Resultant techniques in small dimension:
applications to implicitization and modeling

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Outline

- Toric resultant and subdivision-based matrices
- Compact formulae: Hybrid matrix with toric Jacobian
  - ◦ Modeling examples
- Degeneracy: Linear perturbation
  - ◦ Implicitization and base points
- Rational formula

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Resultant definition

Given $n+1$ Laurent polynomials $f_0, \ldots, f_n \in K[x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}]$ with indeterminate coefficients $\vec{c}$, their toric (or sparse) *resultant* is the unique (up to sign) irreducible polynomial $R(\vec{c}) \in \mathbb{Z}[\vec{c}]$ such that

$$R(\vec{c}) = 0 \Leftrightarrow \exists\, \xi = (\xi_1, \ldots, \xi_n) \,:\, f_0(\xi) = \cdots = f_n(\xi) = 0$$

where ξ lies in the corresponding toric variety X, $(\overline{K}^*)^n \subset X \subset \mathbb{P}^N$. $R(\vec{c})$ is separately homogeneous in the coefficients of each f_i, with *degree* equal to the n-fold mixed volume:

$$\mathrm{MV}_{-i} := \mathrm{MV}(f_0, \ldots, f_{i-1}, f_{i+1}, \ldots, f_n),$$

provided the supports of f_i generate $\mathbb{Z}^n$.

For dense homogeneous f_i : $X = \mathbb{P}^n$ and $\mathrm{MV}_{-i} = \prod_{j \neq i} \deg f_j$ (Bézout).

Generalizations: The toric resultant reduces to:

- the determinant of the coefficient matrix of a *linear* system,
- the Sylvester determinant of 2 *univariate* polynomials,
- the projective resultant for $n+1$ *dense* polynomials.

Resultants

- Projective (classical): over $\mathbb{P}^n$.
- Toric (sparse): over toric variety $X : \overline{K}^n \subset X \subset \mathbb{P}^N$
- Over unirational variety X s.t. $\overline{K}^n \subset X \subset \mathbb{P}^N$ [BusElkMou].
- Residual: over $V(F : G)$ [BEM], $F : G = \{f : fg \in F, \forall g \in G\}$.

Resultant Matrices

- Sylvester, Macaulay, Newton (toric resultant's) etc.
- Bézout
- Hybrid: Dixon, Morley, Jouanolou, [DickD'And00],

corner-cutting [GoldmZhang], with toric Jacobian [D'AndE01] etc.

The u-resultant

$$
\begin{aligned}
f_0 &= & & u_1 x_1 + & & u_2 x_2 + u_0 = 0,\\
f_1 &= x_1^2 + x_1 x_2 + & & 2x_1 \quad + & & x_2 - \; 1 = 0,\\
f_2 &= x_1^2 \qquad\quad + & & 3x_1 - x_2^2 + & & 2x_2 - \; 1 = 0.
\end{aligned}
$$

The Macaulay matrix $=\begin{bmatrix} 1 & 1 & 2 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 2 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 2 & 0 & 0 & 1 & -1 \\ 1 & 0 & 3 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 3 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 3 & 0 & -1 & 2 & -1 \\ 0 & u_1 & 0 & u_2 & u_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & u_1 & 0 & u_2 & u_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & u_1 & 0 & 0 & u_2 & u_0 & 0 \\ 0 & 0 & 0 & 0 & 0 & u_1 & 0 & 0 & u_2 & u_0 \end{bmatrix}$

The u-resultant is $|M| = (u_1 - u_2 + u_0)(-3u_1 + u_2 + u_0)(u_2 + u_0)(u_1 - u_2)$
$\Rightarrow$ the roots are $(1, -1), (-3, 1), (0, 1), (0 : 1 : -1)$.

Deleting row 5, column 7 yields a toric resultant matrix with:

number of rows per polynomial $=$ 4,3,2, whereas the optimal would be $\mathrm{MV}_{-i} = 4, 2, 2$.

Matrix constructions

Algorithms: subdivision [CannE'93,'00], incremental [ECann'95] yield a square Newton matrix M such that

$$\det(M) \not\equiv 0,$$
$$R \mid \det(M),$$
$$\deg_{f_0} \det(M) = \deg_{f_0} R,$$

where R is the toric resultant.
Same properties for the Macaulay matrix of the projective resultant [Mac'02].
In all cases, M exhibits quasi-Toeplitz structure.

Complexity [E'96] $\sim e^n \deg R(\mathrm{vtx} Q_i)^3$, when all n-fold Mixed Volumes > 0, and the Newton polytopes do not differ "too much" (bounded scaling).

Rational form [D'Andr'01] : $R = \det(M)/\det(M')$,
where M' is a submatrix of M, generalizing Macaulay's construction.
By the Schur complement we obtain $R = \det\left(M_{22} - M_{21}(M')^{-1}M_{12}\right)$.

Toric elimination theory: Newton polytopes

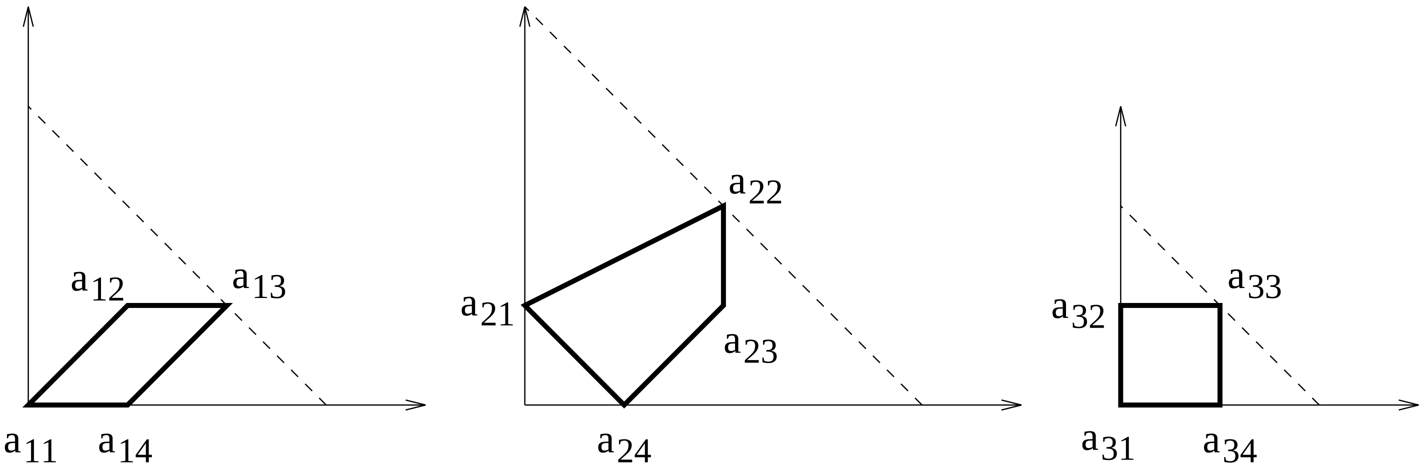
The Newton polytope $N \subset \mathbb{R}^n$ of a polynomial $f(x_1, \ldots, x_n)$ is the Convex Hull of all exponent vectors a_i appearing in

$$f = \sum_i c_i x^{a_i}, \qquad c_i \neq 0.$$

support$(f_i) := \{a_i \in \mathbb{Z}^n : c_i \neq 0\} =: A_i$.

Example:

$$\begin{aligned} f_1 &= c_{11} + c_{12}xy + c_{13}x^2y + c_{14}x \\ f_2 &= c_{21}y + c_{22}x^2y^2 + c_{23}x^2y + c_{24}x + c_{25}xy \\ f_3 &= c_{31} + c_{32}y + c_{33}xy + c_{34}x \end{aligned}$$



Bernstein (BKK) bound

Theorem [Bernstein'75,Kushnirenko'75,Khovanskii'78] [Danilov'78]: Given polynomials $f_1, \ldots, f_n \in K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, $\forall$ field K, the number of common isolated zeros in $(\overline{K} - \{0\})^n$, counting multiplicities, is bounded by the mixed volume of the Newton polytopes $\mathrm{MV}(Q_1, \ldots, Q_n)$ (irrespective of the variety's dimension).

[Canny,Rojas'91]: The bound is exact if the extremal coefficients are generic.

Toric (sparse) resultant: $\deg_{f_i} R(f_0, \ldots, f_n) = \mathrm{MV}(f_0, \ldots, f_{i-1}, f_{i+1}, \ldots, f_n)$

Dense homogeneous: $\mathrm{MV}(Q_1, \ldots, Q_n) = \prod_{i=1}^n d_i$ is the Bézout bound, where $d_i = \deg(f_i)$ and $Q_i = \mathrm{simplex}\{0, (d_i, 0, \ldots, 0), \ldots, (0, \ldots, 0, d_i)\}$.

Extension [Huber,Sturmfels'95]: Let $A'_i = \mathrm{supp}(f_i) \cup \{0\}$. The number of isolated zeros in $\overline{K}^n$, counting multiplicities, is bounded by the stable mixed volume of $A'_1, \ldots, A'_n$ (irrespective of the variety's dimension).
Equality holds for generic extremal coefficients.

Example (cont'd): lifting for the over-constrained problem

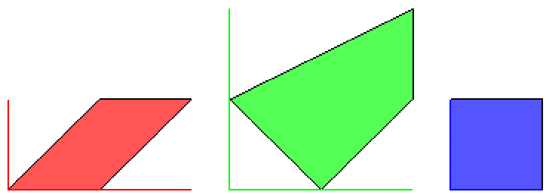


Figure 1: The given polytopes.

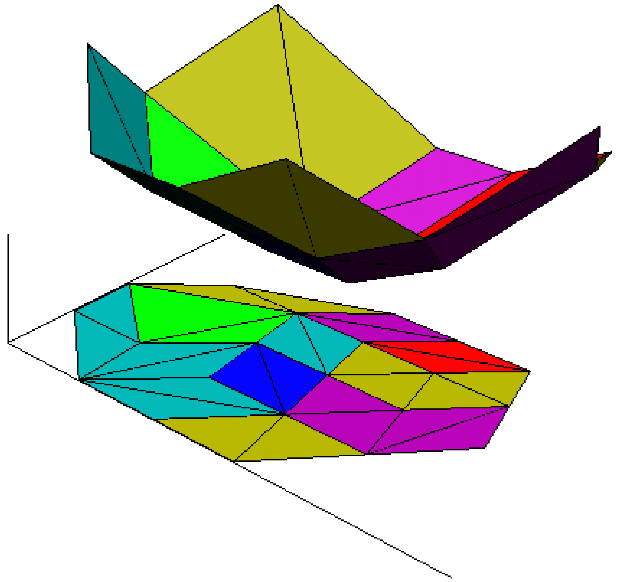


Figure 2: The lower hull of the lifted Minkowski Sum and its planar projection.

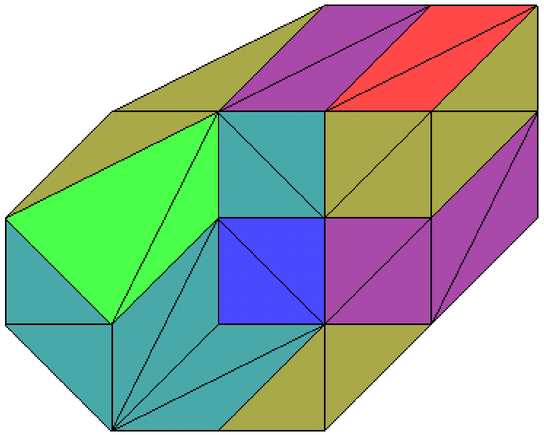
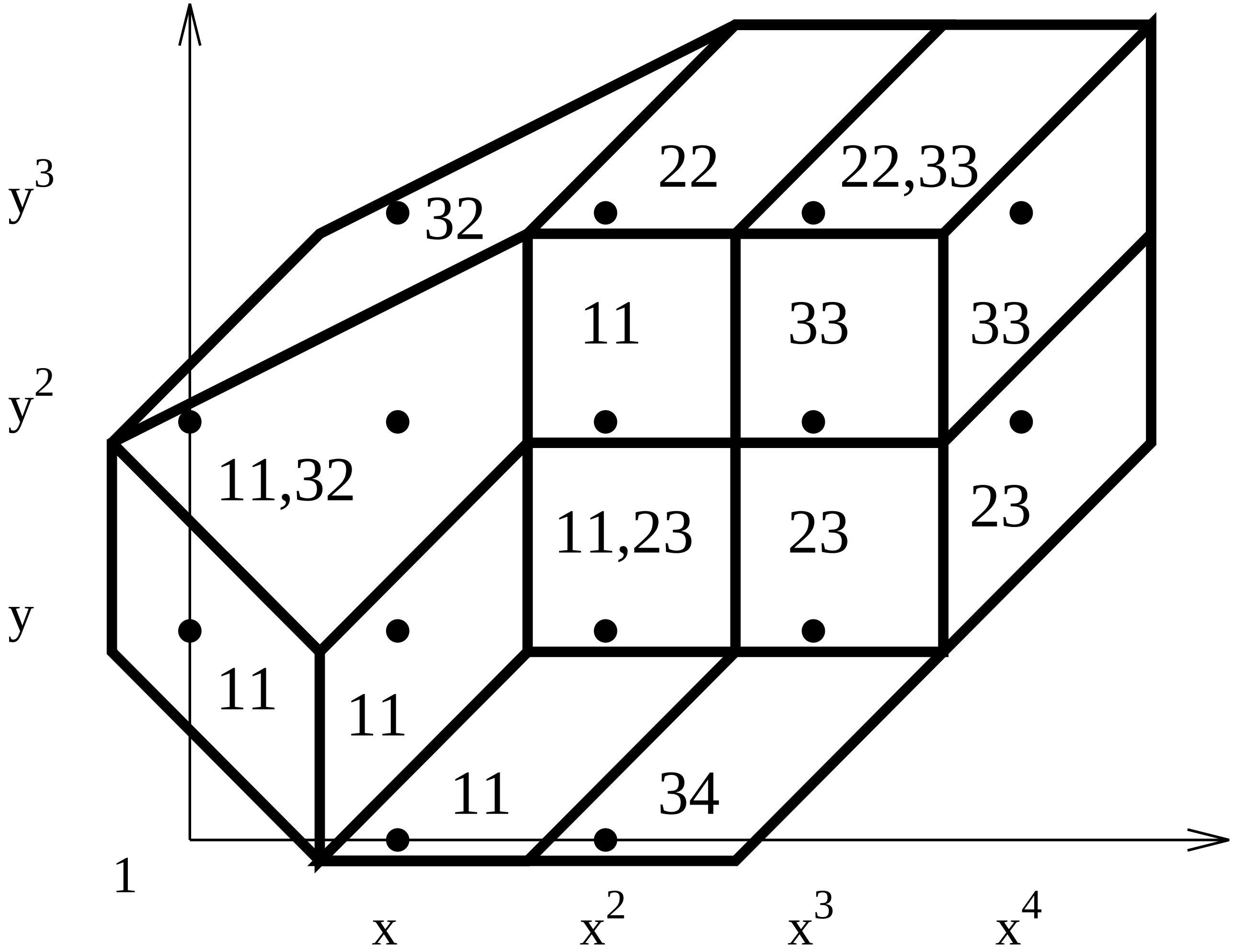


Figure 3: The mixed subdivision.

Example: mixed subdivision for the the over-constrained problem



Eg: $x \mapsto (x,y)^{(1,0)-(0,0)} f_1$, $x^2 y \mapsto (x,y)^{(2,1)-(2,1)} f_2$.

Example: subdivision-based matrix

$$
\begin{aligned}
f_1 &= c_{11} + c_{12}xy + c_{13}x^2y + c_{14}x, \\
f_2 &= c_{21}y + c_{22}x^2y^2 + c_{23}x^2y + c_{24}x, \\
f_3 &= c_{31} + c_{32}y + c_{33}xy + c_{34}x.
\end{aligned}
$$

$$
\begin{array}{l|ccccccccccccccc|}
 & 1,0 & 2,0 & 0,1 & 1,1 & 2,1 & 3,1 & 0,2 & 1,2 & 2,2 & 3,2 & 4,2 & 1,3 & 2,3 & 3,3 & 4,3 \\
\hline
(1,0)x & c_{11} & c_{14} & 0 & 0 & c_{12} & c_{13} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
(2,0)x & c_{31} & c_{34} & 0 & c_{32} & c_{33} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
(0,1)y & 0 & 0 & c_{11} & c_{14} & 0 & 0 & 0 & c_{12} & c_{13} & 0 & 0 & 0 & 0 & 0 & 0 \\
(1,1)xy & 0 & 0 & 0 & c_{11} & c_{14} & 0 & 0 & 0 & c_{12} & c_{13} & 0 & 0 & 0 & 0 & 0 \\
(2,1) & c_{24} & 0 & c_{21} & 0 & c_{23} & 0 & 0 & 0 & c_{22} & 0 & 0 & 0 & 0 & 0 & 0 \\
(3,1)x & 0 & c_{24} & 0 & c_{21} & 0 & c_{23} & 0 & 0 & 0 & c_{22} & 0 & 0 & 0 & 0 & 0 \\
(0,2)y & 0 & 0 & c_{31} & c_{34} & 0 & 0 & c_{32} & c_{33} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
(1,2)xy & 0 & 0 & 0 & c_{31} & c_{34} & 0 & 0 & c_{32} & c_{33} & 0 & 0 & 0 & 0 & 0 & 0 \\
(2,2)x^2y^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{11} & c_{14} & 0 & 0 & 0 & c_{12} & c_{13} \\
(3,2)x^2y & 0 & 0 & 0 & 0 & c_{31} & c_{34} & 0 & 0 & c_{32} & c_{33} & 0 & 0 & 0 & 0 & 0 \\
(4,2)x^2y & 0 & 0 & 0 & 0 & 0 & c_{24} & 0 & 0 & c_{21} & 0 & c_{23} & 0 & 0 & 0 & c_{22} \\
(1,3)xy^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{31} & c_{34} & 0 & 0 & c_{32} & c_{33} & 0 & 0 \\
(2,3)y & 0 & 0 & 0 & c_{24} & 0 & 0 & c_{21} & 0 & c_{23} & 0 & 0 & 0 & c_{22} & 0 & 0 \\
(3,3)x^2y^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{31} & c_{34} & 0 & 0 & c_{32} & c_{33} & 0 \\
(4,3)x^3y^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{31} & c_{34} & 0 & 0 & c_{32} & c_{33}
\end{array}
$$

dim $M = 15$, greedy [CanPed]: 14, incremental [ECan]: 12.

Mixed volumes $= 4, 3, 4 \Rightarrow$ deg $R_{tor} = 11$ while deg(classical resultant) $= 26$.

Residual resultant

Consider homogeneous $g_1, \ldots, g_n \in \mathbb{C}[x_0, \ldots, x_m]$, degrees $k_1 \geq \cdots \geq k_n$.

Assume we have homogeneous

$$f_j = \sum_{i=1}^{n} h_{i,j}(x) g_i(x), \quad j = 0, \ldots, m \quad : \quad \deg f_j = d_j.$$

Let $\phi : \mathbb{P}^m \backslash V(G) \to \mathbb{P}^m \times \mathbb{P}^{n-1} : (x_0 : \cdots : x_m) \mapsto (x_0 : \cdots : x_m) \times (g_1 : \ldots : g_n)$, and $\tilde{\mathbb{P}}^m = \overline{\mathrm{Im}(\phi)}$ be the blow up of $\mathbb{P}^m$ along the ideal $G = \langle g_1, \cdots, g_n \rangle$.

Theorem [BusElkMou'00] If G is a local complete intersection, $\forall d_j \geq k_1, \forall d_j \geq k_n + 1$, then there exists the residual resultant $R_G(f_0, \ldots, f_m) \in \mathbb{C}[\vec{c}_j]$, which is irreducible and vanishes iff

$$(f_0, \ldots, f_m)^{sat} \neq G \Leftrightarrow \exists x \in \tilde{\mathbb{P}}^m : \tilde{f}_j(x) = 0 \Leftrightarrow V(F : G) \neq \emptyset$$

If $V(f_j)$ contains no multiple components, this reduces to $\overline{x \in V(f_j) \setminus V(G)}$.
Moreover, each $\deg_{f_j} R_G$ is known.

Common subexpressions

$$f_0 = c_{00} + c_{01}t_1 + c_{02}t_2 + c_{03}({t_1}^2 + {t_2}^2),$$
$$f_i = c_{i0} + c_{i1}t_1 + c_{i2}t_2 + c_{i3}({t_1}^2 + {t_2}^2) + c_{i4}({t_1}^2 + {t_2}^2)^2.$$

- All homogenized F_i vanish at $(0:1:I)$, $(0:1:-I)$, hence both projective and toric resultants are identically zero.
- Use $t_3 = {t_1}^2 + {t_2}^2$ then $|M| = c_{01}^2 c_{03} R_T$ for the subdivision-based matrix

$$M = \begin{bmatrix}
c_{20} & c_{23} & c_{24} & 0 & c_{22} & 0 & 0 & 0 & 0 & 0 & c_{21} & 0 & 0 & 0 & 0 & 0 & 0 \\
c_{00} & c_{03} & 0 & 0 & c_{02} & 0 & 0 & 0 & 0 & 0 & c_{01} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & c_{00} & c_{03} & 0 & 0 & c_{02} & 0 & 0 & 0 & 0 & 0 & c_{01} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & c_{00} & c_{03} & 0 & 0 & c_{02} & 0 & 0 & 0 & 0 & 0 & c_{01} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & c_{20} & c_{23} & c_{24} & 0 & c_{22} & 0 & 0 & 0 & 0 & c_{21} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & c_{00} & c_{03} & 0 & 0 & c_{02} & 0 & 0 & 0 & 0 & c_{01} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & c_{00} & c_{03} & 0 & 0 & c_{02} & 0 & 0 & 0 & 0 & c_{01} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & c_{20} & c_{23} & c_{24} & 0 & c_{22} & 0 & 0 & 0 & 0 & c_{21} & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
c_{10} & c_{13} & c_{14} & 0 & c_{12} & 0 & 0 & 0 & 0 & 0 & c_{11} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & c_{10} & c_{13} & c_{14} & 0 & c_{12} & 0 & 0 & 0 & 0 & 0 & c_{11} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{20} & c_{23} & c_{24} & c_{22} & 0 & c_{21} & 0 \\
0 & 0 & 0 & 0 & c_{10} & c_{13} & c_{14} & 0 & c_{12} & 0 & 0 & 0 & 0 & c_{11} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & c_{10} & c_{13} & c_{14} & 0 & c_{12} & 0 & 0 & 0 & 0 & c_{11} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{00} & c_{03} & 0 & c_{02} & 0 & c_{01} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{00} & c_{03} & 0 & c_{02} & 0 & c_{01}
\end{bmatrix}$$

Hybrid matrices based on toric Jacobians

Hypothesis: $Q_i = k_i P,\ k_i \in \mathbb{N}$.

Toric Jacobian: $J(f) = \det \begin{pmatrix} f_0 & f_1 & f_2 \\ x_1 \frac{\partial f_0}{\partial x_1} & x_1 \frac{\partial f_1}{\partial x_1} & x_1 \frac{\partial f_2}{\partial x_1} \\ x_2 \frac{\partial f_0}{\partial x_2} & x_2 \frac{\partial f_1}{\partial x_2} & x_2 \frac{\partial f_2}{\partial x_2} \end{pmatrix}$.

Pick vertex $p \in P$, incident edge e. $A_i^2 = \{k_i p\},\ A_i^1 = (k_i\, e \setminus k_i p) \cap \mathbb{Z}^2$.

$f_i^j = \sum_{a \in A_i^j} c_{ia} x^a,\ f_i' = f_i - \sum_{j=1}^2 f_i^j$. $G(f) = \det \begin{pmatrix} f_0 & f_1 & f_2 \\ f_0^1 & f_1^1 & f_2^1 \\ f_0^2 & f_1^2 & f_2^2 \end{pmatrix} = \det \begin{pmatrix} f_0' & f_1' & f_2' \\ f_0^1 & f_1^1 & f_2^1 \\ f_0^2 & f_1^2 & f_2^2 \end{pmatrix}$

Proposition. $G(f) \equiv J(f) \pmod{< f >}$ within a constant. Toric Jacobians $J(f)$ for identical supports [CDS], and $G(f)$ for scaled Newton polytopes, have support in $\mathcal{F}$.

Modules $S_{\mathcal{F}_i} =< x^a : a \in \mathcal{F}_i >,\ S_{Q_i} =< x^a : a \in int(Q_i) \cap \mathbb{Z}^2 >$.

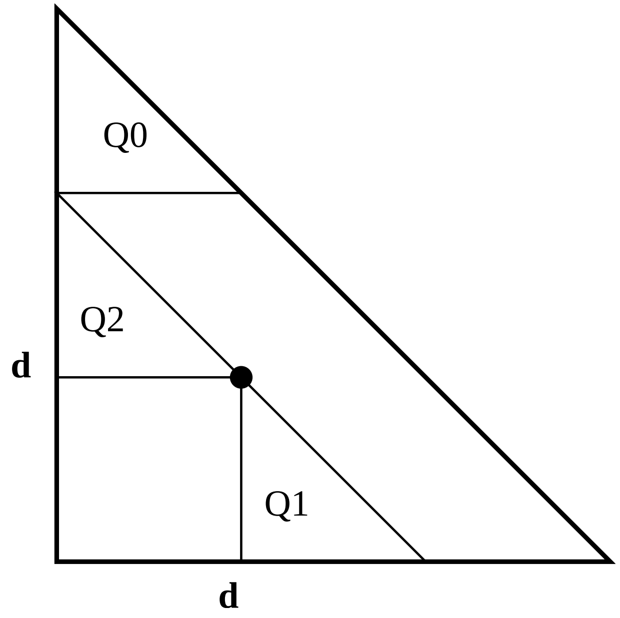
$$0 \to S_{Q_0} \oplus S_{Q_1} \oplus S_{Q_2} \xrightarrow{\psi} S_{\mathcal{F}_0} \oplus S_{\mathcal{F}_1} \oplus S_{\mathcal{F}_2} \oplus \mathbb{Z}[c] \xrightarrow{\phi} S_{\mathcal{F}} \to 0 \ :$$

$$\begin{aligned} \psi(p_0, p_1, p_2) = &\ (p_1 f_2 + p_2 f_1, p_0 f_2 - p_2 f_0, -p_0 f_1 - p_1 f_0, 0), \\ \phi(g_0, g_1, g_2, \lambda) = &\ g_0 f_0 + g_1 f_1 + g_2 f_2 + \lambda J(f). \end{aligned}$$

Proposition [CDS] Generically exact complex and ϕ surjective.

For coefficient specialization, complex is exact iff $R \neq 0$. $R = \gcd(\text{maximal minors of } \phi)$.

Three dense polynomials $Q_i = d_i S$



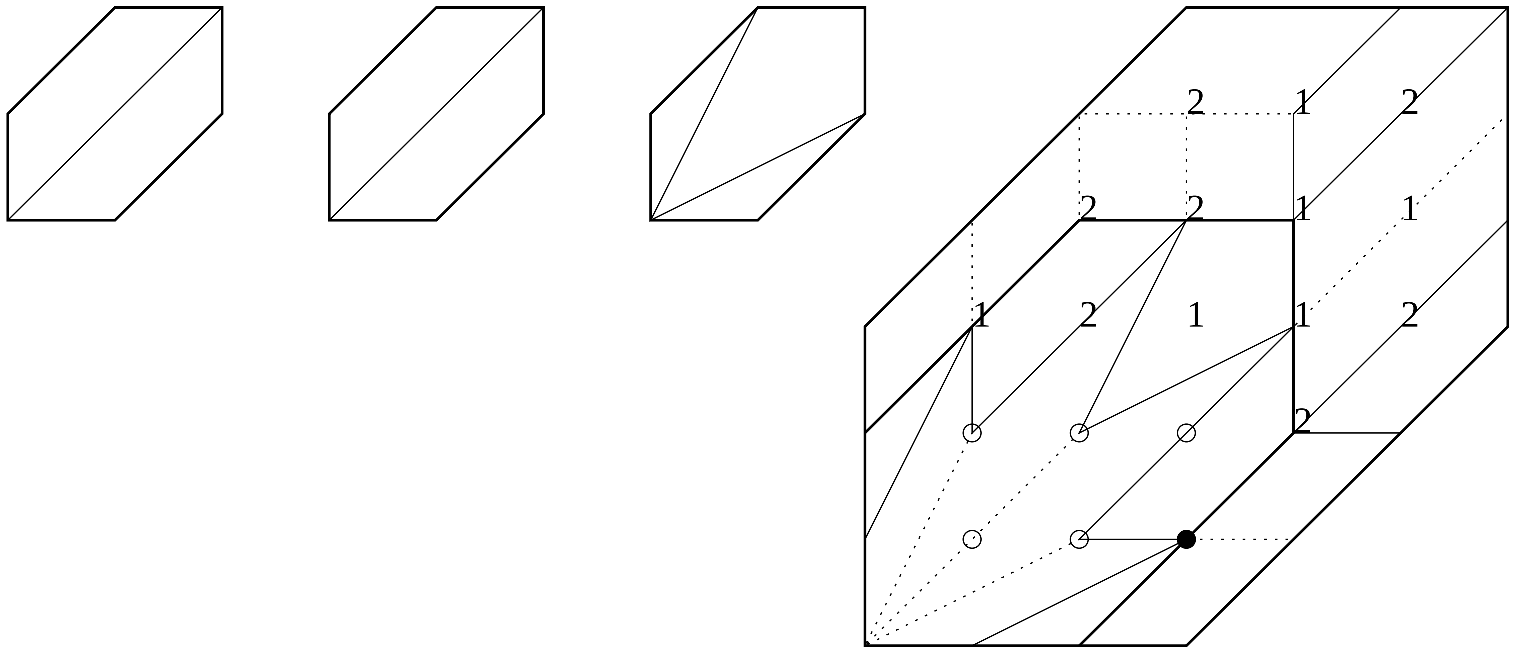
$\delta = (\epsilon, \epsilon)$, $Q_{0,\omega} : (0, 0, -d_0 M), (0, d_0, > d_0), (d_0, 0, d_0 M)$,
$Q_{1,\omega} : (0, 0, 0), (0, d_1, 2d_1), (d_1, 0, d_1)$, $Q_{2,\omega} : (0, 0, 0), (0, d_2, d_2), (d_2, 0, 2d_2)$.

Recover Jouanolou's matrix s.t. $d_0 \le d_1 \le d_2$:
$a_2 \ge d_2 \to$ row contains $x^{a-(0,d_2)} f_2$; $a_2 < d_2, a_1 \ge d_1 \to x^{a-(d_1,0)} f_1$;
$a = (d_0 - 1, d_1 - 1) \to$ row contains $J(f)$; otherwise $x^a f_0$.

Proposition. $d_0 d_1 d_2 R(f_i) = \det M / \det M'$, M' indexed by rows : $a_1 \ge d_1, a_2 \ge d_2$ [Jou97,DA-Dic01].

Example I

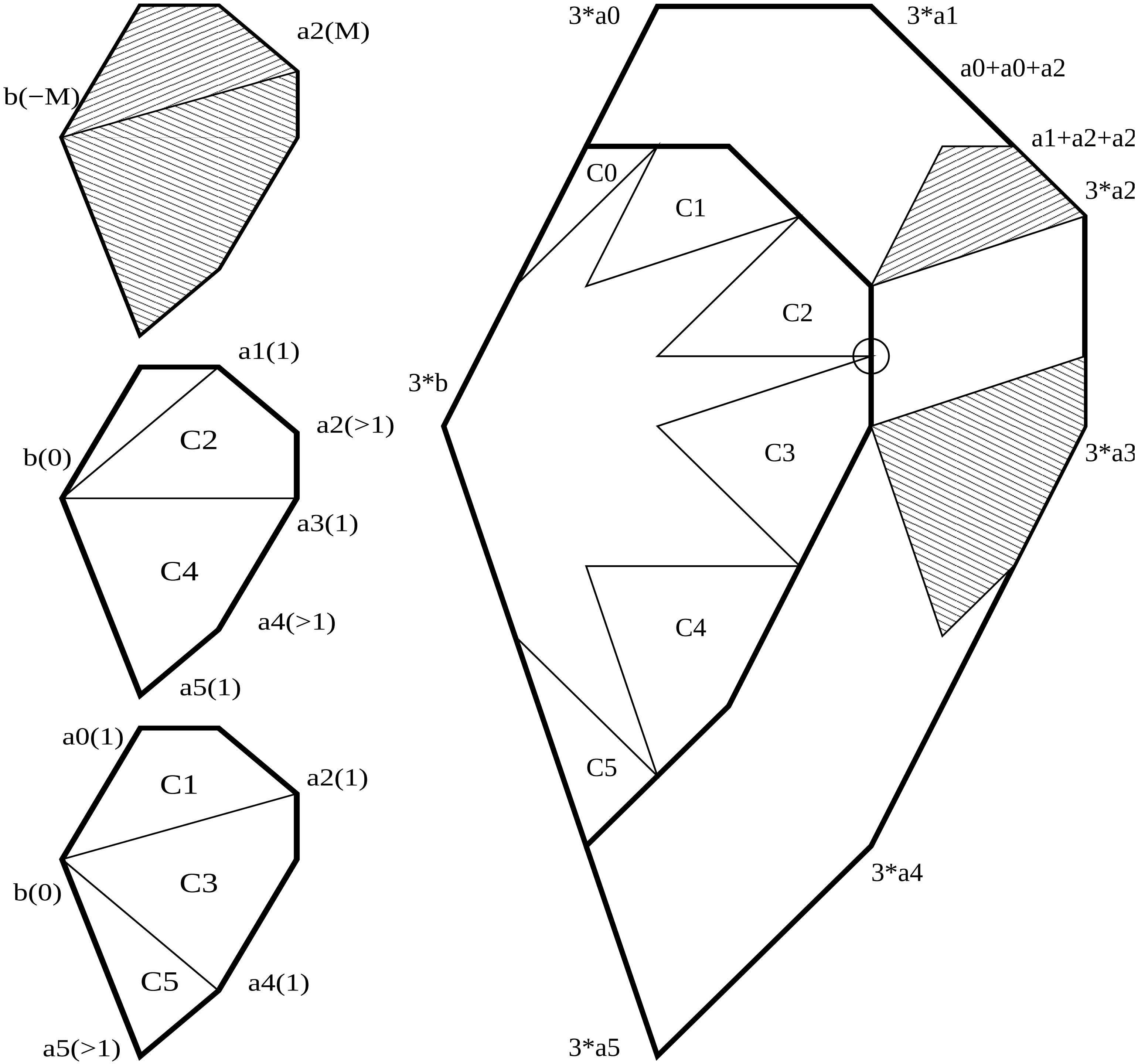
$f_i = c_{i0} + c_{i1}x_1 + c_{i2}x_2 + c_{i3}x_1^2x_2 + c_{i4}x_1^2x_2^2 + c_{i5}x_1x_2^2$
$\delta = (\frac{5}{6}, \frac{1}{3}), b = (0,0), \omega_0(1,0) = \omega_0(0,1) = 1$



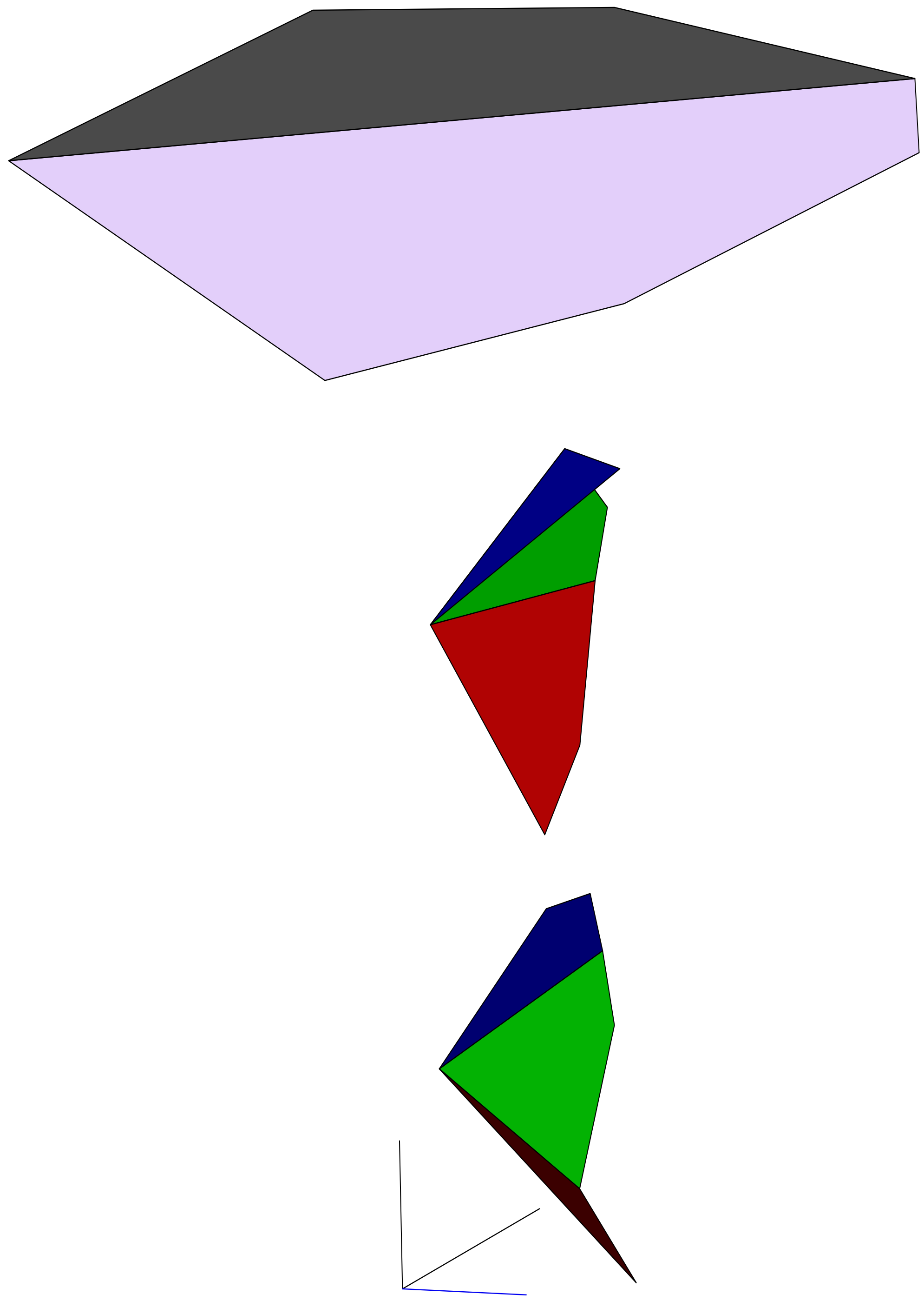
$\dim M = 19, \ \det M = c_{24}(-c_{15}c_{24} + c_{14}c_{25})R$

Example II

$M \gg 1$,
$\delta = (\frac{1}{100}, \frac{1}{5000})$,
Support cardinality of $G(f), J(f) = 5, 30$



Example II:
lifted $Q_{i,\omega} \subset \mathbb{R}^3$



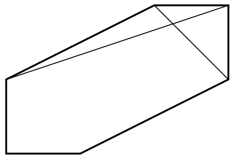
Example: Self-intersections

Self-intersections: the toric resultant

Given parametrized surface, find s, t with more than 1 image.

Locus of $(s+x, t+xy), (s-x, t-xy)$ with the same image for some x, y.
For bicubic surfaces [GallStil'01]:

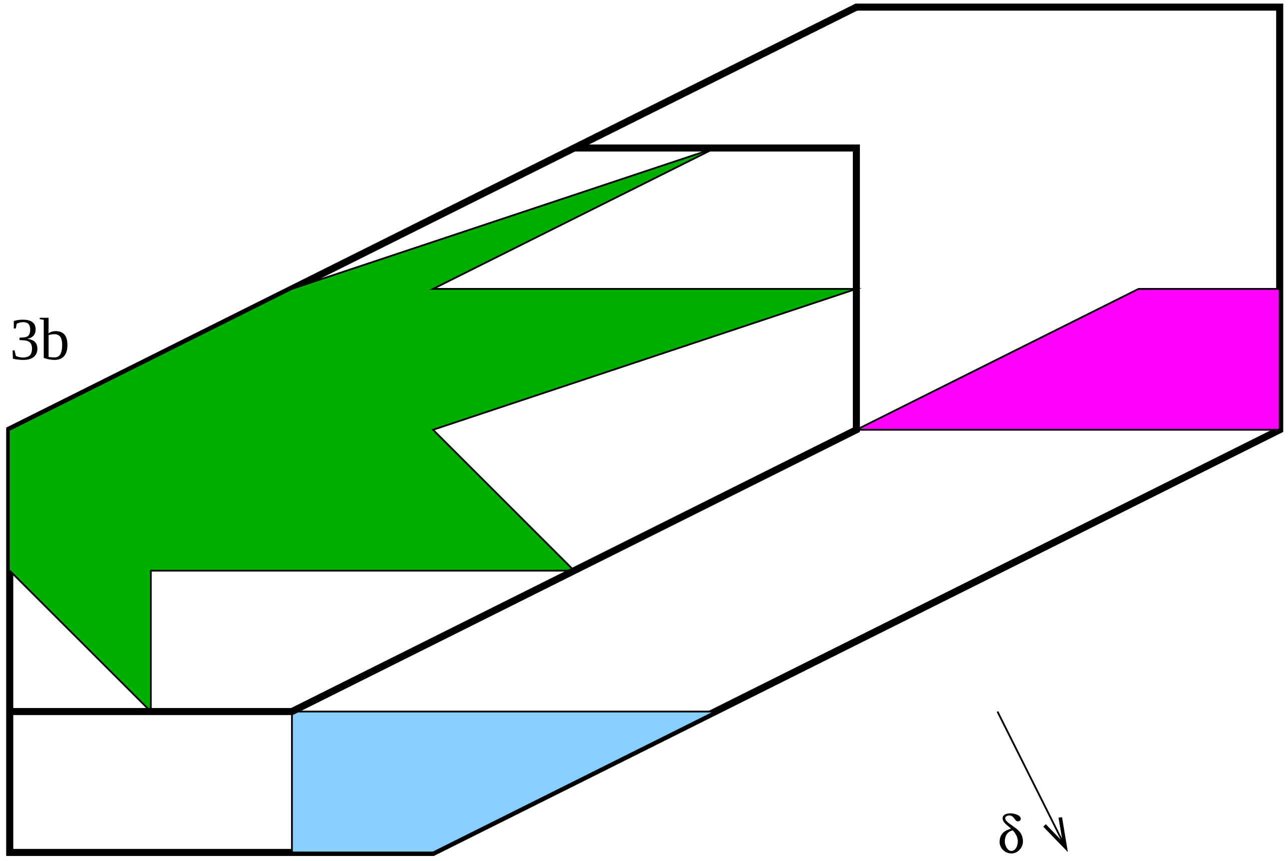
$$a_0 + a_1 x + a_2 y + a_3 xy + a_4 x^2 y + a_5 x^2 y^2 + a_6 x^3 y + a_7 x^3 y^2,$$
$$b_0 + b_1 x + b_2 y + b_3 xy + b_4 x^2 y + b_5 x^2 y^2 + b_6 x^3 y,$$
$$c_0 + c_1 x + c_2 y + c_3 xy + c_4 x^2 y + c_5 x^3 y + c_6 x^3 y^2,$$

with a_i, b_i, c_i functions of s, t. 

Pad with 0's, toric resultant matrix of [D'AndE01], with $\delta = (-0.11, 0.2), b = (0,1), \omega_0(3,1) = 54 \to \dim M = 22$, $\deg_{a,b,c} \det M = 24 = 8+8+8 = \deg R$, $\deg_s \det M = \deg_t \det M = 31$ (all optimal).

Matrices of [CannE00,CanPed93,ZhanGold00] have: $\dim M = 35, 30, 30$, hence extraneous factors of degree = 11,6,6 in a_i, b_i, c_i.

Self-intersections: the mixed subdivision



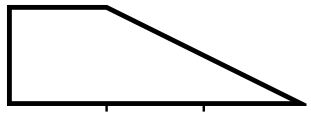
Monomial basis of $K[x_i^{\pm 1}]/I(f_1, f_2)$.

Examples

- Bi-homogeneous $(1,1)$: $f_i = c_{i,0} + c_{i,1}x_1 + c_{i,2}x_2 + c_{i,3}x_1\,x_2$,

$$G(f) = \begin{vmatrix} f_0 & f_1 & f_2 \\ c_{01}x_1 & c_{11}x_1 & c_{21}x_1 \\ c_{00} & c_{10} & c_{20} \end{vmatrix} = G_0x_1\,x_2 + G_1x_1^2\,x_2 + G_2x_1\,x_2^2 + G_3x_1^2\,x_2^2,$$

$$\delta = (\epsilon, < \epsilon),\; b = (0,0) \text{ yield } \det M = \begin{vmatrix} c_{0,0} & c_{0,1} & c_{0,0} & c_{0,1} \\ c_{1,0} & c_{1,1} & c_{1,0} & c_{1,1} \\ c_{2,0} & c_{2,1} & c_{2,0} & c_{2,1} \\ G_0 & G_1 & G_2 & G_3 \end{vmatrix} = \pm R$$

- Chow form of Hirzebruch surface: $\det M = \pm R$. 

$f_i = c_{i0}x_1x_2^3 + c_{i1}x_1x_2^2x_4 + c_{i2}x_1x_2x_4^2 + c_{i3}x_1x_4^3 + c_{i4}x_2x_3 + c_{i5}x_3x_4$.

- Surface implicitization (with base points) [Manocha]: take $\cup A_i$.

$A_i = \{(1,0),(2,0),(0,1)\},\{(1,0),(0,1),(0,2)\},\{(1,0),(0,1),\ (1,1)\}$

- [ZhanGold'00]: $f_0 = 2s+t, f_1 = st + st^2, f_2 = s^2t + 2t$, take $\cup_i A_i$.

Maple code: `http://www-sop.inria.fr/galaad/emiris/`

Comparisons

	$\deg R$	$\lvert\mathcal{E}\rvert$ [CE]	[CaDiSt]	[ZG]	$\lvert\mathcal{F}\rvert = \dim M$
homogeneous	$3d^2$	$\frac{9}{2}d^2 - \frac{3}{2}d$	$6d^2 - 9d + 4$	$\frac{21}{4}d^2$	$\frac{9}{2}d^2 - \frac{9}{2}d + 1$
bi-homogen.	$6d_1d_2$	$9d_1d_2$	$3(2d_1 - 1) \cdot (2d_2 - 1) + 1$	$6d_1d_2$	$(3d_1 - 1)(3d_2 - 1)$
[ZhaGol'00]	5	5	4	6	4
[Manocha]	7	12	7	15	7
Hirzebruch	12	15	10	12	10
example I	18	26	22	18	19
self-$\cap$ [GaSt]	24	35	34	30	22
example II	57	92	97	63	64

Hybrid matrix with the toric Jacobian: Summary

- For scaled copies $Q_i = k_i P$: use $\omega_i(k_i p) = k_i \omega_i(p)$.

- $M \gg 1 \Rightarrow$ monomial basis of $K[x_i^{\pm 1}]/I(f_1, f_2)$ is contiguous and connected to b in the lattice $\mathbb{Z}[A_1, A_2]$. Exhibits quasi-Toeplitz structure.

- Admits Macaulay-type rational formula [D'Andrea'01] by multiplying ω_1 by μ : $1 \ll \mu \ll M$; and linear perturbation [DAnE'01] for extracting projection operator in degenerate cases.

- Bit complexity of construction $\sim \max_i\{\mathrm{MV}_{-i} + |A_i|\}\,(\# vtx(P)) \log d$.

Further work

- Extension of the algorithm to $n \geq 3$.

The problem of Degeneracy

- The (toric) resultant specializes to zero due to lack of genericity, i.e. the coefficients depend on a few parameters.

- The (toric) resultant vanishes because there is a positive-dimensional component in the (toric) variety, forming "excess components".

- The (toric) resultant is nonzero, but the matrix determinant vanishes identically.

- Classical (dense) elimination: diagonal perturbation of Macaulay matrix (Generalized Characteristic Polynomial) [Canny'90].

- Toric (sparse) elimination: random coefficients or Newton polytope fillings ("toric" GCP) [Rojas'99].

Linear perturbations

System p_i : $R(p_i) \neq 0 \Rightarrow R(f_i - \epsilon p_i) = \epsilon^{\deg R} R(p_i) + \cdots + \epsilon^k \phi \not\equiv 0$.

Theorem. Let $\phi(y_1, \ldots, y_m) =$ be the lowest coefficient wrt ϵ in $R(f_i - \epsilon p_i)$. If W is a proper component (dim $= m - 1$) of the variety $V(f_i) \subset \mathbb{C}^m \times \mathcal{T}$, with $\pi : \mathbb{C}^m \times \mathcal{T} \to \mathbb{C}^m$, then

$$w \in W \;\Rightarrow\; \phi(\pi(w)) = 0.$$

Corollary. $\phi(y_1, \ldots, y_m)$ is a suitable projection operator:

- not identically zero (even after specializing the coefficients),
- vanishes at proper solutions.

Observation. In the case of the u-resultant, the additional linear form does not have to be perturbed.

Open: Perturbed eigenproblem $(M_0(\epsilon) + x M_1)\vec{v} = \vec{0}$.

Systems with nonzero resultant

Theorem [D'Andrea,E'00]. For $B_i = \{a \in \text{ supp}_i : \text{RC}(p) = (i,a), p \in \mathcal{E}\}$ and any $\lambda \in \mathbb{C}^*$, except for a finite number of values, the polynomials

$$p_i(\lambda) = \sum_{a \in B_i} \lambda^{\omega_i(a)} x^a, \qquad i = 0, \ldots, n,$$

define a system with nonzero toric resultant. Hence $\det M(f_i - \epsilon p_i) \not\equiv 0$.

Derandomization: $R\,(f_i - p_i(\epsilon)) \neq 0, \det M(f_i - p_i(\epsilon)) \not\equiv 0$.

Corollary. For random c_{ia}, $p_i = \sum_{a \in B_i} c_{ia} x^a$ work with high probability. Same (perturb diagonal monomials) for incremental algorithm of [E,Canny'95].

Theorem [D'Andrea,E'00]. Asymptotic complexity remains polynomial in $\deg R$ and exponential with a linear exponent in n. Interpolate $\phi(x)$ in some (hidden) variable of degree d takes $O^*(|\mathcal{E}|^4 d^2)$.

Code and Examples

Maple function `spares` in `www-sop.inria.fr/galaad/emiris/soft_alg.html`, library `multires`.

Example [Sturmfels'94].

$$\begin{array}{rclcrcl} f_0 & = & -2 + x_1^2 x_2^2 + x_1 x_2^3, & & p_0 & = & \lambda + x_1 x_2^3, \\ f_1 & = & u + (2-u)x_1^2 - 2x_1 x_2^2, & & p_1 & = & [\lambda^{13}]x_1^2, \\ f_2 & = & x_1^3 - x_1 x_2, & & p_2 & = & \lambda^0(x_1^3 + x_1 x_2), \end{array}$$

$\Rightarrow R\,(p_i) = \pm[\lambda^{91}]\lambda^2 \Rightarrow \lambda = 1$ is a possible choice.
Perturbed resultant's constant term $= R(u) \equiv 0 \Rightarrow \dim Z > 0$; indeed, $(1,1,u) \simeq \mathbb{C} \subset Z$.
Linear coefficient $\phi(u)$ yields projections of 6 isolated roots on u.
$\phi(u)$ interpolated on 8 ($\geq \deg_u \phi$) terms, instead of $\dim M = 21$ terms.

Example [Hirukawa'99].
Degenerate dense 3×3 system with degrees $3, 2, 1$; hide x_3.
Optimal perturbation yields

$$\phi(x_3) = \left(x_3 - \frac{1434}{625}\right)\left(x_3 - \frac{12815703325}{21336}\right).$$

$\deg_{x_3} \phi = 2$ bounds number of affine roots; tighter than MV $= 6$.

First solution is unique isolated root.

Surface Implicitization

Given is a parametrization of a rational surface:

$$X = \frac{q_0(x_1,x_2)}{q_3(x_1,x_2)},\ Y = \frac{q_1(x_1,x_2)}{q_3(x_1,x_2)},\ Z = \frac{q_2(x_1,x_2)}{q_3(x_1,x_2)}.$$

Homogenize the q_i to p_i : $\deg p_i = d$, then consider map

$$\theta : \mathbb{P}^2 \to \mathbb{P}^3 : (x_0 : x_1 : x_2) \mapsto (p_0 : p_1 : p_2 : p_3).$$

Problem: compute the smallest algebraic surface $H(X,Y,Z,W)$ containing $\overline{\mathrm{Im}(\theta)}$.
Including the case of base points $x \in \mathbb{P}^2 : p_i(x) = 0$.
Methods: Gröbner bases, moving surfaces, resultants, Newton sums, numeric methods.

- No base points: $R_{\mathbb{P}^2}(p_0 - Xp_3, p_1 - Yp_3, p_2 - Zp_3) = H(X,Y,Z,1)^{\deg\theta}$.
- $\exists$ base points $\Rightarrow R_{\mathbb{P}^2} \equiv 0$.

If the base points are of pure codimension 1, i.e. $p_i = \gcd(p_0,\ldots,p_3)\tilde{p}_i$,

then $R_{\mathbb{P}^2}(\tilde{p}_0 - X\tilde{p}_3, \tilde{p}_1 - Y\tilde{p}_3, \tilde{p}_2 - Z\tilde{p}_3) = H(X,Y,Z,1)^{\deg\theta}$.

Base points in (pure) codimension 2

Hypothesis: The base points are defined by an ideal $G = (g_1, \ldots, g_n)$.

Theorem [Busé'01] Then, if G is a local complete intersection and $d \geq k_1, d \geq k_n + 1$,

$$R_G(p_0 - Xp_3, p_1 - Yp_3, p_2 - Zp_3) = H(X, Y, Z, 1)^\beta.$$

If μ_P is the multiplicity of base point $P \in V(G)$ then $\deg R_G = d^2 - \sum_P \mu_P$.

Matrix formulae [Busé'01] Given $n \times 4$ matrix $P : (p_0, p_1, p_2, p_3) = (g_1, \ldots, g_n)P$,
write $(p_0 - Xp_3, p_1 - Yp_3, p_2 - Zp_3) = (g_1, \ldots, g_n)M$.

Also compute a new $(n-1) \times n$ matrix and $l_i : \sum_{i=1}^{n-1} l_i = \sum_{i=1}^{n} k_i$.
These define a complex of linear maps D_0, D_1, D_2. Then,

- $R_G(p_0 - Xp_3, p_1 - Yp_3, p_2 - Zp_3) = \gcd(\text{of 3 maximal minors of } D_0)$, or
- $R_G(p_0 - Xp_3, p_1 - Yp_3, p_2 - Zp_3) = \Delta_0 \Delta_2 / \Delta_1$ (maximal minors of the D_i).

Moreover $$\deg H(X, Y, Z, 1)^\beta = d^2 - \frac{1}{2}\left(\sum_{i=1}^{n-1} l_i^2 - \sum_{i=1}^{n} k_i^2\right).$$

Example: residual resultant

Input $p_0 = x_0 x_1^2,\ p_1 = x_1^3,\ p_2 = x_0 x_2^2,\ p_3 = x_1^3 + x_2^3$ [Busé01]
Base point of multiplicity 4, $G = (x_1^2, x_2^2)$, with

$$(p_0 - Xp_3, p_1 - Yp_3, p_2 - Zp_3) = (x_1^2, x_2^2) \begin{bmatrix} x_0 - Xx_1 & x_1 - Yx_1 & -Zx_1 \\ -Xx_2 & -Yx_2 & x_0 - Zx_2 \end{bmatrix}$$

- $$D_0 = \begin{bmatrix} 0 & 0 & 0 & -X & 1 & 0 & 0 & 1-Y & 0 & 0 \\ X & -Y & 0 & 0 & -Z & -X & 0 & -Z & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -X & 0 & -Y & -Z \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -X & 0 & 1-Y & -Z \\ 0 & 0 & 0 & 0 & -X & 0 & 1 & 0 & 0 & 0 \\ -Y & 0 & 0 & -Z & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -Y & 0 & 0 & -Z & 0 & 0 & 0 & 1 \\ 0 & X & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & X & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

minors $\Delta_0 = (-Z^3Y^2 + Y^2X^3 - 2X^3Y + X^3)X^2$, $(-1+Y)Y(-Z^3Y^2 + Y^2X^3 - 2X^3Y + X^3)$.

Hence the implicit equation is $H(X, Y, Z, 1) = -Z^3Y^2 + Y^2X^3 - 2X^3Y + X^3$.

- $$R_G = \Delta_0 \cdot 1/\det \begin{bmatrix} -X & 0 \\ 0 & -X \end{bmatrix}.$$

Rational form: numerator

[D'And'01] $f_i = a_i + b_i x_1 + c_i x_2 + d_i x_1 x_2$, $i = 0, 1, 2$.
Linear liftings $(0,1,1,2), (0,0,7,7), (0,14,0,14)$, $\delta = (\frac{2}{3}, \frac{1}{2}) \Rightarrow \dim M = 16$:

$$M = \begin{pmatrix}
a_1 & b_1 & 0 & c_1 & d_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & a_0 & b_0 & 0 & c_0 & d_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & a_1 & b_1 & 0 & c_1 & d_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
a_2 & b_2 & 0 & c_2 & d_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & a_2 & b_2 & 0 & c_2 & d_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & a_0 & 0 & 0 & c_0 & d_0 & b_0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & a_2 & b_2 & 0 & c_2 & d_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & a_2 & b_2 & 0 & c_2 & d_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & a_1 & b_1 & 0 & c_1 & d_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & a_1 & 0 & 0 & c_1 & d_1 & b_1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & a_2 & 0 & 0 & c_2 & d_2 & b_2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & a_2 & 0 & 0 & c_2 & 0 & 0 & 0 & 0 & d_2 & b_2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & a_2 & b_2 & 0 & 0 & 0 & 0 & c_2 & d_2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & a_2 & b_2 & 0 & 0 & 0 & 0 & c_2 & d_2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_2 & b_2 & 0 & 0 & 0 & 0 & c_2 & d_2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_1 & b_1 & 0 & 0 & 0 & 0 & c_1 & d_1
\end{pmatrix}$$

Rational form: denominator

$$M' = \begin{pmatrix} a_1 & 0 & c_1 & d_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b_1 & 0 & c_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_2 & 0 & c_2 & d_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b_2 & 0 & c_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_2 & b_2 & c_2 & d_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_2 & 0 & c_2 & d_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_1 & 0 & c_1 & d_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & c_2 & b_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_2 & b_2 & 0 & c_2 & d_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_2 & 0 & 0 & c_2 \end{pmatrix}$$

$\det(M) = \pm R \cdot \det(M')$, $|M'| = -c_2^3(-c_1a_2+a_1c_2)b_2(c_1d_2-d_1c_2)(-b_2c_1+b_1c_2)$

Algorithm recurses on n [D'And'01].

Open: $\exists$ single lifting yielding both numerator and denominator?

YES for $n = 2$, and for systems with "reduced" (sufficiently different) Newton polytopes. The mixed subdivision provides all info.

Main step: lifting of some $b \in Q_0$ is very negative.