Complexity of Delaunay triangulation for points on lower-dimensional polyhedra

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Abstract

We show that the Delaunay triangulation of a set of points distributed nearly uniformly on a polyhedron (not necessarily convex) of dimension p in d-dimensional space is $O(n^{(d-1)/p})$. For all $2 \leq p \leq d - 1$, this improves on the well-known worst-case bound of $O(n^{\lceil d/2 \rceil})$.

1   Introduction

The Delaunay triangulation of a set of points is a data structure, which in low dimensions has applications in mesh generation, surface reconstruction, molecular modeling, geographic information systems, and many other areas of science and engineering. Like many spatial partitioning techniques, however, it suffers from the "curse of dimensionality": in higher dimensions, the complexity of the Delaunay triangulation increases exponentially. Its worst-case complexity is bounded precisely by the following theorem, known as the Upper Bound Theorem.

Theorem 1 (McMullen [13]) *The number of simplices in the Delaunay triangulation of n points in dimension d is at most*

$$\binom{n - \lfloor \frac{d+1}{2} \rfloor}{n - d} + \binom{n - \lfloor \frac{d+2}{2} \rfloor}{n - d} \;=\; O(n^{\lceil \frac{d}{2} \rceil})$$

This bound is achieved exactly by the vertices of a cyclic polytope, which all lie on a one-dimensional curve known as the *moment curve*. Indeed all of the examples that we have of point sets which have Delaunay triangulations of complexity

$O(n^{\lceil d/2 \rceil})$ are distributed on one-dimensional curves. At the opposite extreme, points distributed uniformly at random inside the sphere have Delaunay triangulations of complexity $O(n)$ in any fixed dimension, with a constant factor which is exponential in the dimension [9]. Our goal in this paper is to begin to fill in the picture for distributions between the two extremes, in which the points lie on manifolds of dimension $2 \leq p \leq d - 1$.

As an easy first case, we consider a fixed polyhedral set (not necessarily convex) $\mathbb{P}$ of dimension p in $d > p$ dimensional space. Our point set S is a *sparse ϵ-sample* from $\mathbb{P}$. Sparse ϵ-sampling is a model, sometimes used in computational geometry, in which the sampling can be neither too dense nor too sparse; we define it precisely below. Let $n = |S|$. We consider how the complexity of the Delaunay triangulation of S grows, as $n \to \infty$, with $\mathbb{P}$ remaining fixed. Our main result is that the number of full-dimensional simplices in the Delaunay triangulation is $O(n^{(d-1)/p})$; this implies that the total number of faces of all dimensions satisfies the same bound. The hidden constant factor depends, among other things, on the geometry of $\mathbb{P}$, which is constant since $\mathbb{P}$ is fixed.

While our result is purely combinatorial, it has both potential and immediate algorithmic implications. The Delaunay triangulation can be computed in optimal worst-case time in dimension $d \geq 3$ by the standard randomized incremental algorithm [8, 16], or deterministically [7]. While our result does not immediately improve these running times for the special case of points distributed on lower-dimensional manifolds [3], it is of course a necessary step towards such an improvement. Our result shows that Seidel's giftwrapping algorithm [15] runs in time $O(n^2 + n^{(d-1)/p} \lg n)$ in our special cases, which can be somewhat improved using more sophisticated data structures [4].

1.1   Prior work

The complexity of the Delaunay triangulation of a set of points on a two-manifold in $\mathbb{R}^3$ has received considerable recent attention, since such point sets arise in practice, and

their Delaunay triangulations are found nearly always to have linear size. Golin and Na [12] proved that the Delaunay triangulation of a large enough set of points distributed uniformly at random on the surface of a fixed convex polytope in $\mathbb{R}^3$ has $O(n)$ size. They later [11] gave an $O(n \lg^4 n)$ upper bound for the case in which the points are distributed uniformly at random on the surface of a non-convex polyhedron.

Attali and Boissonnat considered the problem using a sparse ϵ-sampling model similar to the one we use here, rather than a random distribution. For such a set of points distributed on a polygonal surface $\mathbb{P}$, they showed that the size of the Delaunay triangulation is $O(n)$ [1]. Our proof gives the same bound, and is perhaps a little simpler; but, as we shall describe in a moment, our definition of sparse ϵ-sampling for polyhedra is a little more restrictive. In a subsequent paper with Lieutier [2] they considered "generic" surfaces, and got an upper bound of $O(n \lg n)$. Specifically, a "generic" surface is one for which each medial ball touches the surface in at most a constant number of points.

The genericity assumption is important. Erickson considered more general point distributions, which he characterized by the *spread*: the ratio of the largest inter-point distance to the smallest. The spread of a sparse ϵ-sample of n points from a two-dimensional manifold is $O(\sqrt{n})$. Erickson proved that the Delaunay triangulation of a set of points in $\mathbb{R}^3$ with spread Δ is $O(\Delta^3)$. Perhaps even more interestingly, he showed that this bound is tight for $\Delta = \sqrt{n}$, by giving an example of a sparse ϵ-sample of points from a cylinder that has a Delaunay triangulation of size $\Omega(n^{3/2})$ [10]. Note that this surface is not generic and has a degenerate medial axis.

To the best of our knowledge, there are no prior results for $d > 3$.

1.2 Overview of the proof

Our proof uses two samples, the original sparse ϵ-sample S from the polyhedron $\mathbb{P}$, and a sparse ϵ-sample M of a bounded subset $\mathcal{M}^*$ of the medial axis of $\mathbb{P}$. We prove that any Delaunay ball circumscribing points of S is contained in an enlarged medial ball centered at a point z of $\mathcal{M}^*$, and we assign each z to a medial sample point in M. We then prove that each sample in M is assigned at most a constant number of Delaunay balls.

Since M is a sparse ϵ-sample from a fixed $(d-1)$-dimensional set of constant volume, its cardinality is $m = O(\epsilon^{d-1})$. Similarly, S is a sparse ϵ-sample of $\mathbb{P}$ and we get $n = \Omega(\epsilon^{-p})$. Eliminating ϵ gives $m = O(n^{(d-1)/p})$, and since each sample of M is charged for a constant number of Delaunay balls, this bound applies to the size of the Delaunay triangulation as well. This is the key insight: as a function of ϵ, the number of Delaunay balls depends only on the dimension of the medial axis, which is always $d-1$. The number of samples, n, depends on the dimension p of $\mathbb{P}$. As

p increases, n increases, but the complexity of the Delaunay triangulation stays about the same. If written as a function of n, the complexity of the Delaunay triangulation goes down.

2 Statement of Theorem

In this section, we introduce the setting for our result. We first define simplicial complexes, Delaunay triangulations, polyhedra and medial axes.

2.1 Simplicial complexes

We refer to [14] for more details on simplicial complexes. A *geometric simplex*, σ, is the convex hull of a collection of affinely independent points in $\mathbb{R}^d$. If there are $k+1$ points in the collection, we call σ a k-simplex and k its *dimension*. Any simplex spanned by a subset of the $k+1$ points is called a *face* of σ. A *geometric simplicial complex* is a finite collection of simplices, K, satisfying the two following properties: (1) every face of a simplex in K is in K; (2) the intersection of any two simplices of K is either empty or a face of each of them. Its *underlying space*, $|K|$, is the subset of $\mathbb{R}^d$, covered by the simplices with the subspace topology inherited from $\mathbb{R}^d$. We will also need abstract versions of those geometric notions. An *abstract simplicial complex*, K, is a collection of finite nonempty sets, such that if $\sigma \in K$, so is every nonempty subset of σ. The element σ of K is called an *abstract simplex*, its *dimension* is one less than its cardinality. A *face* of σ is any nonempty subset of σ. The vertex set of σ is the collection one-point element of σ, which we denote as $\mathrm{Vert}\,\sigma$. The *dimension* of a simplicial complex K, geometric or abstract, is the largest dimension of any simplex in K.

2.2 Delaunay triangulations

Let $S \subseteq \mathbb{R}^d$ be a finite set of points. The *Voronoi region* $V(s)$, of $s \in S$ is the set of points $x \in \mathbb{R}^d$ with $\|x - s\| \leq \|x - t\|$ for all $t \in S$. The Delaunay triangulation $\mathrm{Del}(S)$ of S is the nerve of the Voronoi regions. Specifically, an abstract simplex $\sigma = \{s_0, \ldots, s_k\} \subseteq S$ belongs to the Delaunay triangulation iff the Voronoi regions of its vertices have a nonempty common intersection, $\bigcap_{0 \leq i \leq k} V(s_i) \neq \emptyset$. Equivalently, a simplex σ is in the Delaunay triangulation iff there exists of a $(d-1)$-sphere, called *Delaunay sphere*, that passes through $s_0, \ldots, s_k$ and encloses no point of S. Afterwards, any d-ball bounded by a Delaunay sphere is referred to as a *Delaunay ball*. The Delaunay triangulation is an abstract simplicial complex. Notice that using this definition, on degenerate inputs in which $k + 1 > d + 2$ points are co-spherical, every $d + 1$ subset of these points defines a Delaunay simplex. In this paper, we don't assume general position for points in S. The *complexity* (or *size*) of the Delaunay triangulation is the total number of its simplices of all

dimensions. We express it as a function of n, the number of points in S.

2.3 Polyhedron

A *polyhedron* is the underlying space of a geometric simplicial complex. If the dimension of the simplicial complex K is p, we say that $\mathbb{P} = |K|$ is a p-polyhedron. The collection of simplices $\sigma \in K$, whose dimension is i or less is called the *i-skeleton* of K and denoted $K^{(i)}$. Given a point $x \in \mathbb{P}$, we let i be the largest dimension for which a small open d-ball centered at x intersects $|K^{(i)}|$ in an open i-ball $N(x)$ of $|K^i|$ (see Figure 1). The *tangent flat* to $\mathbb{P}$ at x, denoted by $\mathrm{T}\mathbb{P}_x$, is the i-flat spanned by the open i-ball $N(x)$. A *facet F* of $\mathbb{P}$ is a maximal collection of points with identical tangent flat. If the dimension of the tangent flat is i, F is an i-facet. The 0-facets are called *vertices*. Afterwards, f designates the number of facets of $\mathbb{P}$.
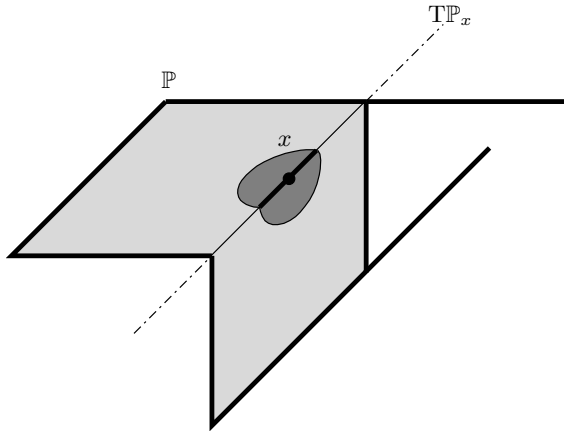

Figure 1: The tangent flat to $\mathbb{P}$ at x is a line. Note that a polyhedron may have smaller-dimensional parts. Points that sample lower-dimensional parts can have a high-complexity Delaunay triangulation but still not damage the overall complexity, because they form a small subset of the set of sample points.

2.4 Medial axis

The medial axis is instrumental in expressing the constant in our main result and crucial for proving it. The *medial axis*, $\mathcal{M} = \mathcal{M}(\mathbb{P})$, is the set of points that have at least two closest points in $\mathbb{P}$. Formally, writing $\Pi(z)$ for the set of points in $\mathbb{P}$ with minimum distance to $z \in \mathbb{R}^d$, we have

$$\mathcal{M} \;=\; \{z \in \mathbb{R}^d \mid \mathrm{card}\,\Pi(z) \geq 2\}.$$

A point z on the medial axis is said to be *medial*. The smallest d-ball centered at z and containing $\Pi(z)$ is called a *medial ball*. The medial axis of a polyhedron $\mathbb{P}$ admits a stratification. In other words, it can be decomposed into a finite number of strata, each a connected i-manifold with boundary, for $i < d$. Furthermore, any point $x \in \mathcal{M}$ belongs to the closure

of at least one stratum of dimension $d-1$. Roughly speaking, the medial axis of a polyhedron is a $(d-1)$-dimensional surface that has a positive $(d-1)$-dimensional volume, possibly infinite.

We introduce a bounded subset of the medial axis, $\mathcal{M}^* = \mathcal{M}^*(\mathbb{P})$, called the essential medial axis and defined as follows. For $z \in \mathcal{M}$, we let $c(z)$ be the center of the smallest d-ball enclosing $\Pi(z)$. A point z for which $z = c(z)$ is called a critical point of the distance-to-polyhedron function [6, 5]. One can prove that $z = c(z)$ iff z lies in the convex hull of $\Pi(z)$. It follows that if z lies outside the convex hull, then $z \neq c(z)$ and the line passing through $c(z)$ and z is well defined. We denote it by $L(z)$. We are now ready to define what it means for a point z to be *essential*. We consider two cases: if z belongs to the convex hull of $\mathbb{P}$, then z is essential; otherwise, z is essential iff it is the center of the smallest medial ball enclosing $\Pi(z)$ and centered on $L(z)$ (see Figure 2). An *essential medial ball* is a d-ball whose center is essential. The *essential medial axis* is defined as the set of essential medial points and, by construction, is bounded. Unlike the medial axis, it has a finite $(d-1)$-dimensional volume, which vanishes iff the polyhedron lies on an hyperplane of $\mathbb{R}^d$.
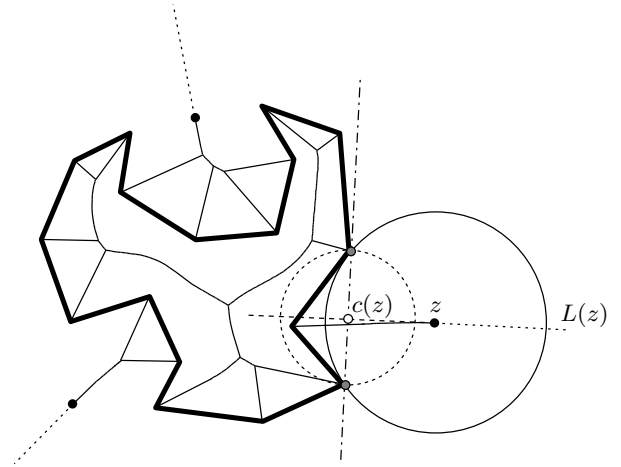
Figure 2: The essential medial axis is solid and includes the black dots. The non-essential part is dotted. The point z is essential. Observe that the circle around z is tangent to one edge incident to a white point.

2.5 Sampling condition

We write $B(x, r)$ for the closed d-ball with center x and radius r. Given a polyhedron $\mathbb{P} \subseteq \mathbb{R}^d$, we say that a set of points $S \subseteq \mathbb{P}$ is a *λ-sparse ϵ-sample* of $\mathbb{P}$ iff it satisfies the following two conditions:

Density: for every point $x \in \mathbb{P}$, $B(x, \epsilon) \cap \mathrm{T}\mathbb{P}_x$ contains at least one point of S;

Sparsity: for every point $y \in \mathbb{R}^d$, $B(y, d\epsilon)$ contains at most λ points of S.

The number of samples on each i-facet is $\Theta(\epsilon^{-i})$. Writing p for the dimension of $\mathbb{P}$, the number of points in the sample is $n = \Theta(\epsilon^{-p})$.

To state our main result, we let v_i be the i-dimensional volume of a unit i-ball. The i-dimensional volume of a i-ball of radius r is $v_i r^i$. Furthermore, writing $\mathrm{vol}_i(X)$ for the i-dimensional volume of $X \subseteq \mathbb{R}^d$, we assume that there exist two positive constants $w_\mathbb{P} > 0$ and $r_0 > 0$ such that for every $0 < r \leq r_0$, for every point $x \in \mathcal{M}^*$

$$w_\mathbb{P} \;\; \leq \;\; \frac{\mathrm{vol}_{d-1}(B(x,r) \cap \mathcal{M})}{r^{d-1}}. \tag{1}$$

To provide an intuition for the constant $w_\mathbb{P}$, suppose x is a point on a smooth hypersurface $\mathbb{M}$ embedded in $\mathbb{R}^d$. In first approximation, the intersection of a d-ball $B(x,r)$ and the hypersurface has the volume of a $(d-1)$-ball with radius r, showing that the ratio on the right side of Inequality (1) tends to v_{d-1} as r tends to 0. To take into account the fact that the closure of the medial axis is not necessarily an hypersurface, we assume that for r small enough, every point of the essential medial axis satisfies Inequality (1). Finally, we introduce the constant

$$C(\mathbb{P}) \;\; = \;\; 2^{2d-1}\frac{\mathrm{vol}_{d-1}(\mathcal{M}^*(\mathbb{P}))}{\mathrm{vol}_p(\mathbb{P})^{\frac{d-1}{p}}} \times \frac{(2v_p)^{\frac{d-1}{p}}}{w_\mathbb{P}}.$$

Theorem 2 (Main theorem) *Let $\mathbb{P}$ be a p-polyhedron in $\mathbb{R}^d$ composed of f facets not all contained in an hyperplane. Let S be a set of n points that forms a λ-sparse ϵ-sample of $\mathbb{P}$. The Delaunay triangulation of S has complexity $O(n^{\frac{d-1}{p}})$. More precisely, for n large enough, the number of Delaunay k-simplices is bounded from above by*

$$C(\mathbb{P})\binom{\lambda f}{k+1}n^{\frac{d-1}{p}} \;\; + \;\; \binom{f}{k+1}n.$$

Our proof assumes that neither the p-dimensional volume of the polyhedron nor the $(d-1)$-dimensional volume of the essential medial axis vanishes, which ensures that $0 < C(\mathbb{P}) < \infty$.

3 Contact of Delaunay balls

In this section, we establish preliminary results that bound the intersection of a Delaunay ball and the polyhedron. For this, we use a shape obtained by enlarging either a medial ball or a point on the polyhedron. Those results will be used in Section 4 to prove our bound on the complexity of Delaunay triangulations.

3.1 Almost tangent balls

We first give a crucial property of Delaunay balls induced by our sampling condition. For this, we need notations and definitions. Recall that $B(z, r)$ is the closed d-ball with center z and radius r. We define the *penetration* $h(b)$ of $b = B(z, r)$ as the difference of square radii between b and the largest d-ball centered at z and whose interior does not intersect $\mathbb{P}$

$$h(b) \;\; = \;\; r^2 - d(z, \mathbb{P})^2.$$

Note that b intersects the polyhedron iff $h(b) \geq 0$. A ball b for which $h(b) = 0$ is said to be *tangent* to the polyhedron. Calling a ball b for which $0 \leq h(b) \leq \epsilon^2$ an ϵ-*almost tangent* ball, we have the fundamental property:

FUNDAMENTAL PROPERTY. Delaunay balls are ϵ-almost tangent.

PROOF. Any Delaunay ball b intersects the polyhedron in at least one sample point $s \in S$, showing that $h(b) \geq 0$. Let x be a point in $\mathbb{P}$ with minimum distance to the center of b and t be a sample point on $\mathbb{TP}_x$ with minimum distance to x. Because of our sampling condition, $\|x - t\| \leq \epsilon$. Because b contains no sample point in its interior, b intersects $\mathbb{TP}_x$ in a $(d-1)$-ball of radius $\sqrt{h(b)} \leq \|x - t\|$. It follows that $h(b) \leq \epsilon^2$. $\;\;\;\;\;\;\;\;\;\;\;\;\;\;\;\;\;\;\;$ ⊟

3.2 Contact

We now introduce the notion of ϵ-contact of a ball ν and use it to cover the intersection of ϵ-almost tangent balls with the polyhedron. Given a d-ball ν with center z and radius r, we let $\nu^{\oplus a} = B(z, \sqrt{r^2 + a^2})$ be the d-ball with center z and radius $\sqrt{r^2 + a^2}$. Recalling that $\Pi(z)$ is the set of points in $\mathbb{P}$ with minimum distance to z, we define the ϵ-*contact* of ν as

$$\mathrm{Contact}(\nu, \epsilon) \;\; = \;\; \left(\nu^{\oplus\epsilon} \cap \mathbb{P}\right) \cup \bigcup_{x \in \Pi(z)} B(x, \epsilon).$$

Suppose b is an ϵ-almost tangent ball. We show in the next lemma that there exists a ball ν tangent to the polyhedron whose ϵ-contact contains $b \cap \mathbb{P}$. In addition, this tangent ball is either medial or a point of $\mathbb{P}$. To distinguish between the two cases, we need definitions. If a ball b is not medial, its center y has a unique closest point $x \in \mathbb{P}$. The half-line L begining at x and passing through the center y of b either intersects the medial axis in a point z, or extends to infinity (for instance, when x is on the boundary of the convex hull of $\mathbb{P}$, in the interior of a p-facet). In the first case we say that b is *ordinary*, and in the second we say that b is *trivial*. A medial ball is said to be ordinary. Notice that when b is ordinary, y lies in the interior of segment xz (since x is the closest point on $\mathbb{P}$ to y).

Lemma 3 *Suppose b is an ϵ-almost tangent ball to $\mathbb{P}$.*

(i) If b is trivial, then there exists a point $x \in \mathbb{P}$ such that $b \cap \mathbb{P} \subseteq B(x, \epsilon)$.

(ii) If b is ordinary, then there exists a medial ball ν such that $b \cap \mathbb{P} \subseteq \mathrm{Contact}(\nu, \epsilon)$.

PROOF. If b is medial or if its center lies on $\mathbb{P}$, the result follows immediately by setting $\nu = b$. Assume b is not medial and its center y has a unique closest point $x \neq y$ on $\mathbb{P}$ (see Figure 3). Consider the half-line, L, with origin x and

Figure 3: Notations for the proof of Lemma 3. Dotted spheres are tangent to $\mathbb{P}$. Solid spheres penetrate $\mathbb{P}$ from r and belong to the same pencil of spheres passing through the boundary of τ.

passing through y. The half-line L intersects the medial axis (possibly at infinity) and we let $z \in \mathcal{M}$ be the point on L closest to y. Because b is ϵ-almost tangent, the hyperplane H that passes through x and is orthogonal to the straight-line xy intersects b in a $(d-1)$-ball τ with center x and radius $0 \leq r \leq \epsilon$. Consider the pencil of $(d-1)$-spheres that pass through the boundary of τ and let b_u be the d-ball bounded by a sphere in this pencil with center $u \in L$. Because the interval defined by x and z contains y, we have $b = b_y \subseteq b_x \cup b_z$. First, assume that z is at infinity (*i.e.* b is trivial). The ball b_z is the one half-space that H bounds and which contains y. Its interior does not intersect $\mathbb{P}$ and it follows that

$$b \cap \mathbb{P} \quad \subseteq \quad b_x = B(x, r) \quad \subseteq \quad B(x, \epsilon).$$

Second, assume that z belongs to the medial axis (*i.e.* b is ordinary) and let ν be the medial ball centered at z. This medial ball passes through x, and therefore $b_z = \nu^{\oplus r} \subseteq \nu^{\oplus \epsilon}$. Observing that $x \in \Pi(z)$, it follows that

$$b \cap \mathbb{P} \quad \subseteq \quad \left[\nu^{\oplus \epsilon} \cup B(x, \epsilon) \right] \cap \mathbb{P}$$
$$\subseteq \quad \text{Contact}(\nu, \epsilon).$$

$\square$

We strenghten the second part of Lemma 3 by showing that the medial ball ν whose ϵ-contact contains $b \cap \mathbb{P}$ can be chosen essential.

Lemma 4 *For every non-essential medial ball μ, there exists an essential medial ball ν such that*

$$\text{Contact}(\mu, \epsilon) \quad \subseteq \quad \text{Contact}(\nu, \epsilon)$$

PROOF. See Figure 4; let y be the center of μ. Let $c = c(y)$ be the center of the smallest ball enclosing $\Pi(y)$, the set of points in $\mathbb{P}$ with minimum distance to y. Because y is not essential, it does not lie in the convex hull of $\mathbb{P}$ and $y \neq c$. Consider H, the hyperplane passing through c and orthogonal to the vector $c-y$. $\mathbb{P}$ and y lie on opposite sides of the hyperplane H. Let ω and ω' be the two closed half-spaces that H bounds and assume that $y \in w$ and $\mathbb{P} \in w'$. The $(d-1)$-spheres bounding μ and w generate a pencil of spheres, whose centers lie on the straight-line passing through c and y. They intersect the hyperplane H in a common $(d-1)$-ball τ, whose boundary contains $\Pi(y)$. Consider the smallest medial sphere in the pencil and let ν be the ball that this sphere bounds. The center z of ν lies on the segment cy. Note that $\Pi(y) \subseteq \Pi(z)$ and points in $\Pi(z) - \Pi(y)$ lie in the smallest ball centered at c and passing through $\Pi(y)$. It follows that the smallest ball enclosing $\Pi(z)$ is also centered at $c = c(z)$. Therefore, ν is an essential medial ball. Now, observe that $\nu^{\oplus \epsilon}$ belongs to the pencil of spheres generated by $\mu^{\oplus \epsilon}$ and w. Since the center z of the former lie on the

Figure 4: Notations for the proof of Lemma 4. The point z is essential and the point y is not. Inner spheres of gray annuli are medial. Outer spheres penetrate $\mathbb{P}$ from ϵ.

half-line defined by the centers of the latters, we have $\mu^{\oplus \epsilon} \subseteq \nu^{\oplus \epsilon} \cup \omega$. It follows that

$$\mu^{\oplus \epsilon} \cap \mathbb{P} \quad \subseteq \quad \nu^{\oplus \epsilon} \cap \mathbb{P},$$

and because $\Pi(y) \subseteq \Pi(z)$, the ϵ-contacts of μ and ν are nested as claimed. $\square$

3.3 Extended contact

It will be convenient to bound the intersection of Delaunay balls and the polyhedron, using a slightly different notion of

ϵ-contact. For $x \in \mathbb{P}$, we let $\pi_x(z)$ be the point of $\mathrm{T}\mathbb{P}_x$ closest to z. Given $z \in \mathbb{R}^d$, we set $\chi(z) = \{\pi_x(z) \mid x \in \mathbb{P}\}$ and define the *extended ϵ-contact* of the d-ball ν with center z as

$$\mathrm{ExtendedContact}(\nu, \epsilon) = \bigcup_{x \in \chi(z)} B(x, \epsilon).$$

The extended contact of ν is formed of f balls, centered at the orthogonal projections of its center z on the tangent planes of $\mathbb{P}$. For any two medial balls whose centers are less than ϵ apart, we show that the ϵ-contact of one is contained in the extended $d\epsilon$-contact of the other. This result will be used in the next section to reduce the amount of essential medial balls necessary to cover intersections of Delaunay balls and the polyhedron.

Lemma 5 *For any two medial balls d-balls μ and ν whose centers y and z satisfy $\|y - z\| \leq \epsilon$,*

$$\mathrm{Contact}(\mu, \epsilon) \quad \subseteq \quad \mathrm{ExtendedContact}(\nu, d\epsilon).$$

PROOF. Let $x \in \mathrm{Contact}(\mu, \epsilon)$. First, suppose that $x \in B(q, \epsilon)$ for some $q \in \Pi(y)$. Applying the triangle inequality and using $\pi_q(y) = q$, we get

$$
\begin{aligned}
\|x - \pi_q(z)\| &\leq \|x - q\| + \|\pi_q(y) - \pi_q(z)\| \\
&\leq \epsilon + \|y - z\| \\
&\leq 2\epsilon,
\end{aligned}
$$

and $x \in \mathrm{ExtendedContact}(\nu, 2\epsilon)$. Suppose now that $x \in \mu^{\oplus \epsilon}$ and consider the annulus $\alpha = \mu^{\oplus \epsilon} - \mu$. We distinguish two cases:

1. if $\pi_x(y) \in \alpha$, then $\|x - \pi_x(y)\| \leq \epsilon$ and we apply again triangular inequality. We get

$$
\begin{aligned}
\|x - \pi_x(z)\| &\leq \|x - \pi_x(y)\| + \|\pi_x(y) - \pi_x(z)\| \\
&\leq \epsilon + \|y - z\| \\
&\leq 2\epsilon.
\end{aligned}
$$

2. if $\pi_x(y) \notin \alpha$, we consider the point $q \in \mathbb{P}$ on the segment $x\pi_x(y)$, which is closest to x and does not have the same tangent flat as x. The dimension of $\mathrm{T}\mathbb{P}_x$ is strictly greater than the dimension of $\mathrm{T}\mathbb{P}_q$ and $\|x - q\| \leq \epsilon$, since the segment xq is contained in α. We now iterate using q instead of x.

After a finite number of steps, we find a point $q \in \mathbb{P}$ such that $\pi_q(y) \in \alpha$. If q is a vertex of the polyhedron, $q = \pi_q(y)$ and at most $d - 1$ steps were necessary, showing that $\|x - q\| \leq (d-1)\epsilon$. If q is not a vertex, $\|q - \pi_q(y)\| \leq \epsilon$ and at most $d - 2$ steps were necessary, showing that $\|x - q\| \leq (d - 2)\epsilon$. We apply again triangular inequality and get

$$
\begin{aligned}
\|x - \pi_q(z)\| &\leq \|x - q\| + \|q - \pi_q(y)\| + \|\pi_q(y) - \pi_q(z)\| \\
&\leq (d - 1)\epsilon + \|y - z\| \\
&\leq d\,\epsilon,
\end{aligned}
$$

$\square$

4 Bounding the number of simplices

In this section, we bound the number of Delaunay k-simplices induced by λ-sparse ϵ-samples S of the polyhedron $\mathbb{P}$. A key step in our proof is to introduce a sample of the essential medial axis $M \subseteq \mathcal{M}^*$ consisting of $m = O(n^{\frac{d-1}{p}})$ points.

4.1 Sampling the medial axis

An *ϵ-sample* of the essential medial axis is a subset $M \subseteq \mathcal{M}^*$ such that every point $x \in \mathcal{M}^*$ has a point $z \in M$ at distance no more than ϵ, $\|x - z\| \leq \epsilon$. We construct such a sample by considering a maximal collection of m non-overlapping d-balls $b_i = B(z_i, \frac{\epsilon}{2})$ whose centers z_i lie on the essential medial axis $\mathcal{M}^*$. Because the collection is maximal, no ball with center $z \in \mathcal{M}^*$ and radius $\frac{\epsilon}{2}$ can be added to the collection without overlapping $\bigcup b_i$. This implies that the set of centers z_i is an ϵ-sample of the essential medial axis, which we denote by M. On the other hand, the patches $\gamma_i = B(z_i, \frac{\epsilon}{2}) \cap \mathcal{M}$ pack the enlarged essential medial axis $\mathcal{M}^*_{\epsilon/2} = \{x \in \mathcal{M} \mid d(x, \mathcal{M}^*) \leq \epsilon/2\}$ and

$$m \min_i \mathrm{vol}_{d-1}(\gamma_i) \leq \mathrm{vol}_{d-1}(\mathcal{M}^*_{\epsilon/2}).$$

As ϵ tends to 0, the ratio between the two $(d-1)$-dimensional volume of $\mathcal{M}^*$ and $\mathcal{M}^*_{\epsilon/2}$ tends to 1 and for ϵ small enough, $\mathrm{vol}_{d-1}(\mathcal{M}^*_{\epsilon/2}) \leq 2\,\mathrm{vol}_{d-1}(\mathcal{M}^*)$. Applying Inequality (1) yields to an upper bound on the number of points m in M:

$$m \leq 2^d \frac{\mathrm{vol}_{d-1}(\mathcal{M}^*)}{w_\mathbb{P}} \epsilon^{-(d-1)}$$

In short, $m = O(\epsilon^{-(d-1)})$. We now establish that size of S is at least some constant times the p-th power of one over ϵ, $n = \Omega(\epsilon^{-p})$.

Lemma 6 *Let S be a λ-sparse ϵ-sample of a p-polyhedron $\mathbb{P}$. For ϵ small enough, the number of points n in S satisfies*

$$2^{-(p+1)} \frac{\mathrm{vol}_p(\mathbb{P})}{v_p} \epsilon^{-p} \leq n$$

PROOF. Let F be a facet of $\mathbb{P}$ that spans a k-flat H. Let δ be the unit k-ball centered at the origin of $\mathbb{R}^d$ and parallel to H. Consider a maximal collection of l non-overlapping k-balls $\delta_i = B(x_i, \epsilon) \cap H$ contained in F. Each ball δ_i contains at least a sample point of S. Thus, $l \leq \mathrm{card}\, S \cap F$. On the other hand, the k-balls $B(x_i, 2\epsilon) \cap H$ cover

$$F - \epsilon\delta = \{x \in F \mid B(x, \epsilon) \cap H \subseteq F\}.$$

Translating this in terms of k-dimensional volume, we get $\mathrm{vol}_k(F - \epsilon\delta) \leq l\, v_k(2\epsilon)^k$. Eliminating l and summing over all facets F of $\mathbb{P}$, we get

$$\sum_F \frac{\mathrm{vol}_k(F - \epsilon\delta)}{v_k 2^k} \epsilon^{-k} \leq n.$$

The claim follows for ϵ small enough. $\square$

We combine our upper bound on the number of points m in M and our lower bound on the number of sample points n and get the following lemma:

Lemma 7 *For ϵ small enough, there exists an ϵ-sample of the essential medial axis with no more than $C(\mathbb{P})\,n^{\frac{d-1}{p}}$ points.*

4.2 Proof of the Main Theorem

We now give the proof of our main theorem. For this, we map each Delaunay k-simplex σ to a point $z \in S \cup M$ and use z to locate the vertices of σ. To explain this, let b be one of the Delaunay ball that circumscribes σ. The vertices of σ are located on the boundary of b. By the fundamental property of Delaunay balls, b has the property to be ϵ-almost tangent. We apply Lemma 3:

1. If b is trivial, by (i) of Lemma 3, there exists a point $x \in \mathbb{P}$ such that $b \cap \mathbb{P} \subseteq B(x, \epsilon)$. Because S is an ϵ-sample of $\mathbb{P}$, there exists a point $z \in S$ with $\|x - z\| \le \epsilon$ and
$$\text{Vert}\,\sigma \;\subseteq\; B(z, 2\epsilon).$$

2. If b is ordinary, we apply (ii) of Lemma 3, combined with Lemma 4 and Lemma 5 to find that there exists a point $z \in M$ such that its associated medial ball ν satisfies
$$\text{Vert}\,\sigma \;\subseteq\; \text{ExtendedContact}(\nu, d\epsilon).$$

To summarize, any simplex in the Delaunay triangulation has its vertices contained either in a ball $B(z, 2\epsilon)$ with $z \in S$ or in the extended $(d\epsilon)$-contact of a medial ball whose center belongs to M. Because S is λ-sparse, $B(z, 2\epsilon)$ contains at most λ sample points. The number of k-simplices σ that we can form by picking $k + 1$ vertices among those λ points is $\binom{\lambda}{k+1}$. The extended $(d\epsilon)$-contact is the union of f balls of radius $(d\epsilon)$ and therefore contains at most λf points. The number of k-simplices σ that we can form by picking $k + 1$ vertices among those $f\lambda$ points is $\binom{f\lambda}{k+1}$. Therefore, the number of Delaunay k-simplices is therefore bounded by

$$\binom{\lambda}{k+1}\,n \;+\; \binom{f\lambda}{k+1}\,m,$$

which concludes the proof of our main theorem.

5 Conclusion

In this paper, we prove that the Delaunay triangulation of a sparse ϵ-sample of a fixed p-dimensional polyhedron in $\mathbb{R}^d$ has size $O(n^{(d-1)/p})$ as the number n of points in the sample goes to infinity (ϵ goes to zero). This gives a linear bound in the important case of a full dimensional polyhedron in $\mathbb{R}^d$. This result is, to our knowledge, the first result of this kind for dimension greater than three. We see several directions in which this result could be improved.

The sparse ϵ-sample definition we use here is more restrictive than the definition in previous papers for the three dimensional case, in that we require the sampling hypothesis to be satisfied by all the faces of all dimensions, rather than only for the full-dimensional polyhedron itself. If we sample only the polyhedron, a first observation is that a Delaunay sphere is no longer ϵ-almost tangent but only $\sqrt{\epsilon}$-almost tangent, which yields easily to a $O(n^{2(d-1)/p})$ bound on the number of Delaunay spheres. We conjecture, however, that the $O(n^{(d-1)/p})$ bound is still achievable in this setting.

Similar results in the random sampling model should be attainable, or when noise is added to the sample points.

Another question of interest would be to bound the size of the Delaunay triangulation of sets of points sampled on or near smooth manifolds of dimension $2 \le p \le d - 1$. Our results, and the prior work in dimension three, suggest that it may be possible to improve on the worst case bounds.

Acknowledgments

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