Theorem 1. *Suppose we have n sensors in the plane and we must broadcast a message that reaches each of the sensors. We are to design a tour which travels in the vicinity of a sensor, or group of sensors, and then broadcasts the message. The cost function for sending out messages is $t + c\sum r_i$ where t is the length of a tour, and r_i is the broadcast radius from the i-th broadcast location. If $c < 4$, the least cost way to send a message to all sensors is to broadcast from their circum-center at tour cost $t = 0$.*

Proof. The crux of the argument is the following lemma:

Lemma 1. *For 3 points in the plane, the cost of a tour visiting each of the points is at least $4r$ where r is the radius of the smallest enclosing circle.*

Proof. Either (i) the smallest containing circle is determined by 2 of the points which are diametrically opposite each other with the third point in the interior, or (ii) the circum-circle is determined by all 3 points, and not all 3 of the points lie on the same half-circle.

In case (i) the lemma follows by the triangle inequality. In case (ii) there must be a point amongst the three which has no point within 90 degrees of arc length from it. To see this, place a first point on the circle. If this first point has a point within 90 degrees of arc length from itself, place such a second point. Now if the third point is within 90 degrees of either of the first two points, then all three points lie on the same half-circle, contradicting the assumptions of case (ii).

Hence we have the situation in Figure 1

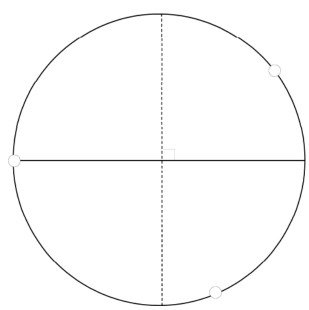

FIGURE 1. Case (ii) where all 3 points lie on the circum-circle and not all points lie on the same half-circle

Now we consider a trip $p \to q \to r \to p$ and compare it to a trip back and forth along the diameter pp' as illustrated in Figure 2.

Letting $d(p, q)$ denote the distance from p to q, we see that $d(p, q) > d(p, q')$ and $d(q, t) > d(q, q') > d(q', p')$. The latter inequality, $d(q, q') > d(q', p')$, holds since $\angle p'qq' < \frac{\pi}{4}$ and $\frac{\pi}{4} < \angle qp'q'$ so $\angle p'qq' < \angle qp'q'$. Putting $d(p, q) > d(p, q')$ together with $d(q, t) > d(q', p')$ gives $d(p, q) + d(q, t) > d(p, p')$ and analogously $d(p, r) + d(r, t) > d(p, p')$, establishing the lemma. □

Using the triangle inequality, the above lemma extends to:

Corollary 1. *For n points in the plane, the cost of a tour visiting each of the points is at least $4r$ where r is the radius of the smallest enclosing circle.* □

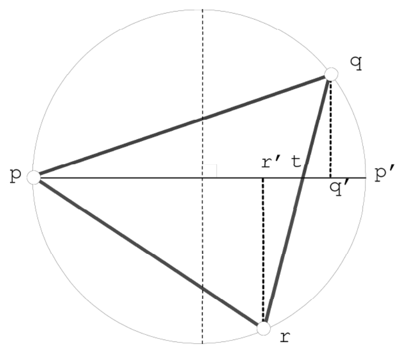

FIGURE 2. Comparison of the trip $p \to q \to r \to p$ to a trip back and forth along the diameter pp'.

Finally suppose we have a tour of total travel distance t that stops in l locations and broadcasts to a radius r_i for $i = 1, ..., l$. Then

$$Cost = t + c\sum_{i=1}^{l} r_i$$

If we let r denote the circum-radius of the tour, then we have

$$\begin{aligned} Cost &\geq 4r + c\sum_{i=1}^{l} r_i \\ &\geq 4r + c\max_i r_i \end{aligned}$$

and since $c < 4$ it is cheaper to broadcast from the circum-center of the tour to a radius of $r + \max_i r_i$ at cost $c(r + \max_i r_i)$ and no tour cost. The theorem is thus established. □