On universal set of points for planar graphs

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Abstract

We prove in this paper that any planar graph, G_n, with n vertices can be drawn on a universal set $\mathcal{S}$ of $\Theta(n^2/\log n)$ points with at most 1 bend by edge and with the vertices and the bend points in $\mathcal{S}$. If 2 bends per edge is allowed, we show that $\Theta(n \log n)$ points are sufficient, and if 3 bends per edge is allowed, $\Theta(n)$ points are sufficient.

1 Universal set of $\Theta(n^2/\log n)$ points for drawing planar graphs with one bend per edge

The construction is similar as in [1]. We recall briefly this construction, refering to Figures 1 and 2. Given a graph G with n vertices, we embed the graph on vertices $p_i = (i, -2^i)$ for $i = 0, \ldots, n-1$ with at most one bend per edge, as follows. We first compute a proper monotone topological book embedding, Γ of G. We relabel the vertices of that book embedding from right to left, as $v_0, \ldots, v_{n-1}$. We then map these vertices to $p_0, \ldots, p_{n-1}$, respectively. All the edges below the spine are drawn as line segments. The others are drawn with a bend point as follows. Consider an edge whose rightmost vertex is v_i and that intersects the spine on the interval $(v_u, v_{u+1}]$ (inclusive v_{u+1} for the case where the leftmost endpoint of the edge is v_{u+1}). Such an edge is drawn with a bend point at the same height as v_u, and in the vertical strip delimited by v_i and v_{i+1}. A universal set of points for the bend location can easily be determined in this construction. However, this construction requires a set of size $\Theta(n^3)$ for the bend points. Indeed, There can be n bend points on each of n bend lines, and in each of $n-1$ vertical strips (delimited by v_i and v_{i+1}).

We show how this construction can be modified to only have a subquadratic universal set of points for the bends, while preserving a linear size universal set of points for the vertices.

We consider as before a proper monotone topological book embedding, Γ of our input graph G. We then add on the spine extra isolated vertices so that there is at most one edge crossing the spine between two vertices. Since the number of edges of a planar graph is at most $3n-6$, we add at most that number of isolated vertices and the total number of vertices is less than $4n$. Let G' be the resulting graph. Note that, if we use the construction of [1] with this augmented graph G', there is at most one bend point on each bend line; this yields that, for every bend line, we can consider only one candidate location for the bend points in each vertical strip, leading to a quadratic universal set of points for the bends. We obtain a subquadratic size as follows

First we construct a set of $4n$ points $p_0, \ldots, p_{4n-1}$ as in [1] that will support the vertices of G'. Now on the bend line at height k, we place $k/\log k$ candidate vertices for the bend points of the edges that intersect the spine through the window $p_k p_{k+1}$.

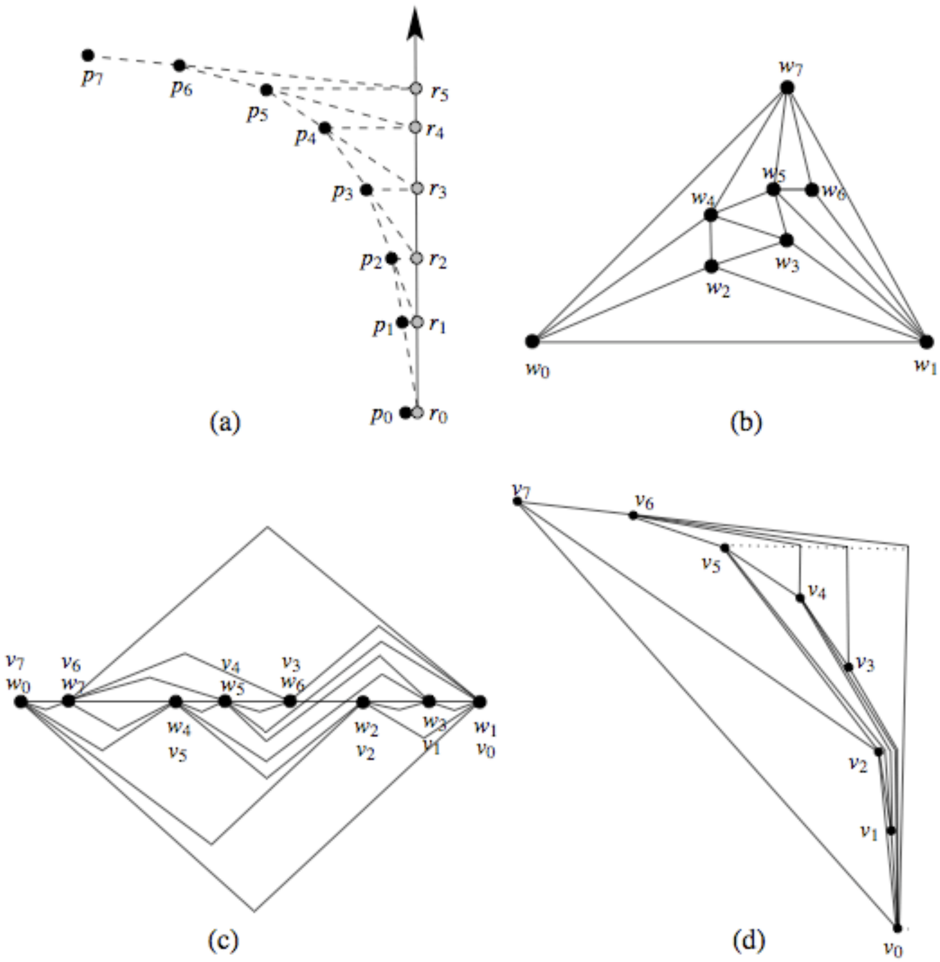

Figure 1: (a) A universal set, S, of 8 points, $p_0, \ldots, p_7$. (b) A graph G with 8 vertices. (c) A proper monotone topological book embedding, Γ, of G. The vertices are labeled both by their labels in graph G and drawing $\hat{\Gamma}$. (d) The drawing, $\hat{\Gamma}$, of G on the universal point set S.

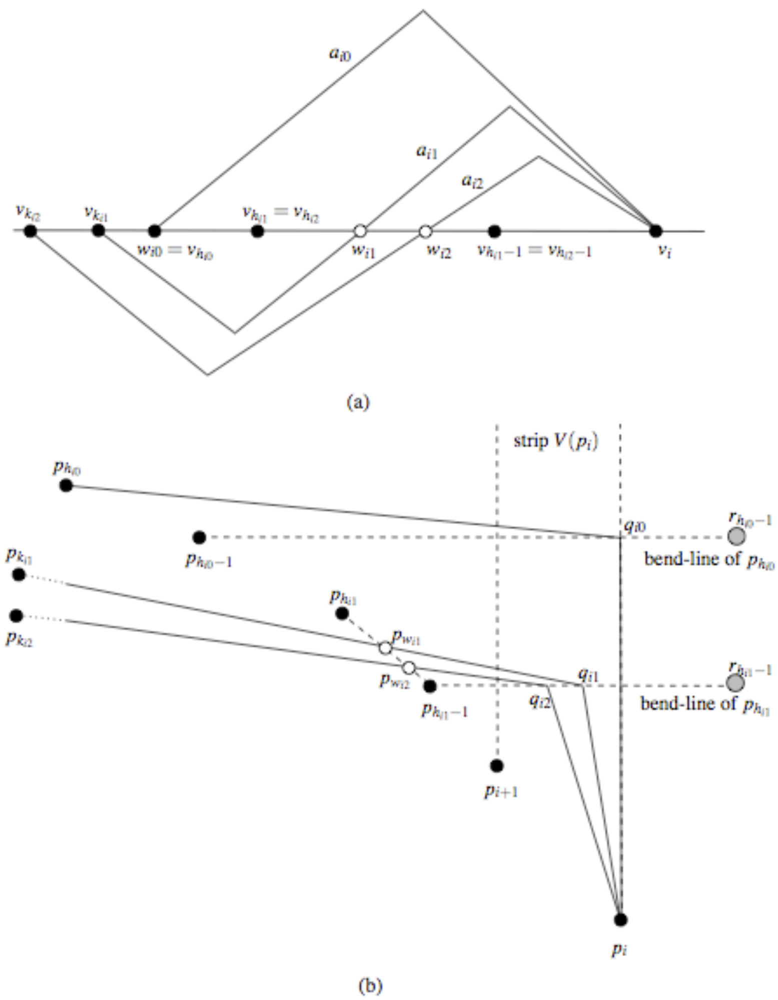

Figure 2: (a) Part of an proper monotone topological book embedding. (b) Sketch of the corresponding drawing Points $p_{k_{i1}}$ and $p_{k_{i2}}$ are much farther to the left than shown.

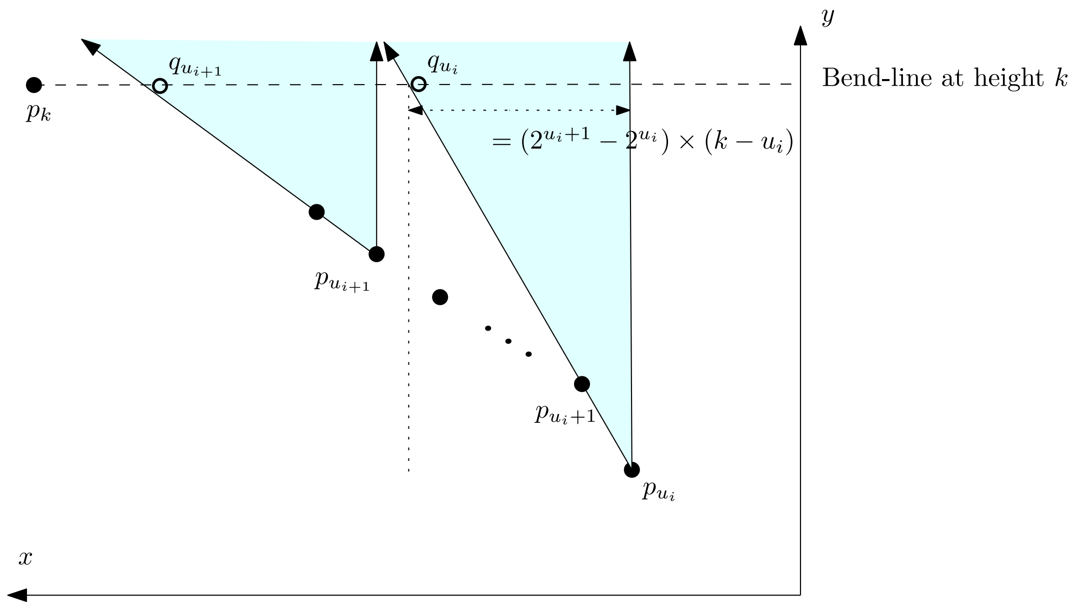

Figure 3: Universal point set for the bend points.

The points on the bend-line at height k are defined as follows. See Figure 3. The first (rightmost) point q_1 lies infinitesimaly to the right of the line p_0p_1. Let p_{u_1} be the rightmost point of $p_0, \dots, p_{4n-1}$ that is left of q_1. The second point q_2 is infinitesimaly to the right of the line $p_{u_1}p_{u_1+1}$ (at height k). Etc.

Because we added some dummy points on the spine, we ensured that there is at most one edge that intersect the spine between two vertices on the spine. Hence, for any given graph, there is at most one bend point used on any bend line. This essentially implies that the construction of [1] works here.

How many bend points is there on a bend-line? The points p_i have coordinates $(2^i, i)$. We first prove that the index u_i of the p_{u_i} is roughly $u_i = u_{i-1} + \log(n - u_{i-1})$. Here $u_0 = 0$ because p_{u_0} is initialized to p_0.

By definition $p_{u_{i+1}}$ is the rightmost point of $p_0, \dots, p_{4n-1}$ that is left of q_{u_i}. The x-distance between p_{u_i} and q_{u_i} is, as shown on the figure, $2^{u_i}(k - u_i)$ (up to some ϵ). Thus the x coordinate of q_{u_i} is $2^{u_i} + 2^{u_i}(k - u_i)$. By definition of $p_{u_{i+1}}$ we have $2^{u_{i+1}-1} \leqslant 2^{u_i} + 2^{u_i}(k - u_i) \leqslant 2^{u_{i+1}}$. Thus u_{i+1} is the ceiling of $u_i + \log(k - u_i + 1)$, or simply u_{i+1} is of order $u_i + \log(k - u_i)$.

The number of bend points at height k is thus the smallest i such that $u_i \geqslant k$. One way to look at it is that we start with k vertices and at each step we remove log of the remaining number of vertices. The number of bend points at height k is the number of iterations until there is no vertex left. Let $T(k)$ be this number of iterations on k vertices. As long as there is at least $k/2$ vertices left, we remove between $\log k$ and $\log k/2 = (\log k) - 1$ vertices. Hence, $T(k) = T(k/2) + k/\log k$. The Master theorem (cf. Cormen) yields that $T(k) = \Theta(k/\log k)$.

Hence there are $\Theta(k/\log k)$ candidate bend points on the bend line at height k, and $\Theta(n^2/\log n)$ candidate bend points in total.

2 Universal set of $\Theta(n)$ points for drawing planar graphs with n vertices and three bends per edge

Consider a proper monotone topological book embedding our input graph. Cut all the edges that intersect the spine at the intersection point. Draw the vertices on the spine on a horizontal line. Consider all the arcs on the top page as two half arcs. These arcs can be naturally ordered from left to right. Consider $\Theta(n)$ vertices on a convex curve above the spine, and draw all the half arcs to these point in order. Then connect these vertices by a line segments when they are on a same edge. Considering the spine on a very slightly concave curve does not change anything, except that now all the edges on the bottom page can be drawn as line segments. Hence the result.

3 Universal set of $\Theta(n \log n)$ points for drawing planar graphs with n vertices and two bends per edge

Same as above except that for every pairs of (disjoint) consecutive points on the spine we place a candidate bend at height 1 in between them. For every three disjoint consecutive points on the spine we place a candidate bend high enough. We end up with n/i candidate bends at height i, that will be used for connecting to vertices that are distance i apart. Roughly speaking that is...

References

[1] H. Everett, S. Lazard, G. Liotta, and S. Wismath. Universal sets of n points for one-bend drawings of planar graphs with n vertices. *Discrete and Computational Geometry*, 43(2):272–288, 2010.