarrow \mathbf{C}_0 \quad \partial^+ : \mathbf{C}_1 \rightarrow \mathbf{C}_0 \quad c : \mathbf{C}_1 \times_0 \mathbf{C}_1 \rightarrow \mathbf{C}_1$$

together with equations

$$\partial^\epsilon(\text{id}(x)) = x \quad c(\text{id}(x), g) = g \quad c(f, \text{id}(y)) = f \quad c(c(f, g), h) = c(f, c(g, h))$$

## A more concrete definition

Example: the ess. alg. theory of small categories.

2 sorts:

$$\mathbf{C}_0 \quad \text{and} \quad \mathbf{C}_1$$

4 operations:

$$\text{id} : \mathbf{C}_0 \rightarrow \mathbf{C}_1 \quad \partial^- : \mathbf{C}_1 \rightarrow \mathbf{C}_0 \quad \partial^+ : \mathbf{C}_1 \rightarrow \mathbf{C}_0 \quad c : \mathbf{C}_1 \times_0 \mathbf{C}_1 \rightarrow \mathbf{C}_1$$

A model  $\mathcal{M}$  is then the data of

- ▶ two sets  $\mathcal{M}(\mathbf{C}_0)$  and  $\mathcal{M}(\mathbf{C}_1)$ ,
- ▶ functions  $\mathcal{M}(\text{id}) : \mathcal{M}(\mathbf{C}_0) \rightarrow \mathcal{M}(\mathbf{C}_1)$  and *etc.* satisfying the equations.

### Proposition

*The category of models (i.e., small categories) is a locally presentable category.*

## A more concrete definition

So there are a lot of locally finitely presentable categories:

- ▶ category of sets
- ▶ categories of groups, rings, *etc.*
- ▶ categories of presheaves, sheaves
- ▶ categories of strict  $n$ -categories, (algebraic) weak  $n$ -categories
- ▶ *etc.*

## Another description of locally presentable categories

Consider again the ess. alg. theory of monoids:

1 sort:

$\mathbf{M}$

2 generating operations:

$$e: 1 \rightarrow \mathbf{M}$$

$$c: \mathbf{M}^2 \rightarrow \mathbf{M}$$

Let's build a category out of this.

## Another description of locally presentable categories

$\mathbf{M}$

Start with sorts as objects.

## Another description of locally presentable categories

$$1 \quad M \quad M^2$$

Add objects for the domains of the operations.

$$e: 1 \rightarrow M \quad c: M^2 \rightarrow M$$

## Another description of locally presentable categories

$$\mathbf{1} \xrightarrow{e} \mathbf{M} \xleftarrow{c} \mathbf{M}^2$$

Add the arrows for these operations.

## Another description of locally presentable categories

$$\mathbf{1} \xrightarrow{e} \mathbf{M} \begin{array}{c} \xleftarrow{\pi_L} \\ \xleftarrow{c} \\ \xleftarrow{\pi_R} \end{array} \mathbf{M}^2$$

Add arrows for the cone projections.

## Another description of locally presentable categories

$$\mathbf{1} \xleftarrow{e} \mathbf{M} \begin{array}{c} \xrightarrow{\pi_L} \\ \xleftarrow{c} \\ \xrightarrow{\pi_R} \end{array} \mathbf{M}^2$$

Reverse all arrows.

## Another description of locally presentable categories

$$\mathbf{1} \xleftarrow{e} \mathbf{M} \begin{array}{c} \xrightarrow{\pi_L} \\ \xleftarrow{c} \\ \xrightarrow{\pi_R} \end{array} \mathbf{M}^2$$

A model  $\mathcal{M}$  of  $\mathbf{T}$  is then a particular **presheaf** on the above category  $C$ , *i.e.*, a functor

$$X: C^{\text{op}} \rightarrow \mathbf{Set}$$

## Another description of locally presentable categories

$$\mathbf{1} \xleftarrow{e} \mathbf{M} \begin{array}{c} \xrightarrow{\pi_L} \\ \xleftarrow{c} \\ \xrightarrow{\pi_R} \end{array} \mathbf{M}^2$$

A model  $\mathcal{M}$  of  $\mathbf{T}$  is then a particular **presheaf** on the above category  $C$ , *i.e.*, a functor

$$X: C^{\text{op}} \rightarrow \mathbf{Set}$$

Which presheaf  $X \in \widehat{C}$  are actual models, *i.e.*, monoids?

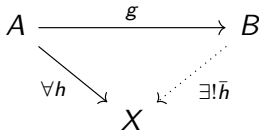
- ▶  $X(\mathbf{1})$  must be a terminal set
- ▶  $(X(\mathbf{M}^2), X(\pi_L), X(\pi_R))$  must be the product of  $X(\mathbf{M})$  and  $X(\mathbf{M})$
- ▶ the equations of monoids must hold:  $X(c)(X(e)(x), y) = y$ , *etc.*

These conditions can be expressed through **orthogonality conditions**.

# Orthogonality

Let  $\mathcal{C}$  be a category,  $g: A \rightarrow B$  and  $X \in \mathcal{C}$ .

$X$  is **orthogonal** to  $g$  when, for all  $h: A \rightarrow X$ , there is a unique  $\bar{h}: B \rightarrow X$  such that  $h = \bar{h} \circ g$ .



## Orthogonality

Let  $O^c \subseteq \mathcal{C}_1$  be a chosen set of **orthogonality morphisms**.

## Orthogonality

Let  $O^{\mathcal{C}} \subseteq \mathcal{C}_1$  be a chosen set of **orthogonality morphisms**.

$\mathcal{C}^{\perp}$ : full subcategory of objects of  $\mathcal{C}$  orthogonal to the arrows of  $O^{\mathcal{C}}$ .

## Orthogonality

Let  $O^{\mathcal{C}} \subseteq \mathcal{C}_1$  be a chosen set of **orthogonality morphisms**.

$\mathcal{C}^{\perp}$ : full subcategory of objects of  $\mathcal{C}$  orthogonal to the arrows of  $O^{\mathcal{C}}$ .

There is then a canonical inclusion functor

$$J: \mathcal{C}^{\perp} \rightarrow \mathcal{C}.$$

## Orthogonality

Let  $O^{\mathcal{C}} \subseteq \mathcal{C}_1$  be a chosen set of **orthogonality morphisms**.

$\mathcal{C}^{\perp}$ : full subcategory of objects of  $\mathcal{C}$  orthogonal to the arrows of  $O^{\mathcal{C}}$ .

There is then a canonical inclusion functor

$$J: \mathcal{C}^{\perp} \rightarrow \mathcal{C}.$$

### Proposition (Adámek, Rosický)

*If  $\mathcal{C}$  is loc. fin. presentable, the canonical inclusion functor  $J: \mathcal{C}^{\perp} \rightarrow \mathcal{C}$  has a left adjoint  $(-)^{\perp}$ :*

$$\begin{array}{ccc} & \xrightarrow{(-)^{\perp}} & \\ \mathcal{C} & \perp & \mathcal{C}^{\perp} \\ & \xleftarrow{J} & \end{array}$$

## Orthogonality conditions

The restrictions on presheaves can be expressed as orthogonality conditions.

## Orthogonality conditions

The restrictions on presheaves can be expressed as orthogonality conditions.

Example for monoids:

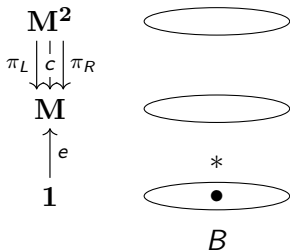
$$\mathbf{1} \xleftarrow{e} \mathbf{M} \begin{array}{c} \xrightarrow{\pi_R} \\ \xleftarrow{c} \\ \xrightarrow{\pi_L} \end{array} \mathbf{M}^2$$

# Orthogonality conditions

The restrictions on presheaves can be expressed as orthogonality conditions.

Example for monoids:  $1 \xleftarrow{e} M \xrightleftharpoons[\pi_L]{\pi_R} M^2$

Let  $B$  be the presheaf freely generated from one element  $*$  in  $B(1)$ .



## Orthogonality conditions

The restrictions on presheaves can be expressed as orthogonality conditions.

Example for monoids:  $\mathbf{1} \xleftarrow{e} \mathbf{M} \begin{matrix} \xrightarrow{\pi_R} \\ \xleftarrow{c} \\ \xrightarrow{\pi_L} \end{matrix} \mathbf{M}^2$

Let  $B$  be the presheaf freely generated from one element  $*$  in  $B(\mathbf{1})$ .

Let  $X$  in  $\widehat{C}$ . Then,  $X(\mathbf{1})$  is a terminal set when  $X$  is orthogonal to  $\emptyset \rightarrow B$

$$\begin{array}{ccc} \emptyset & \xrightarrow{\quad \emptyset \quad} & B \\ & \searrow \scriptstyle \forall H & \nearrow \scriptstyle \exists! \bar{H} \\ & X & \end{array}$$

Indeed,  $\widehat{C}(B, X) \simeq X(\mathbf{1})$ , so that the condition says  $X(\mathbf{1}) \simeq \{*\}$ .

## Orthogonality conditions

The restrictions on presheaves can be expressed as orthogonality conditions.

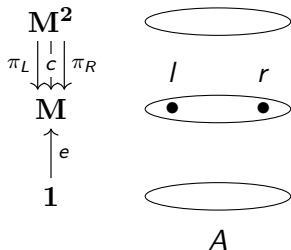
Let

## Orthogonality conditions

The restrictions on presheaves can be expressed as orthogonality conditions.

Let

- ▶  $A \in \widehat{C}$  freely gen. from two element  $l, r$  in  $B(\mathbf{M})$

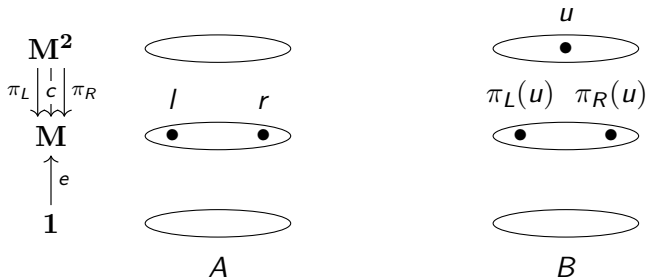


# Orthogonality conditions

The restrictions on presheaves can be expressed as orthogonality conditions.

Let

- ▶  $A \in \widehat{C}$  freely gen. from two element  $l, r$  in  $B(\mathbf{M})$
- ▶  $B \in \widehat{C}$  freely gen. from an element  $u \in B(\mathbf{M}^2)$

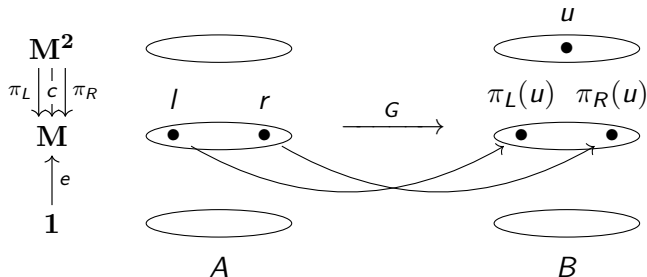


# Orthogonality conditions

The restrictions on presheaves can be expressed as orthogonality conditions.

Let

- ▶  $A \in \widehat{C}$  freely gen. from two element  $l, r$  in  $B(\mathbf{M})$
- ▶  $B \in \widehat{C}$  freely gen. from an element  $u \in B(\mathbf{M}^2)$
- ▶  $G: A \rightarrow B$  such that  $G(l) = \pi_L(u)$  and  $G(r) = \pi_R(u)$ .



## Orthogonality conditions

The restrictions on presheaves can be expressed as orthogonality conditions.

Let

- ▶  $A \in \widehat{C}$  freely gen. from two element  $l, r$  in  $B(\mathbf{M})$
- ▶  $B \in \widehat{C}$  freely gen. from an element  $u \in B(\mathbf{M}^2)$
- ▶  $G: A \rightarrow B$  such that  $G(l) = \pi_L(u)$  and  $G(r) = \pi_R(u)$ .

$(X(\mathbf{M}^2), X(\pi_L), X(\pi_R))$  is a product iff  $X$  is orthogonal to  $G: A \rightarrow B$ .

Indeed,  $\widehat{C}(A, X) \simeq X(\mathbf{M}) \times X(\mathbf{M})$  and  $\widehat{C}(B, X) \simeq X(\mathbf{M}^2)$ .

## Orthogonality conditions

The restrictions on presheaves can be expressed as orthogonality conditions.

The equations of monoids can also be expressed as orthogonality conditions.

$$A^L \xrightarrow{G^L} B^L \quad A^R \xrightarrow{G^R} B^R \quad A^A \xrightarrow{G^A} B^A$$

Thus,  $\mathbf{Mon} \simeq \widehat{\mathcal{C}}^\perp$  for a set  $O^C \subseteq \widehat{\mathcal{C}}_1$  of orthogonality morphisms.

$$\mathcal{C} = \quad 1 \xleftarrow{e} \mathbf{M} \begin{array}{c} \xrightarrow{\pi_L} \\ \xleftarrow{c} \\ \xrightarrow{\pi_R} \end{array} \mathbf{M}^2$$

## Orthogonality conditions

The restrictions on presheaves can be expressed as orthogonality conditions.

More generally,

### Proposition

*Every loc. fin. pres. category  $\mathcal{C}$  can be described as*

$$\mathcal{C} \simeq \widehat{C}^\perp$$

*for some  $C \in \mathbf{Cat}$  and  $O^C \subseteq (\widehat{C})_1$ .*

# Summary

- ▶ A lot of categories of interest are locally presentable categories.
- ▶ Such categories can be seen as orthogonality classes of presheaf categories.

# Outline

Locally presentable categories

Computational descriptions of functors

Method for left adjointness

Applications

Playing a game

Proof of the criterion

## Describe functors

$$\mathcal{F}: \mathcal{C} \rightarrow \mathcal{D}$$

Goal: describe (some) functors between two loc. pres. categories  $\mathcal{C}$  and  $\mathcal{D}$ .

We will need to filter some out.

## Describe functors

$$F: \quad \widehat{\mathcal{C}}^\perp \quad \rightarrow \quad \widehat{\mathcal{D}}^\perp$$

First, we use the characterization:  $\mathcal{C} \simeq \widehat{\mathcal{C}}^\perp$  and  $\mathcal{D} \simeq \widehat{\mathcal{D}}^\perp$ .

## Describe functors

$$\bar{F}': \hat{C} \rightarrow \hat{D}^\perp$$

Then, let's actually define a functor  $\bar{F}'$  on a larger domain.

In good cases,  $F$  can then be recovered by precomposition with  $J: \hat{C}^\perp \rightarrow \hat{C}$ .

## Describe functors

$$\bar{F}: \hat{C} \rightarrow \hat{D}$$

Also, let's actually define a functor  $\bar{F}$  on a larger domain.

In good cases,  $\bar{F}'$  can be recovered by post-composition with  $(-)^{\perp}$ .

## Describe functors

$$\tilde{F}: \mathcal{C} \rightarrow \hat{\mathcal{D}}$$

Then, let's actually only define  $\bar{F} \circ y$  where  $y$  is the Yoneda embedding

$$y: \mathcal{C} \mapsto \text{Hom}(-, \mathcal{C})$$

## Describe functors

$$\tilde{F}: C \rightarrow \hat{D}$$

If  $\bar{F}$  is nice enough, it can be recovered using a **left Kan extension**:

$$\begin{array}{ccc} \hat{C} & & \\ \uparrow y & \searrow \bar{F} & \\ C & \xrightarrow{\tilde{F}} & \hat{D} \end{array}$$

$\Uparrow \alpha$

## Describe functors

$$\tilde{F}: C \rightarrow \hat{D}$$

Under some finiteness hypothesis on  $C$ ,  $D$  and  $\tilde{F}$ , the latter can be described computationally.

## Describe functors

Summary: nice functors  $\mathcal{F}$  between presentable categories  $\mathcal{C} \simeq \widehat{C}^\perp$  and  $\mathcal{D} \simeq \widehat{D}^\perp$  can be described computationally by a functor

$$\tilde{F}: C \rightarrow \widehat{D}$$

and recovered using the diagram

$$\begin{array}{ccccc} \widehat{C}^\perp & & & & \\ J \downarrow & \searrow F & & & \\ \widehat{C} & \xrightarrow{\bar{F}} & \widehat{D} & \xrightarrow{(-)^\perp} & \widehat{D}^\perp \\ y \uparrow & \nearrow \tilde{F} & & & \\ C & & & & \end{array}$$

## Kan extensions

What is actually a Kan extension doing?

Some intuition with a particular case but essential for the following.

## Kan extensions

$$\begin{array}{ccc} & \widehat{C} & \\ y \uparrow & & \\ C & \xrightarrow{\tilde{F}} & \widehat{D} \end{array}$$

Given  $\tilde{F}: C \rightarrow \widehat{D}$  and  $y: C \rightarrow \widehat{C}$  the Yoneda embedding,

## Kan extensions

A commutative triangle diagram illustrating the relationship between Kan extensions. The vertices are  $\widehat{C}$  at the top,  $C$  at the bottom-left, and  $\widehat{D}$  at the bottom-right. A vertical arrow labeled  $y$  points from  $C$  to  $\widehat{C}$ . A horizontal arrow labeled  $\tilde{F}$  points from  $C$  to  $\widehat{D}$ . A diagonal arrow labeled  $F$  points from  $\widehat{C}$  to  $\widehat{D}$ . A curved arrow labeled  $\alpha$  points from  $\tilde{F}$  to  $F$ , indicating a natural transformation.

a left Kan extension of  $\tilde{F}$  along  $y$  is a pair  $(F, \alpha)$  which is universal in some sense.

## Kan extensions

$$\begin{array}{ccc} \widehat{C} & & \\ \uparrow y & \searrow F & \\ C & \xrightarrow{\tilde{F}} & \widehat{D} \end{array} \quad \begin{array}{c} \uparrow \alpha \end{array}$$

Concretely:

$$F(X) = \int^{c \in C} \tilde{F}(c) \otimes X(c)$$

Idea: for each  $e \in X(c)$ , there is one copy of  $\tilde{F}(c)$  in  $F(X)$ , adequately glued to other copies.

## Kan extensions

$$\begin{array}{ccc} \widehat{C} & & \\ \uparrow y & \searrow F & \\ C & \xrightarrow{\tilde{F}} & \widehat{D} \end{array} \quad \uparrow \alpha$$

Even more concretely:

$$F(X) = \left( \coprod_{c \in C, e \in X(c)} \tilde{F}(c) \right) / \sim$$

where

$$(c', e', \tilde{F}(g)(u)) \sim (c, X(g)(e), u)$$

for every  $g: c \rightarrow c' \in C$ ,  $e' \in X(c')$ ,  $u \in \tilde{F}(c)$ .

Note: under finiteness conditions, this is computable.

## Examples of functor descriptions

Taking

- ▶  $\mathbf{Set} \simeq \widehat{1}^\perp$  with  $O^{\mathbf{Set}} = \emptyset$
- ▶  $\mathbf{Set} \times \mathbf{Set} \simeq \widehat{1 \coprod 1}^\perp$  with  $O^{\mathbf{Set} \times \mathbf{Set}} = \emptyset$

## Examples of functor descriptions

Taking

- ▶  $\mathbf{Set} \simeq \widehat{1}^\perp$  with  $O^{\mathbf{Set}} = \emptyset$
- ▶  $\mathbf{Set} \times \mathbf{Set} \simeq \widehat{1 \amalg 1}^\perp$  with  $O^{\mathbf{Set} \times \mathbf{Set}} = \emptyset$

the functor

$$\mathcal{F}: (X, Y) \in \mathbf{Set} \times \mathbf{Set} \mapsto X \in \mathbf{Set}$$

can be described by  $\tilde{F}: 1 \amalg 1 \rightarrow \widehat{1}$  where  $\tilde{F}(0_L) = \{*\}$  and  $\tilde{F}(0_R) = \emptyset$ .

$$\begin{array}{ccc} \mathbf{Set} \times \mathbf{Set} & & \\ y \uparrow & \searrow \mathcal{F} & \\ 1 \amalg 1 & \xrightarrow[\{*\}, \emptyset]{\uparrow \alpha} & \mathbf{Set} \end{array}$$

## Examples of functor descriptions

Taking

►  $\mathbf{Set} \simeq \widehat{1}^\perp$  with  $O^{\mathbf{Set}} = \emptyset$

►  $\mathbf{Set} \times \mathbf{Set} \simeq \widehat{1 \amalg 1}^\perp$  with  $O^{\mathbf{Set} \times \mathbf{Set}} = \emptyset$

the functor

$$\mathcal{F}: (X, Y) \in \mathbf{Set} \times \mathbf{Set} \mapsto X \in \mathbf{Set}$$

can be described by  $\tilde{F}: 1 \amalg 1 \rightarrow \widehat{1}$  where  $\tilde{F}(0_L) = \{*\}$  and  $\tilde{F}(0_R) = \emptyset$ .

$$\begin{array}{ccc} \mathbf{Set} \times \mathbf{Set} & & \\ y \uparrow & \searrow \mathcal{F} & \\ 1 \amalg 1 & \xrightarrow[\llbracket \{*\}, \emptyset \rrbracket]{\uparrow \alpha} & \mathbf{Set} \end{array}$$

Idea: in  $\mathbf{Set} \times \mathbf{Set}$ ,  $0_L \rightsquigarrow (\{*\}, \emptyset)$ ,  $0_R \rightsquigarrow (\emptyset, \{*\})$

## Examples of functor descriptions

Taking

- ▶  $\mathbf{Set} \simeq \widehat{1}^\perp$  with  $O^{\mathbf{Set}} = \emptyset$
- ▶  $\mathbf{Mon} \simeq \widehat{C}^\perp$  with  $O^{\mathbf{Mon}} = \{G^T, G^P, G^L, G^R, G^A\}$  and

$$C = \mathbf{1} \xleftarrow{e} \mathbf{M} \begin{array}{c} \xrightarrow{\pi_L} \\ \xleftarrow{c} \\ \xrightarrow{\pi_R} \end{array} \mathbf{M}^2$$

## Examples of functor descriptions

Taking

- ▶  $\mathbf{Set} \simeq \hat{1}^\perp$  with  $O^{\mathbf{Set}} = \emptyset$
- ▶  $\mathbf{Mon} \simeq \hat{C}^\perp$  with  $O^{\mathbf{Mon}} = \{G^T, G^P, G^L, G^R, G^A\}$  and

$$C = 1 \xleftarrow{e} M \begin{array}{c} \xrightarrow{\pi_L} \\ \xleftarrow{c} \\ \xrightarrow{\pi_R} \end{array} M^2$$

the free monoid functor

$$\mathcal{F}: S \in \mathbf{Set} \mapsto S^* \in \mathbf{Mon}$$

can be described by  $\tilde{F}: 1 \rightarrow \hat{C}$  where  $\tilde{F}(0) = y(\mathbf{M})$ .

$$\begin{array}{ccccc} & \mathbf{Set} & & & \\ & \uparrow y & \searrow \mathcal{F} & & \\ 1 & \xrightarrow[y(\mathbf{M})]{\uparrow \alpha} \hat{C} & \xrightarrow{(-)^\perp} & \mathbf{Mon} & \end{array}$$

## Examples of functor descriptions

Taking

- ▶  $\mathbf{Set} \simeq \widehat{1}^\perp$  with  $O^{\mathbf{Set}} = \emptyset$
- ▶  $\mathbf{Mon} \simeq \widehat{\widehat{C}}^\perp$  with  $O^{\mathbf{Mon}} = \{G^T, G^P, G^L, G^R, G^A\}$  and

$$C = 1 \xleftarrow{e} M \begin{array}{c} \xrightarrow{\pi_L} \\ \xleftarrow{c} \\ \xrightarrow{\pi_R} \end{array} M^2$$

the free monoid functor

$$\mathcal{F}: S \in \mathbf{Set} \mapsto S^* \in \mathbf{Mon}$$

can be described by  $\tilde{F}: 1 \rightarrow \widehat{C}$  where  $\tilde{F}(0) = y(\mathbf{M})$ .

Idea:

- ▶ in  $\mathbf{Set}$ ,  $0 \rightsquigarrow \{*\}$
- ▶ in  $\mathbf{Mon}$ ,  $y(\mathbf{M})$  corresponds to the free monoid  $\{*\}^*$

# Outline

Locally presentable categories

Computational descriptions of functors

**Method for left adjointness**

Applications

Playing a game

Proof of the criterion

# Problem

Given a functor

$$\mathcal{F}: \mathcal{C} \rightarrow \mathcal{D}$$

described by a functor

$$\tilde{F}: C \rightarrow \hat{D}$$

how can we check that  $\mathcal{F}$  is a left adjoint?

## A solution

### Proposition (Adámek, Rosický)

*A functor  $\mathcal{F}: \mathcal{C} \rightarrow \mathcal{D}$  between loc. fin. pres. cat. is a left adjoint if and only if it preserves all small colimits.*

So: when is  $\mathcal{F}$  preserving all small colimits?

## A solution

Using  $\mathcal{C} \simeq \widehat{C}^\perp$  and  $\mathcal{D} \simeq \widehat{D}^\perp$ ,

### Theorem

*If the functor  $(-)^{\perp} \circ \bar{F}: \widehat{C} \rightarrow \widehat{D}^\perp$  sends the elements of  $O^{\mathcal{C}}$  to isomorphisms, then  $\mathcal{F}: \mathcal{C} \rightarrow \mathcal{D}$  preserves all colimits (and thus is a left adjoint).*

## A solution

Using  $\mathcal{C} \simeq \widehat{C}^\perp$  and  $\mathcal{D} \simeq \widehat{D}^\perp$ ,

### Theorem

*If the functor  $(-)^{\perp} \circ \bar{F}: \widehat{C} \rightarrow \widehat{D}^\perp$  sends the elements of  $O^{\mathcal{C}}$  to isomorphisms, then  $\mathcal{F}: \mathcal{C} \rightarrow \mathcal{D}$  preserves all colimits (and thus is a left adjoint).*

The above property is very computational in nature

## A solution

Using  $\mathcal{C} \simeq \widehat{\mathcal{C}}^\perp$  and  $\mathcal{D} \simeq \widehat{\mathcal{D}}^\perp$ ,

### Theorem

*If the functor  $(-)^{\perp} \circ \bar{F}: \widehat{\mathcal{C}} \rightarrow \widehat{\mathcal{D}}^\perp$  sends the elements of  $O^{\mathcal{C}}$  to isomorphisms, then  $\mathcal{F}: \mathcal{C} \rightarrow \mathcal{D}$  preserves all colimits (and thus is a left adjoint).*

The above property is very computational in nature

- $\mathcal{C}$ ,  $\mathcal{D}$ ,  $O^{\mathcal{C}}$ ,  $O^{\mathcal{D}}$  can be described to a computer

## A solution

Using  $\mathcal{C} \simeq \widehat{C}^\perp$  and  $\mathcal{D} \simeq \widehat{D}^\perp$ ,

### Theorem

*If the functor  $(-)^{\perp} \circ \bar{F}: \widehat{C} \rightarrow \widehat{D}^\perp$  sends the elements of  $O^C$  to isomorphisms, then  $\mathcal{F}: \mathcal{C} \rightarrow \mathcal{D}$  preserves all colimits (and thus is a left adjoint).*

The above property is very computational in nature

- ▶  $C, D, O^C, O^D$  can be described to a computer
- ▶ the images of  $G: A \rightarrow B \in O^C$  by the functor  $\bar{F}: \widehat{C} \rightarrow \widehat{D}^\perp$  can be computed

## A solution

Using  $\mathcal{C} \simeq \widehat{\mathcal{C}}^\perp$  and  $\mathcal{D} \simeq \widehat{\mathcal{D}}^\perp$ ,

### Theorem

*If the functor  $(-)^{\perp} \circ \bar{F}: \widehat{\mathcal{C}} \rightarrow \widehat{\mathcal{D}}^\perp$  sends the elements of  $O^{\mathcal{C}}$  to isomorphisms, then  $\mathcal{F}: \mathcal{C} \rightarrow \mathcal{D}$  preserves all colimits (and thus is a left adjoint).*

The above property is very computational in nature

- ▶  $\mathcal{C}$ ,  $\mathcal{D}$ ,  $O^{\mathcal{C}}$ ,  $O^{\mathcal{D}}$  can be described to a computer
- ▶ the images of  $G: A \rightarrow B \in O^{\mathcal{C}}$  by the functor  $\bar{F}: \widehat{\mathcal{C}} \rightarrow \widehat{\mathcal{D}}^\perp$  can be computed

$$\bar{F}(A) = (\coprod_{c \in \mathcal{C}, e \in X(c)} \tilde{F}(c)) / \sim$$

## A solution

Using  $\mathcal{C} \simeq \widehat{\mathcal{C}}^\perp$  and  $\mathcal{D} \simeq \widehat{\mathcal{D}}^\perp$ ,

### Theorem

*If the functor  $(-)^{\perp} \circ \bar{F}: \widehat{\mathcal{C}} \rightarrow \widehat{\mathcal{D}}^\perp$  sends the elements of  $O^{\mathcal{C}}$  to isomorphisms, then  $\mathcal{F}: \mathcal{C} \rightarrow \mathcal{D}$  preserves all colimits (and thus is a left adjoint).*

The above property is very computational in nature

- ▶  $\mathcal{C}$ ,  $\mathcal{D}$ ,  $O^{\mathcal{C}}$ ,  $O^{\mathcal{D}}$  can be described to a computer
- ▶ the images of  $G: A \rightarrow B \in O^{\mathcal{C}}$  by the functor  $\bar{F}: \widehat{\mathcal{C}} \rightarrow \widehat{\mathcal{D}}$  can be computed
- ▶ checking that a functor  $G': A' \rightarrow B' \in \widehat{\mathcal{D}}$  is sent to an isomorphism by  $(-)^{\perp}$  can be done by **playing a game**

# Outline

Locally presentable categories

Computational descriptions of functors

Method for left adjointness

## Applications

Playing a game

Proof of the criterion

## Non-example

Consider the functor

$$\begin{array}{rcl} \mathcal{F}: & \mathbf{Set} \times \mathbf{Set} & \rightarrow \mathbf{Set} \\ & (X, Y) & \mapsto X \times Y \end{array}$$

It is not a left adjoint. Let's see where the criterion fails.

## Non-example

Consider the functor

$$\mathcal{F}: \begin{array}{ccc} \mathbf{Set} \times \mathbf{Set} & \rightarrow & \mathbf{Set} \\ (X, Y) & \mapsto & X \times Y \end{array}$$

It is not a left adjoint. Let's see where the criterion fails.

First, let's get a description for  $\mathcal{F}$ :

- ▶  $\mathbf{Set} \simeq \widehat{\mathbf{1}}$
- ▶  $\mathbf{Set} \times \mathbf{Set} \simeq \widehat{\mathbf{1} \coprod \mathbf{1}}$

## Non-example

Consider the functor

$$\begin{array}{ccc} \mathcal{F}: & \mathbf{Set} \times \mathbf{Set} & \rightarrow \mathbf{Set} \\ & (X, Y) & \mapsto X \times Y \end{array}$$

It is not a left adjoint. Let's see where the criterion fails.

First, let's get a description for  $\mathcal{F}$ :

- ▶  $\mathbf{Set} \simeq \widehat{\mathbf{1}}$
- ▶  $\mathbf{Set} \times \mathbf{Set} \simeq \widehat{\mathbf{1} \coprod \mathbf{1}}$

But,  $\mathcal{F}$  cannot be expressed by  $\tilde{F}: \mathbf{1} \coprod \mathbf{1} \rightarrow \widehat{\mathbf{1}}$ .

## Non-example

Consider the functor

$$\begin{array}{ccc} \mathcal{F}: & \mathbf{Set} \times \mathbf{Set} & \rightarrow \mathbf{Set} \\ & (X, Y) & \mapsto X \times Y \end{array}$$

It is not a left adjoint. Let's see where the criterion fails.

First, let's get a description for  $\mathcal{F}$ :

- ▶  $\mathbf{Set} \simeq \widehat{\mathbf{1}}$
- ▶  $\mathbf{Set} \times \mathbf{Set} \simeq \widehat{\mathbf{1} \coprod \mathbf{1}}$

But,  $\mathcal{F}$  cannot be expressed by  $\tilde{F}: \mathbf{1} \coprod \mathbf{1} \rightarrow \widehat{\mathbf{1}}$ .

Indeed,

- ▶  $0_L \rightsquigarrow (\{*\}, \emptyset), \quad 0_R \rightsquigarrow (\emptyset, \{*\})$
- ▶  $(\{*\}, \emptyset)$  and  $(\emptyset, \{*\})$  are mapped to  $\emptyset$  by  $\mathcal{F}$ .
- ▶ but  $\tilde{F} = \emptyset$  describes the functor  $(X, Y) \mapsto \emptyset$ .

## Non-example

Another try: we add a (useless) product in the description of  $\mathbf{Set} \times \mathbf{Set}$

- ▶  $\mathbf{Set} \simeq \widehat{\mathbf{1}}$
- ▶  $\mathbf{Set} \times \mathbf{Set} \simeq \widehat{C}^\perp$

where

$$C = \begin{array}{ccc} & \pi_L \nearrow & p \\ 0_L & & 0_R \nwarrow \pi_R \end{array}$$

Idea:  $0_L \rightsquigarrow (\{*\}, \emptyset), \quad 0_R \rightsquigarrow (\emptyset, \{*\}), \quad p \rightsquigarrow (\{*\}, \{*\})$

## Non-example

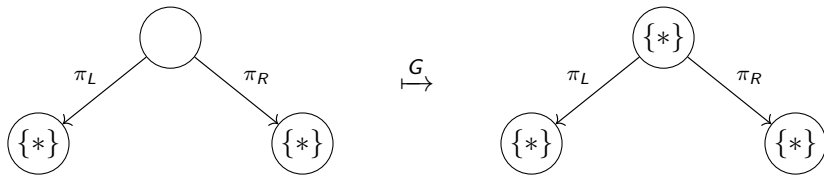
Another try: we add a (useless) product in the description of  $\mathbf{Set} \times \mathbf{Set}$

- ▶  $\mathbf{Set} \simeq \hat{\mathbf{1}}$
- ▶  $\mathbf{Set} \times \mathbf{Set} \simeq \hat{\mathbf{C}}^\perp$

where

$$C = \begin{array}{ccc} & p & \\ \pi_L \nearrow & & \nwarrow \pi_R \\ 0_L & & 0_R \end{array}$$

and where we require orthogonality to  $G: A \rightarrow B$ :



i.e., given  $X \in \hat{\mathbf{C}}^\perp$ ,  $X(p)$  must be the product of  $X(0_L)$  and  $X(0_R)$ .

## Non-example

Another try: we add a (useless) product in the description of  $\mathbf{Set} \times \mathbf{Set}$

- ▶  $\mathbf{Set} \simeq \widehat{\mathbf{1}}$
- ▶  $\mathbf{Set} \times \mathbf{Set} \simeq \widehat{C}^\perp$

where

$$C = \begin{array}{ccc} & \pi_L \nearrow & p \\ 0_L & & \nwarrow \pi_R \\ & 0_R & \end{array}$$

Now, we can describe  $\mathcal{F}: (X, Y) \mapsto X \times Y$  with

$$\begin{array}{rclcl} \tilde{F}: & C & \rightarrow & \widehat{\mathbf{1}} \\ & 0_L & \mapsto & \emptyset \\ & 0_R & \mapsto & \emptyset \\ & p & \mapsto & \{*\} \end{array}$$

## Non-example

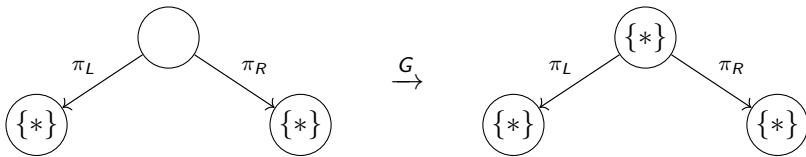
$\mathcal{F}: (X, Y) \mapsto X \times Y$  is not a left adjoint (coproducts are not preserved), so the criterion should not be satisfied.

We thus check that  $(-)^{\perp} \circ \bar{F}: \hat{C} \rightarrow \hat{D}^{\perp}$  does not map  $G: A \rightarrow B$  to an isomorphism.

## Non-example

$\mathcal{F}: (X, Y) \mapsto X \times Y$  is not a left adjoint (coproducts are not preserved), so the criterion should not be satisfied.

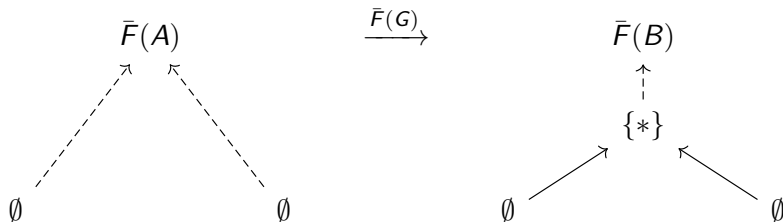
We thus check that  $(-)^{\perp} \circ \bar{F}: \hat{C} \rightarrow \hat{D}^{\perp}$  does not map  $G: A \rightarrow B$  to an isomorphism.



## Non-example

$\mathcal{F}: (X, Y) \mapsto X \times Y$  is not a left adjoint (coproducts are not preserved), so the criterion should not be satisfied.

We thus check that  $(-)^{\perp} \circ \bar{F}: \hat{C} \rightarrow \hat{D}^{\perp}$  does not map  $G: A \rightarrow B$  to an isomorphism.



## Non-example

$\mathcal{F}: (X, Y) \mapsto X \times Y$  is not a left adjoint (coproducts are not preserved), so the criterion should not be satisfied.

We thus check that  $(-)^{\perp} \circ \bar{F}: \hat{C} \rightarrow \hat{D}^{\perp}$  does not map  $G: A \rightarrow B$  to an isomorphism.

$$\emptyset \xrightarrow{\bar{F}(G)} \{*\}$$

## A small application

We recover the following well-known property using our criterion:

### Proposition

*Every functor  $F: \mathbf{Set} \rightarrow \mathcal{D}$  of the form  $F(X) = \coprod_X B$  is a left adjoint.*

## A small application

We recover the following well-known property using our criterion:

### Proposition

*Every functor  $F: \mathbf{Set} \rightarrow \mathcal{D}$  of the form  $F(X) = \coprod_X B$  is a left adjoint.*

Indeed,

- ▶ functors as above are described by functors  $\mathbf{1} \rightarrow \widehat{\mathcal{D}}$ ,
- ▶  $\mathbf{Set} \simeq \widehat{\mathbf{1}}^\perp$  with an empty set of orthogonality morphisms

so that our criterion is verified automatically.

## A bigger example

Let's show that this functor is a left adjoint:

$$\begin{array}{ccc} \mathcal{F}: & \mathbf{Cat} & \rightarrow \mathbf{Set} \\ & D & \mapsto D_0 \end{array}$$

## A bigger example

Let's show that this functor is a left adjoint:

$$\begin{array}{ccc} \mathcal{F}: \mathbf{Cat} & \rightarrow & \mathbf{Set} \\ D & \mapsto & D_0 \end{array}$$

Consider the presentations of  $\mathbf{Cat} \simeq \widehat{\mathbf{C}}^\perp$  and  $\mathbf{Set} \simeq \widehat{\mathbf{1}}$  with

$$\mathcal{C} = \mathbf{C}_0 \begin{array}{c} \xrightarrow{\partial^+} \\ \xleftarrow{\text{id}} \\ \xrightarrow{\partial^-} \end{array} \mathbf{C}_1 \begin{array}{c} \xrightarrow{\pi_L} \\ \xleftarrow{c} \\ \xrightarrow{\pi_R} \end{array} \mathbf{C}_1^2$$

## A bigger example

Let's show that this functor is a left adjoint:

$$\begin{array}{ccc} \mathcal{F}: \mathbf{Cat} & \rightarrow & \mathbf{Set} \\ D & \mapsto & D_0 \end{array}$$

Consider the presentations of  $\mathbf{Cat} \simeq \widehat{\mathbf{C}}^\perp$  and  $\mathbf{Set} \simeq \widehat{\mathbf{1}}$  with

$$C = \mathbf{C}_0 \begin{array}{c} \xrightarrow{\partial^+} \\ \xleftarrow{\text{id}} \\ \xrightarrow{\partial^-} \end{array} \mathbf{C}_1 \begin{array}{c} \xrightarrow{\pi_L} \\ \xleftarrow{c} \\ \xrightarrow{\pi_R} \end{array} \mathbf{C}_1^2$$

Consider the functor  $\tilde{F}: C \rightarrow \mathbf{Set}$  where

$$\begin{aligned} \tilde{F}(\mathbf{C}_0) &= \{*\} \\ \tilde{F}(\mathbf{C}_1) &= \{*_0, *_1\} \\ \tilde{F}(\mathbf{C}_1^2) &= \{*_0, *_1, *_2\} \end{aligned}$$

## A bigger example

Let's show that this functor is a left adjoint:

$$\begin{array}{ccc} \mathcal{F}: \mathbf{Cat} & \rightarrow & \mathbf{Set} \\ D & \mapsto & D_0 \end{array}$$

Consider the presentations of  $\mathbf{Cat} \simeq \widehat{\mathbf{C}}^\perp$  and  $\mathbf{Set} \simeq \widehat{\mathbf{1}}$  with

$$C = \mathbf{C}_0 \begin{array}{c} \xrightarrow{\partial^+} \\ \xleftarrow{\text{id}} \\ \xrightarrow{\partial^-} \end{array} \mathbf{C}_1 \begin{array}{c} \xrightarrow{\pi_L} \\ \xleftarrow{c} \\ \xrightarrow{\pi_R} \end{array} \mathbf{C}_1^2$$

Consider the functor  $\tilde{F}: C \rightarrow \mathbf{Set}$  where

$$\begin{aligned} \tilde{F}(\mathbf{C}_0) &= \{*\} \\ \tilde{F}(\mathbf{C}_1) &= \{*_0, *_1\} \\ \tilde{F}(\mathbf{C}_1^2) &= \{*_0, *_1, *_2\} \end{aligned}$$

### Proposition

*The functor  $\mathcal{F}$  is presented by  $\tilde{F}$ .*

## A bigger example

Let's show that this functor is a left adjoint:

$$\begin{array}{ccc} \mathcal{F}: & \mathbf{Cat} & \rightarrow \mathbf{Set} \\ & D & \mapsto D_0 \end{array}$$

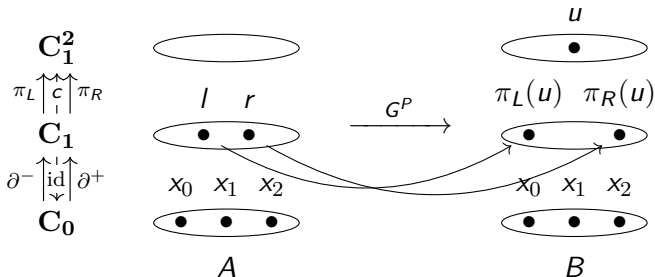
Let's compute whether  $O^C = \{G^P, G^L, G^R, G^A\}$  is sent to isomorphisms by  $\bar{F}: \hat{C} \rightarrow \mathbf{Set}$

## A bigger example

Let's show that this functor is a left adjoint:

$$\begin{aligned} \mathcal{F}: \mathbf{Cat} &\rightarrow \mathbf{Set} \\ D &\mapsto D_0 \end{aligned}$$

Let's compute whether  $O^C = \{G^P, G^L, G^R, G^A\}$  is sent to isomorphisms by  $\bar{F}: \hat{C} \rightarrow \mathbf{Set}$



## A bigger example

Let's show that this functor is a left adjoint:

$$\begin{array}{ccc} \mathcal{F}: & \mathbf{Cat} & \rightarrow \mathbf{Set} \\ & D & \mapsto D_0 \end{array}$$

Let's compute whether  $O^C = \{G^P, G^L, G^R, G^A\}$  is sent to isomorphisms by  $\bar{F}: \hat{C} \rightarrow \mathbf{Set}$

The diagram illustrates the mapping of a set of three elements to another set of three elements via a functor. On the left, a set  $\bar{F}A$  is represented by an oval containing three black dots, with labels  $x_0$ ,  $x_1$ , and  $x_2$  above each dot. An arrow labeled  $\bar{F}(G^P)$  points to the right. On the right, a set  $\bar{F}B$  is represented by an oval containing three black dots, with labels  $x_0$ ,  $x_1$ , and  $x_2$  above each dot.

## A bigger example

Let's show that this functor is a left adjoint:

$$\begin{array}{ccc} \mathcal{F}: & \mathbf{Cat} & \rightarrow \mathbf{Set} \\ & D & \mapsto D_0 \end{array}$$

Let's compute whether  $O^C = \{G^P, G^L, G^R, G^A\}$  is sent to isomorphisms by  $\bar{F}: \hat{C} \rightarrow \mathbf{Set}$

Similarly, we have

The diagram illustrates the mapping of a 2-object category to a set. On the left, a horizontal oval contains two black dots. Above the left dot is the label  $x_0$  and above the right dot is the label  $x_1$ . Below the oval is the label  $\bar{F}A^L$ . An arrow points from this oval to another oval on the right. Above the arrow is the label  $\bar{F}(G^L)$ . The oval on the right also contains two black dots, with  $x_0$  above the left dot and  $x_1$  above the right dot. Below this oval is the label  $\bar{F}B$ .

## A bigger example

Let's show that this functor is a left adjoint:

$$\begin{array}{ccc} \mathcal{F}: & \mathbf{Cat} & \rightarrow \mathbf{Set} \\ & D & \mapsto D_0 \end{array}$$

Let's compute whether  $O^C = \{G^P, G^L, G^R, G^A\}$  is sent to isomorphisms by  $\bar{F}: \hat{C} \rightarrow \mathbf{Set}$

Similarly, we have

The diagram illustrates the mapping of a 2-object category to a set. On the left, an oval contains two black dots representing objects. Above the dots are the labels  $x_0$  and  $x_1$ . Below the oval is the label  $\bar{F}A^R$ . An arrow points from this oval to another oval on the right. Above the arrow is the label  $\bar{F}(G^R)$ . The right oval also contains two black dots with labels  $x_0$  and  $x_1$  above them, and is labeled  $\bar{F}B^R$  below it.

$$\begin{array}{ccc} \begin{array}{c} x_0 \quad x_1 \\ \bullet \quad \bullet \\ \bar{F}A^R \end{array} & \xrightarrow{\bar{F}(G^R)} & \begin{array}{c} x_0 \quad x_1 \\ \bullet \quad \bullet \\ \bar{F}B^R \end{array} \end{array}$$

## A bigger example

Let's show that this functor is a left adjoint:

$$\begin{array}{ccc} \mathcal{F}: & \mathbf{Cat} & \rightarrow \mathbf{Set} \\ & D & \mapsto D_0 \end{array}$$

Let's compute whether  $O^C = \{G^P, G^L, G^R, G^A\}$  is sent to isomorphisms by  $\bar{F}: \hat{C} \rightarrow \mathbf{Set}$

Similarly, we have

$$\begin{array}{ccc} \begin{array}{c} x_0 \ x_1 \ x_2 \ x_3 \\ \bullet \quad \bullet \quad \bullet \quad \bullet \\ \bar{F}A^A \end{array} & \xrightarrow{\bar{F}(G^A)} & \begin{array}{c} x_0 \ x_1 \ x_2 \ x_3 \\ \bullet \quad \bullet \quad \bullet \quad \bullet \\ \bar{F}B^A \end{array} \end{array}$$

## A bigger example

Let's show that this functor is a left adjoint:

$$\begin{array}{ccc} \mathcal{F}: & \mathbf{Cat} & \rightarrow \mathbf{Set} \\ & D & \mapsto D_0 \end{array}$$

Let's compute whether  $O^C = \{G^P, G^L, G^R, G^A\}$  is sent to isomorphisms by  $\bar{F}: \widehat{C} \rightarrow \mathbf{Set}$

### Proposition

*The functor  $\mathcal{F}$  is a left adjoint.*

## Product functors

Product functors can be given as inputs to the criterion:

### Proposition

*Given  $\mathcal{C} \simeq \widehat{\mathcal{C}}^\perp$  and  $A \in \mathcal{C}$ , the functor*

$$X \mapsto A \times X$$

*can be described by a functor  $\mathcal{C} \rightarrow \widehat{\mathcal{C}}$ .*

Thus, our criterion can be used to show that functors  $A \times (-): \mathcal{A} \rightarrow \mathcal{C}$  are left adjoints.

## A criterion for closedness?

A category  $\mathcal{C}$  is **closed** when, for every  $A, B \in \mathcal{C}$ , there is  $B^A$  such that

$$\mathrm{Hom}(A \times X, B) \simeq \mathrm{Hom}(X, B^A)$$

## A criterion for closedness?

A category  $\mathcal{C}$  is **closed** when, for every  $A, B \in \mathcal{C}$ , there is  $B^A$  such that

$$\mathrm{Hom}(A \times X, B) \simeq \mathrm{Hom}(X, B^A)$$

### Proposition

*A category  $\mathcal{C}$  is closed when the functors*

$$A \times (-): \mathcal{C} \rightarrow \mathcal{C}$$

*are left adjoint for all  $A \in \mathcal{C}$ .*

## A criterion for closedness?

A category  $\mathcal{C}$  is **closed** when, for every  $A, B \in \mathcal{C}$ , there is  $B^A$  such that

$$\mathrm{Hom}(A \times X, B) \simeq \mathrm{Hom}(X, B^A)$$

### Proposition

*A category  $\mathcal{C}$  is closed when the functors*

$$A \times (-): \mathcal{C} \rightarrow \mathcal{C}$$

*are left adjoint for all  $A \in \mathcal{C}$ .*

This suggests that closedness could be a computable property by the earlier criterion.

Problem: the above quantification on  $A$  is infinite.

Future work: how can we change that?

## Example

We can use the criterion to show that  $\mathbf{2} \times (-): \mathbf{Cat} \rightarrow \mathbf{Cat}$  is a left adjoint where  $\mathbf{Cat} \simeq \widehat{\mathbf{C}}^\perp$  with

$$\mathcal{C} = \mathbf{C}_0 \begin{array}{c} \xrightarrow{\partial^+} \\ \xleftarrow{\text{id}} \\ \xrightarrow{\partial^-} \end{array} \mathbf{C}_1 \begin{array}{c} \xrightarrow{\bar{\pi}_R} \\ \xleftarrow{\bar{c}} \\ \xrightarrow{\bar{\pi}_L} \end{array} \mathbf{C}_1^2$$

Indeed, by computation, we check that every orthogonality morphism is sent to an isomorphism.

# Outline

Locally presentable categories

Computational descriptions of functors

Method for left adjointness

Applications

Playing a game

Proof of the criterion

# The reflection construction

Recall the adjunction

$$\begin{array}{ccc} & \xrightarrow{(-)^\perp} & \\ \hat{D} & \perp & \hat{D}^\perp \\ & \xleftarrow{J} & \end{array}$$

Given  $H: X \rightarrow Y$ , we have

$$\begin{array}{ccc} X & \xrightarrow{H} & Y \\ \eta_X \downarrow & & \downarrow \eta_Y \\ JX^\perp & \xrightarrow{JH^\perp} & JY^\perp \end{array}$$

# The reflection construction

Recall the adjunction

$$\begin{array}{ccc} & \xrightarrow{(-)^\perp} & \\ \widehat{D} & \perp & \widehat{D}^\perp \\ & \xleftarrow{J} & \end{array}$$

Given  $H: X \rightarrow Y$ , we have

$$\begin{array}{ccc} X & \xrightarrow{H} & Y \\ \eta_X \downarrow & & \downarrow \eta_Y \\ X^\perp & \xrightarrow{H^\perp} & Y^\perp \end{array}$$

How to compute whether  $H^\perp$  is an isomorphism?

# The reflection construction

Recall the adjunction

$$\begin{array}{ccc} & \xrightarrow{(-)^\perp} & \\ \widehat{D} & \perp & \widehat{D}^\perp \\ & \xleftarrow{J} & \end{array}$$

Given  $H: X \rightarrow Y$ , we have

$$\begin{array}{ccc} X & \xrightarrow{H} & Y \\ \eta_X \downarrow & & \downarrow \eta_Y \\ X^\perp & \xrightarrow{H^\perp} & Y^\perp \end{array}$$

First: given  $X \in \widehat{D}$ , what is  $\eta_X: X \rightarrow X^\perp$ ?

Idea: if  $X$  is not orthogonal,  $\eta_X$  is adding and merging the elements as required.

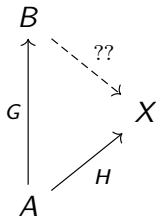
## The reflection construction

Let  $G: A \rightarrow B \in \mathcal{O}^D$  be an orthogonality morphism.

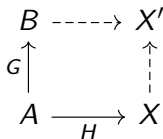
## The reflection construction

Let  $G: A \rightarrow B \in O^D$  be an orthogonality morphism.

If some liftings are missing, as in



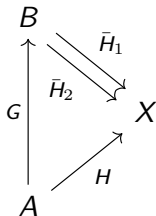
we correct that using a pushout:



## The reflection construction

Let  $G: A \rightarrow B \in \mathcal{O}^D$  be an orthogonality morphism.

If some liftings are non-unique, as in



we correct that using a coequalizer:

$$B \begin{array}{c} \xrightarrow{\bar{H}_1} \\ \xrightarrow{\bar{H}_2} \end{array} X \dashrightarrow X'$$

# The reflection construction

$\eta_X$  is then the transfinite composition

$$X = X_0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow \cdots \longrightarrow X^\perp$$

# The game

Given  $H: X \rightarrow Y \in \widehat{D}$ , how can we check that  $H^\perp: X^\perp \rightarrow Y^\perp$  is an isomorphism?

Idea: progressively apply the moves of the reflection procedure until an isomorphism is obtained.

# The game

$$H: X \rightarrow Y \in \hat{D}$$

Four possible moves

# The game

$$H: X \rightarrow Y \in \hat{D}$$

Four possible moves

- ▶ add elements to  $X$  using a pushout with  $G \in O^D$

$$H': X' \rightarrow Y$$

# The game

$$H: X \rightarrow Y \in \hat{D}$$

Four possible moves

- ▶ add elements to  $X$  using a pushout with  $G \in O^D$
- ▶ merge elements in  $X$  using a coequalizer of liftings of  $G \in O^D$

$$H': X' \rightarrow Y$$

# The game

$$H: X \rightarrow Y \in \hat{D}$$

Four possible moves

- ▶ add elements to  $X$  using a pushout with  $G \in O^D$
- ▶ merge elements in  $X$  using a coequalizer of liftings of  $G \in O^D$
- ▶ add elements to  $Y$  using a pushout with  $G \in O^D$

$$H': X \rightarrow Y'$$

# The game

$$H: X \rightarrow Y \in \hat{D}$$

Four possible moves

- ▶ add elements to  $X$  using a pushout with  $G \in O^D$
- ▶ merge elements in  $X$  using a coequalizer of liftings of  $G \in O^D$
- ▶ add elements to  $Y$  using a pushout with  $G \in O^D$
- ▶ merge elements in  $Y$  using a coequalizer of liftings of  $G \in O^D$

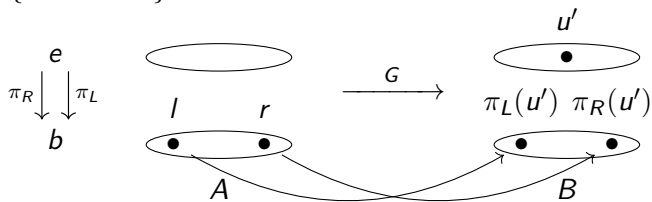
$$H': X \rightarrow Y'$$

# Play the game

Consider the category  $D$  where

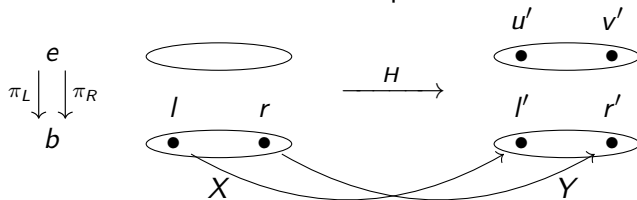
$$D = \begin{array}{c} e \\ \uparrow \uparrow \\ \pi_l \quad \pi_r \\ \downarrow \downarrow \\ b \end{array}$$

and with  $O^D = \{G: A \rightarrow B\} \subseteq \hat{D}$  with



## Play the game

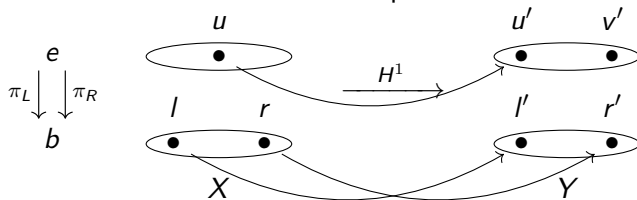
Show that  $H: X \rightarrow Y \in \hat{D}$  is sent to an isomorphism:



with  $l' = \pi_l(u') = \pi_l(v')$  and  $r' = \pi_r(u') = \pi_r(v')$

## Play the game

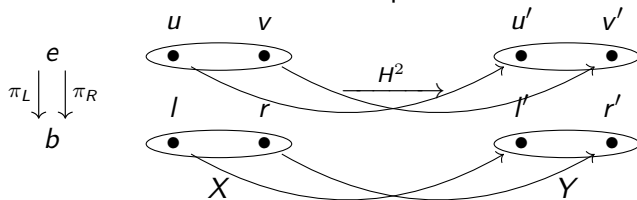
Show that  $H: X \rightarrow Y \in \hat{D}$  is sent to an isomorphism:



First, create a preimage for  $u'$ .

## Play the game

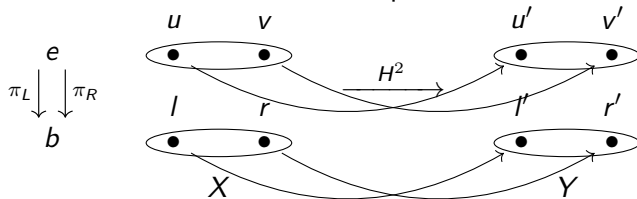
Show that  $H: X \rightarrow Y \in \hat{D}$  is sent to an isomorphism:



Then, create a preimage for  $v'$ .

## Play the game

Show that  $H: X \rightarrow Y \in \hat{D}$  is sent to an isomorphism:

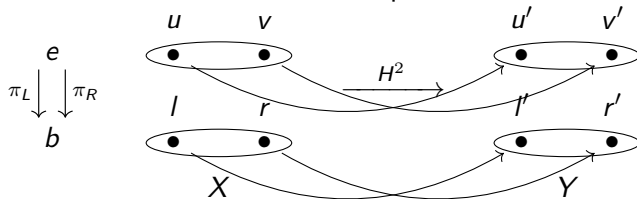


Then, create a preimage for  $v'$ .

We thus get an isomorphism.

## Play the game

Show that  $H: X \rightarrow Y \in \hat{D}$  is sent to an isomorphism:



Then, create a preimage for  $v'$ .

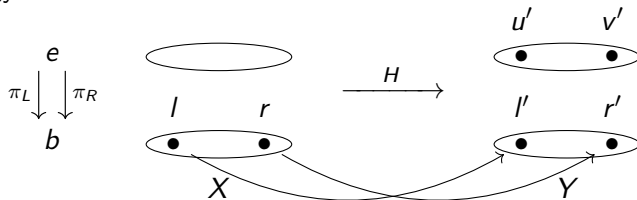
We used a “greedy strategy”: add/merge when required and possible.

### Proposition

*The greedy strategy can decide whether  $H^\perp$  is an isomorphism for finite  $H: X \rightarrow Y \in \hat{D}$ .*

# Play the game

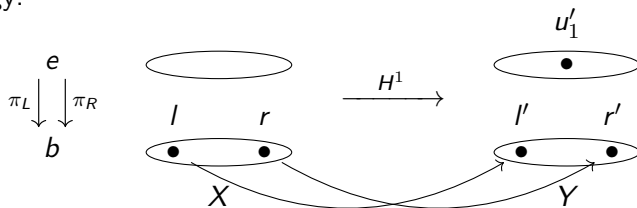
Another strategy:



with  $l' = \pi_l(u') = \pi_l(v')$  and  $r' = \pi_r(u') = \pi_r(v')$

# Play the game

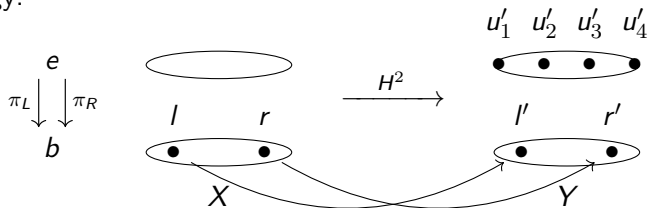
Another strategy:



First, merge  $u'$  and  $v'$ , since they lift the same morphism.

# Play the game

Another strategy:

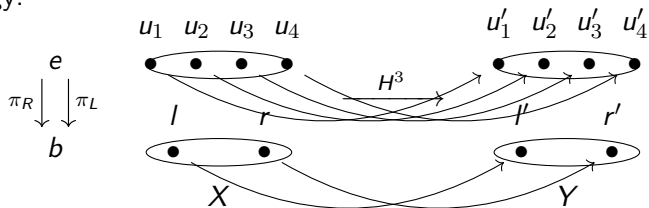


Then, create all the possible liftings in  $Y$ .

$$u'_1 = (l', r') \quad u'_2 = (l', l') \quad u'_3 = (r', r') \quad u'_4 = (r', l')$$

# Play the game

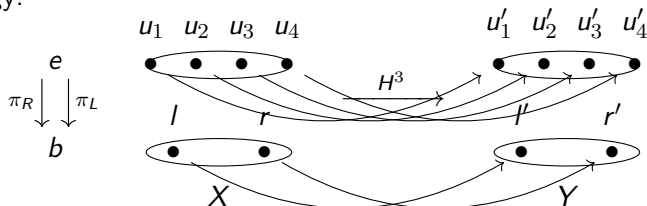
Another strategy:



Then, create all the possible liftings in  $X$ .

# Play the game

Another strategy:

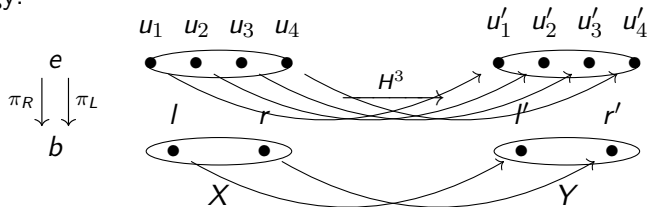


Then, create all the possible liftings in  $X$ .

We thus get an isomorphism.

# Play the game

Another strategy:



Then, create all the possible liftings in  $X$ .

We used an “exhaustive strategy”: add/merge whenever possible.

## Proposition

*The exhaustive strategy can decide whether  $H^\perp$  is an isomorphism for finite  $H: X \rightarrow Y \in \widehat{D}$ .*

## Strategies in general

Winning the game can answer positively whether a morphism is sent to an isomorphism.

However,

- ▶ greedy strategies can be too stupid and miss some winnable games
- ▶ exhaustive strategies might not terminate

Future work: characterize the categories  $D$  and sets  $O^D$  for which these strategies terminate.

In any case: one can enter “manual mode” and provide a winning play.

# Outline

Locally presentable categories

Computational descriptions of functors

Method for left adjointness

Applications

Playing a game

**Proof of the criterion**

## Colimit preservation

Recall the definition of  $F$ :

$$\begin{array}{ccccc} \widehat{C}^\perp & & & & \\ J \downarrow & \searrow F & & & \\ \widehat{C} & \xrightarrow{\bar{F}} & \widehat{D} & \xrightarrow{(-)^\perp} & \widehat{D}^\perp \\ y \uparrow & \nearrow \tilde{F} & & & \\ C & & & & \end{array}$$

### Proposition

The functor  $\bar{F}: \widehat{C} \rightarrow \widehat{D}$  preserves colimits.

### Proof.

Indeed we have

$$\bar{F}(\operatorname{colim}_i X_i) \simeq \int^{c \in C_0} \tilde{F}(c) \otimes (\operatorname{colim}_i X_i)(c)$$

## Colimit preservation

Recall the definition of  $F$ :

$$\begin{array}{ccccc} \widehat{C}^\perp & & & & \\ J \downarrow & \searrow F & & & \\ \widehat{C} & \xrightarrow{\bar{F}} & \widehat{D} & \xrightarrow{(-)^\perp} & \widehat{D}^\perp \\ y \uparrow & \nearrow \tilde{F} & & & \\ C & & & & \end{array}$$

### Proposition

The functor  $\bar{F}: \widehat{C} \rightarrow \widehat{D}$  preserves colimits.

### Proof.

Indeed we have

$$\bar{F}(\operatorname{colim}_i X_i) \simeq \int^{c \in C_0} \tilde{F}(c) \otimes \operatorname{colim}_i (X_i(c))$$

## Colimit preservation

Recall the definition of  $F$ :

$$\begin{array}{ccccc} \widehat{C}^\perp & & & & \\ J \downarrow & \searrow F & & & \\ \widehat{C} & \xrightarrow{\bar{F}} & \widehat{D} & \xrightarrow{(-)^\perp} & \widehat{D}^\perp \\ y \uparrow & \nearrow \tilde{F} & & & \\ C & & & & \end{array}$$

### Proposition

The functor  $\bar{F}: \widehat{C} \rightarrow \widehat{D}$  preserves colimits.

### Proof.

Indeed we have

$$\bar{F}(\operatorname{colim}_i X_i) \simeq \int^{c \in C_0} \operatorname{colim}_i (\tilde{F}(c) \otimes X_i(c))$$

## Colimit preservation

Recall the definition of  $F$ :

$$\begin{array}{ccccc} \widehat{C}^\perp & & & & \\ J \downarrow & \searrow F & & & \\ \widehat{C} & \xrightarrow{\bar{F}} & \widehat{D} & \xrightarrow{(-)^\perp} & \widehat{D}^\perp \\ y \uparrow & \nearrow \tilde{F} & & & \\ C & & & & \end{array}$$

### Proposition

The functor  $\bar{F}: \widehat{C} \rightarrow \widehat{D}$  preserves colimits.

### Proof.

Indeed we have

$$\bar{F}(\operatorname{colim}_i X_i) \simeq \operatorname{colim}_i \left( \int^{c \in C_0} \tilde{F}(c) \otimes X_i(c) \right)$$

## Colimit preservation

Recall the definition of  $F$ :

$$\begin{array}{ccccc} \widehat{C}^\perp & & & & \\ J \downarrow & \searrow F & & & \\ \widehat{C} & \xrightarrow{\bar{F}} & \widehat{D} & \xrightarrow{(-)^\perp} & \widehat{D}^\perp \\ y \uparrow & \nearrow \tilde{F} & & & \\ C & & & & \end{array}$$

### Proposition

The functor  $\bar{F}: \widehat{C} \rightarrow \widehat{D}$  preserves colimits.

### Proof.

Indeed we have

$$\bar{F}(\operatorname{colim}_i X_i) \simeq \operatorname{colim}_i \left( \int^{c \in C_0} \tilde{F}(c) \otimes X_i(c) \right) \simeq \operatorname{colim}_i \bar{F}(X_i)$$

## Colimit preservation

$$\begin{array}{ccc} \widehat{C}^\perp & & \\ J \downarrow & \searrow F & \\ \widehat{C} & \xrightarrow{\bar{F}'} & \widehat{D}^\perp \end{array}$$

Knowing that  $\bar{F}' \hat{=} (-)^\perp \circ \bar{F}$  is preserving colimits, when  $F$  is?

## Colimit preservation

$$\begin{array}{ccc} \widehat{C}^\perp & & \\ J \downarrow & \searrow F & \\ \widehat{C} & \xrightarrow{\bar{F}'} & \widehat{D}^\perp \end{array}$$

### Proposition (A-R)

*The colimits in  $\widehat{C}^\perp$  are the reflection of the ones computed in  $\widehat{C}$ :*

$$\operatorname{colim}_i^{\widehat{C}^\perp} A_i \simeq (\operatorname{colim}_i^{\widehat{C}} J(A_i))^\perp$$

Thus, the unit of the reflection gives a canonical morphism

$$\eta: \operatorname{colim}_i^{\widehat{C}} JA_i \rightarrow J(\operatorname{colim}_i^{\widehat{C}^\perp} A_i)$$

## Colimit preservation

$$\begin{array}{ccc} \widehat{C}^\perp & & \\ J \downarrow & \searrow F & \\ \widehat{C} & \xrightarrow{\bar{F}'} & \widehat{D}^\perp \end{array}$$

### Proposition (A-R)

*The colimits in  $\widehat{C}^\perp$  are the reflection of the ones computed in  $\widehat{C}$ :*

$$\operatorname{colim}_i^{\widehat{C}^\perp} A_i \simeq (\operatorname{colim}_i^{\widehat{C}} J(A_i))^\perp$$

Thus, the unit of the reflection gives a canonical morphism

$$\bar{F}'\eta: \bar{F}'(\operatorname{colim}_i^{\widehat{C}} JA_i) \rightarrow \bar{F}'J(\operatorname{colim}_i^{\widehat{C}^\perp} A_i)$$

## Colimit preservation

$$\begin{array}{ccc} \widehat{C}^\perp & & \\ J \downarrow & \searrow F & \\ \widehat{C} & \xrightarrow{\bar{F}'} & \widehat{D}^\perp \end{array}$$

### Proposition (A-R)

*The colimits in  $\widehat{C}^\perp$  are the reflection of the ones computed in  $\widehat{C}$ :*

$$\operatorname{colim}_i^{\widehat{C}^\perp} A_i \simeq (\operatorname{colim}_i^{\widehat{C}} J(A_i))^\perp$$

Thus, the unit of the reflection gives a canonical morphism

$$\bar{F}'\eta: \bar{F}'(\operatorname{colim}_i^{\widehat{C}} JA_i) \rightarrow F(\operatorname{colim}_i^{\widehat{C}^\perp} A_i)$$

## Colimit preservation

$$\begin{array}{ccc} \widehat{C}^\perp & & \\ J \downarrow & \searrow F & \\ \widehat{C} & \xrightarrow{\bar{F}'} & \widehat{D}^\perp \end{array}$$

### Proposition (A-R)

*The colimits in  $\widehat{C}^\perp$  are the reflection of the ones computed in  $\widehat{C}$ :*

$$\operatorname{colim}_i^{\widehat{C}^\perp} A_i \simeq (\operatorname{colim}_i^{\widehat{C}} J(A_i))^\perp$$

Thus, the unit of the reflection gives a canonical morphism

$$\bar{F}'\eta: \operatorname{colim}_i^{\widehat{D}^\perp} (\bar{F}'JA_i) \rightarrow F(\operatorname{colim}_i^{\widehat{C}^\perp} A_i)$$

## Colimit preservation

$$\begin{array}{ccc} \widehat{C}^\perp & & \\ J \downarrow & \searrow F & \\ \widehat{C} & \xrightarrow{\bar{F}'} & \widehat{D}^\perp \end{array}$$

### Proposition (A-R)

*The colimits in  $\widehat{C}^\perp$  are the reflection of the ones computed in  $\widehat{C}$ :*

$$\operatorname{colim}_i^{\widehat{C}^\perp} A_i \simeq (\operatorname{colim}_i^{\widehat{C}} J(A_i))^\perp$$

Thus, the unit of the reflection gives a canonical morphism

$$\bar{F}'\eta: \operatorname{colim}_i^{\widehat{D}^\perp} (FA_i) \rightarrow F(\operatorname{colim}_i^{\widehat{C}^\perp} A_i)$$

## Colimit preservation

$$\begin{array}{ccc} \widehat{C}^\perp & & \\ J \downarrow & \searrow F & \\ \widehat{C} & \xrightarrow{\bar{F}'} & \widehat{D}^\perp \end{array}$$

### Proposition

The functor  $F: \widehat{C}^\perp \rightarrow \widehat{D}^\perp$  preserves colimits (and is a left adjoint) if and only if  $\bar{F}'\eta_{\operatorname{colim}_i \widehat{C} JA_i}$  is an isomorphism for all diagrams  $i \mapsto A_i$  in  $\widehat{C}^\perp$ .

# Colimit preservation

$$\begin{array}{ccc} \widehat{C}^\perp & & \\ J \downarrow & \searrow F & \\ \widehat{C} & \xrightarrow{\bar{F}'} & \widehat{D}^\perp \end{array}$$

## Proposition

The functor  $F: \widehat{C}^\perp \rightarrow \widehat{D}^\perp$  preserves colimits (and is a left adjoint) if and only if  $\bar{F}'\eta_{\text{colim}_i \widehat{C} J A_i}$  is an isomorphism for all diagrams  $i \mapsto A_i$  in  $\widehat{C}^\perp$ .

## Corollary

If  $\bar{F}'\eta$  is an isomorphism, then  $F$  preserves colimits (and is a left adjoint).

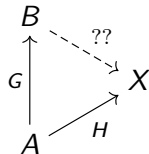
## Theorem

Suppose now that, for every orthogonality morphism  $G \in O^C$ ,  $\bar{F}(G)$  is an isomorphism.

## Theorem

Suppose now that, for every orthogonality morphism  $G \in O^C$ ,  $\bar{F}(G)$  is an isomorphism.

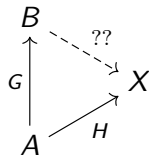
If some liftings are missing for  $X$ , as in



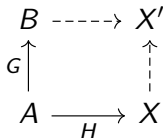
## Theorem

Suppose now that, for every orthogonality morphism  $G \in O^C$ ,  $\bar{F}(G)$  is an isomorphism.

If some liftings are missing for  $X$ , as in



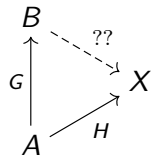
we correct that using a pushout:



## Theorem

Suppose now that, for every orthogonality morphism  $G \in O^C$ ,  $\bar{F}(G)$  is an isomorphism.

If some liftings are missing for  $X$ , as in



...and we obtain the pushout

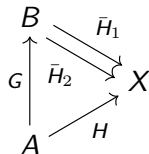
$$\begin{array}{ccc} \bar{F}B & \dashrightarrow & \bar{F}X' \\ \bar{F}(G) \uparrow & & \uparrow \\ \bar{F}A & \xrightarrow{\bar{F}(H)} & \bar{F}X \end{array}$$

where  $\bar{F}(G)$  is an isomorphism. Thus,  $\bar{F}X \simeq \bar{F}X'$ .

## Theorem

Suppose now that, for every orthogonality morphism  $G$ ,  $\bar{F}(G)$  is an isomorphism.

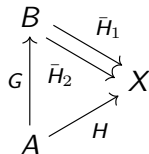
If liftings are non-unique, as in



## Theorem

Suppose now that, for every orthogonality morphism  $G$ ,  $\bar{F}(G)$  is an isomorphism.

If liftings are non-unique, as in



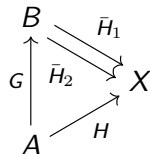
we correct that using a coequalizer:

$$B \begin{array}{c} \xrightarrow{\bar{H}_1} \\ \xrightarrow{\bar{H}_2} \end{array} X \dashrightarrow X'$$

## Theorem

Suppose now that, for every orthogonality morphism  $G$ ,  $\bar{F}(G)$  is an isomorphism.

If liftings are non-unique, as in



...and we obtain the coequalizer:

$$\bar{F}B \begin{array}{c} \xrightarrow{\bar{F}(\bar{H}_1)} \\ \xrightarrow{\bar{F}(\bar{H}_2)} \end{array} \bar{F}X \dashrightarrow \bar{F}X'$$

with  $\bar{F}(\bar{H}_1) \circ \bar{F}(G) = \bar{F}(\bar{H}_2) \circ \bar{F}(G)$ , thus  $\bar{F}(\bar{H}_1) = \bar{F}(\bar{H}_2)$  and  $\bar{F}X \simeq \bar{F}X'$

## Theorem

Thus,  $\bar{F}\eta_X$  is a transfinite composition of isomorphism

$$\bar{F}X = \bar{F}X_0 \xrightarrow{\sim} \bar{F}X_1 \xrightarrow{\sim} \bar{F}X_2 \xrightarrow{\sim} \cdots \xrightarrow{\sim} \bar{F}X^\perp$$

## Theorem

Thus,  $\bar{F}\eta_X$  is a transfinite composition of isomorphism

$$\bar{F}X = \bar{F}X_0 \xrightarrow{\sim} \bar{F}X_1 \xrightarrow{\sim} \bar{F}X_2 \xrightarrow{\sim} \cdots \xrightarrow{\sim} \bar{F}X^\perp$$

## Theorem

*If, for all  $G \in O^C$ ,  $\bar{F}(G)$  is an isomorphism, then  $\bar{F}\eta$  is an isomorphism.*

## Theorem

Thus,  $\bar{F}\eta_X$  is a transfinite composition of isomorphism

$$\bar{F}X = \bar{F}X_0 \xrightarrow{\sim} \bar{F}X_1 \xrightarrow{\sim} \bar{F}X_2 \xrightarrow{\sim} \cdots \xrightarrow{\sim} \bar{F}X^\perp$$

## Theorem

*If, for all  $G \in O^C$ ,  $\bar{F}(G)$  is an isomorphism, then  $\bar{F}\eta$  is an isomorphism.*

## Corollary

*With the same hypothesis,  $F$  preserves colimits and is a left adjoint.*

The end

Thank you!

