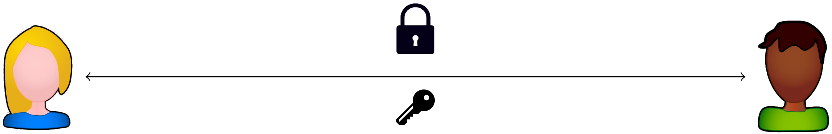
Algorithmique des courbes destinées au contexte de la cryptographie bilinéaire et post-quantique

Simon Masson

4 Décembre 2020

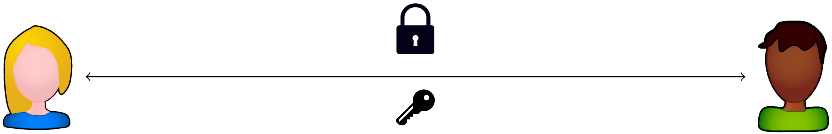
https://members.loria.fr/smasson/slides.pdf

Symmetric and asymmetric cryptography



Symmetric and asymmetric cryptography

Symmetric cryptography
(communication with 🔒 and 🔑)



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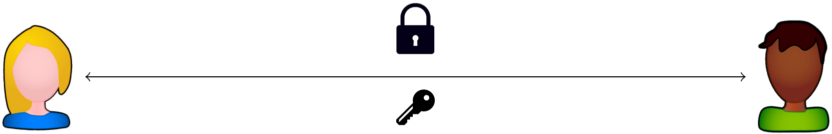

Asymmetric cryptography
(generation of the secret 🔑)

Key exchange with strawberry and mint syrups

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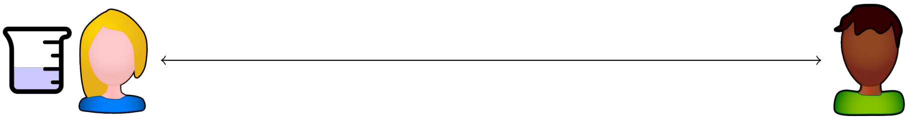

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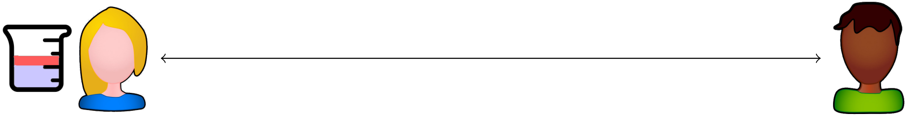

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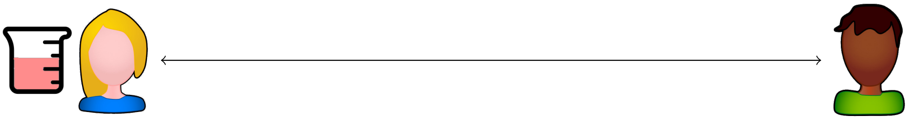

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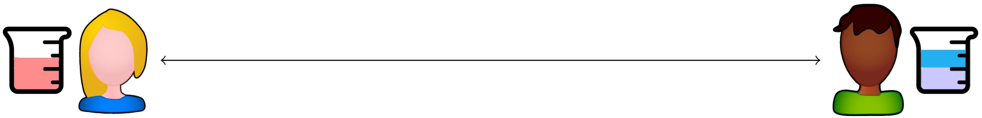

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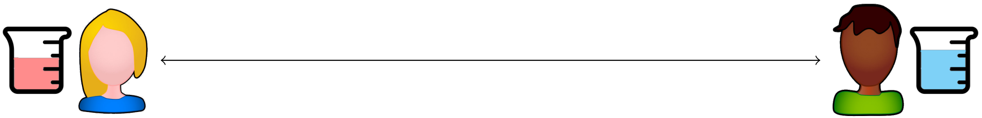

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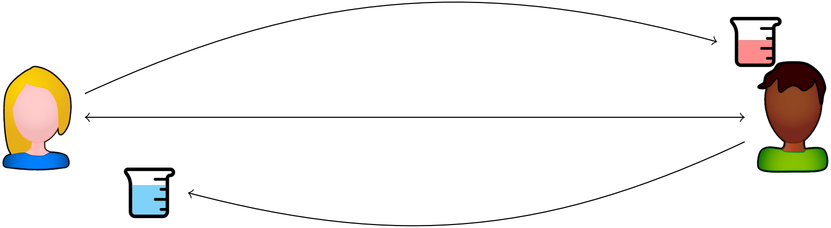

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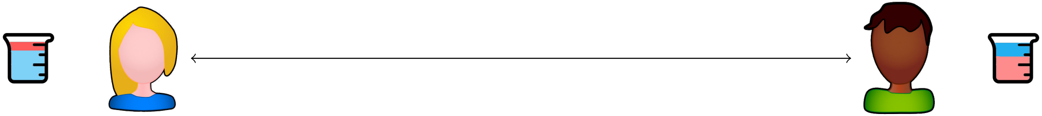


Key exchange with strawberry and mint syrups

Key exchange with strawberry and mint syrups

Common secret

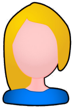
Key exchange with strawberry and mint syrups

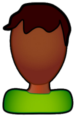
Common secret

Secure if we consider that splitting the syrups is hard.

Diffie–Hellman key exchange (1976)

Let $G = \langle g \rangle$ be a cyclic group,





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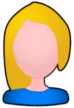

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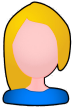

secret integer s_B

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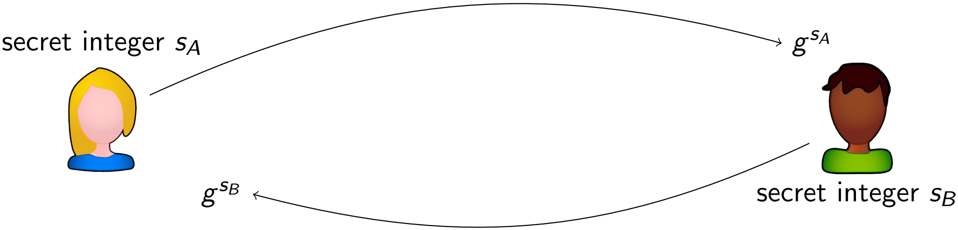

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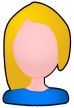

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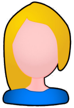

$$(g^{s_B})^{s_A} = g^{s_A s_B} = (g^{s_A})^{s_B}$$

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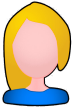

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Definition (DLP over G)

Given $h \in G = \langle g \rangle$, find s such that $h = g^s$.

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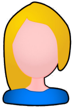

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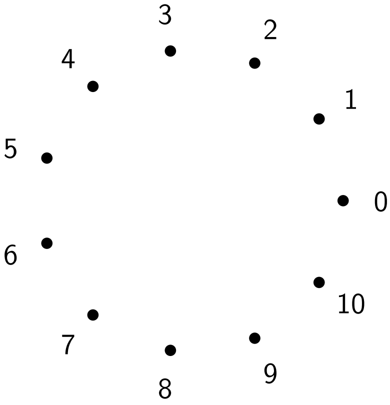
The different choices of group lead to different security levels.

Finite fields

Diffie–Hellman in $\mathbb{F}_p$

Let p be a prime, and $\mathbb{F}_p$ the finite field with p elements.



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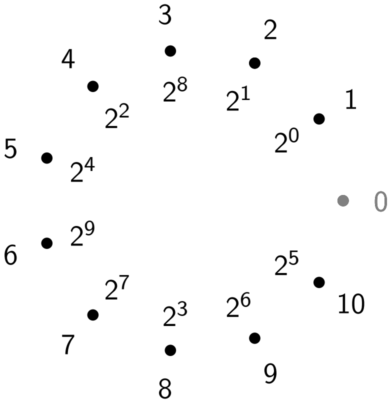

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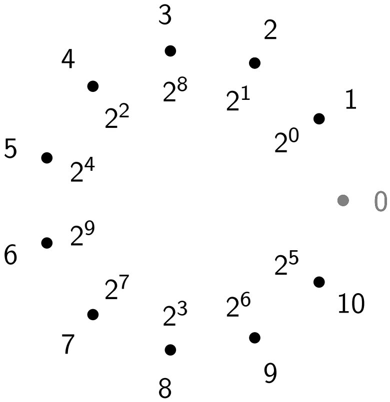

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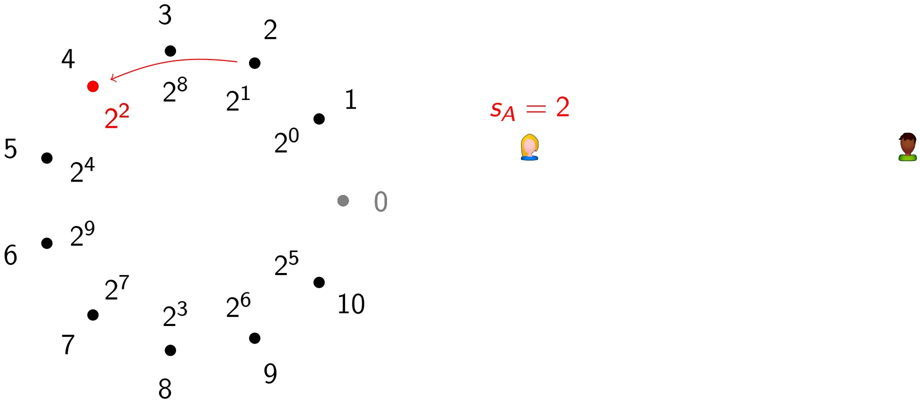

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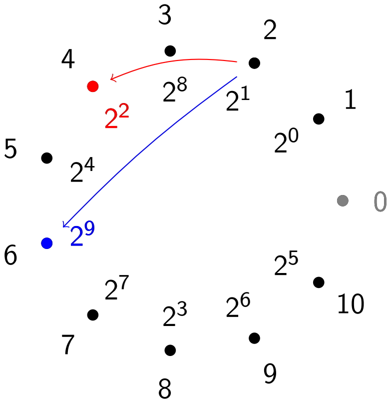

$s_A = 2$

$s_B = 9$

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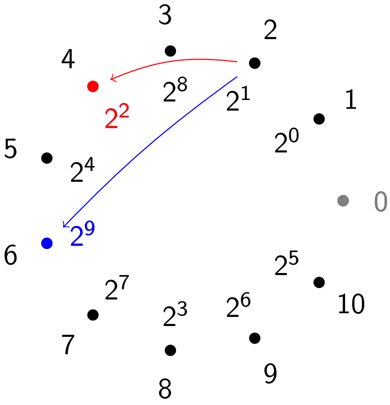

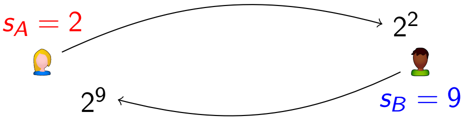

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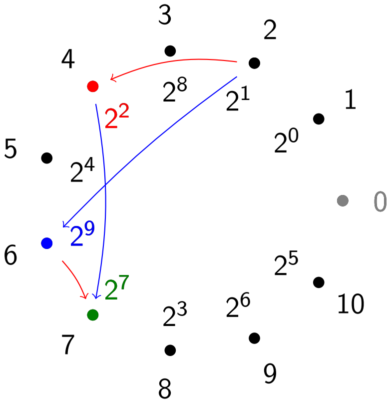

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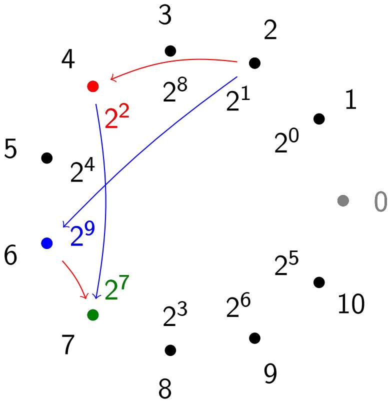

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$\mathrm{DL}_2(6)$?

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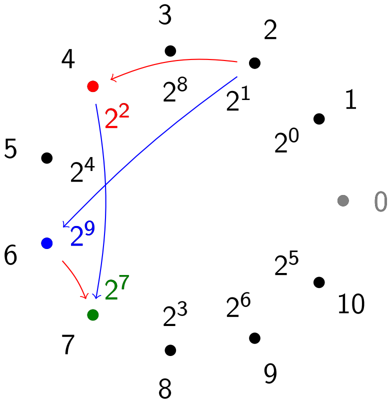

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$DL_2(6)$? $2^1 = 2 \bmod 11$

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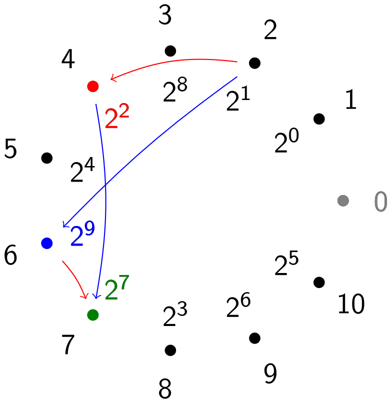

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$\mathrm{DL}_2(6)$? $2^2 = 4 \bmod 11$

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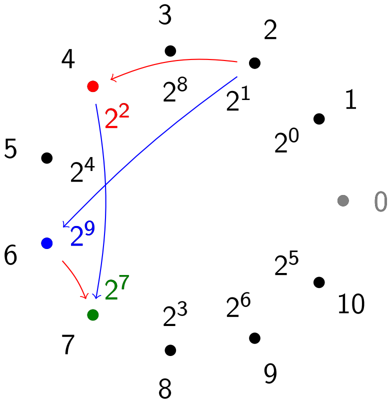

$s_A = 2$

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$\mathrm{DL}_2(6)$? $2^3 = 8 \bmod 11$

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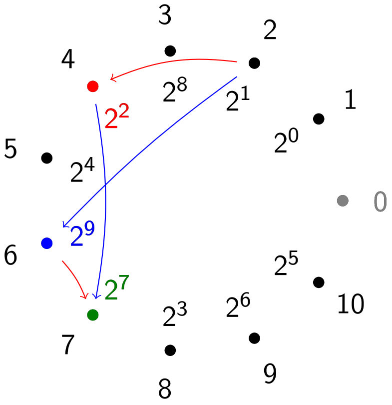

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$DL_2(6)$? $2^4 = 5 \bmod 11$

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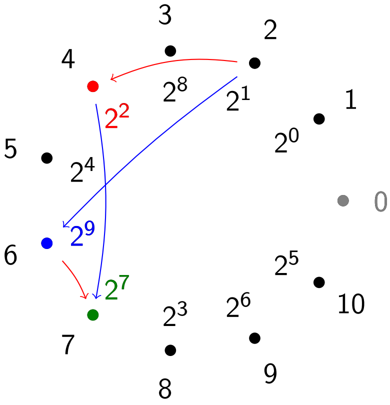

$s_A = 2$

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$\mathrm{DL}_2(6)$? $2^5 = 10 \bmod 11$

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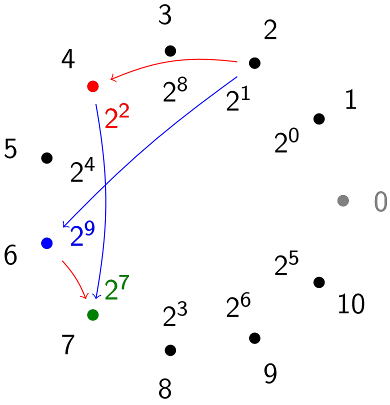

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$\mathrm{DL}_2(6)$? $2^6 = 9 \bmod 11$

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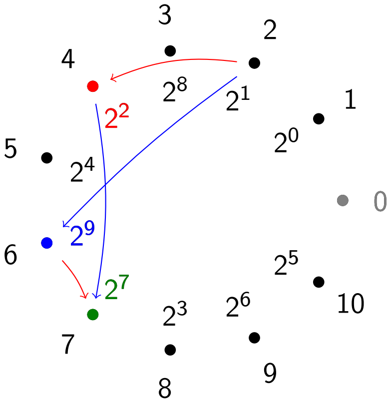

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$\mathrm{DL}_2(6)$? $2^7 = 7 \bmod 11$

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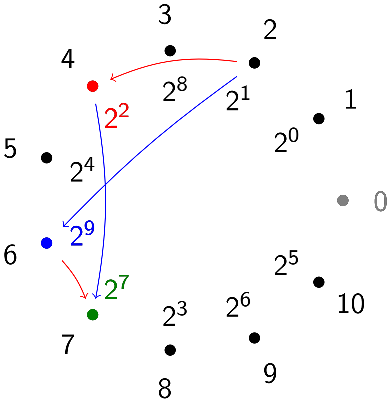

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$\mathrm{DL}_2(6)$? $2^8 = 3 \bmod 11$

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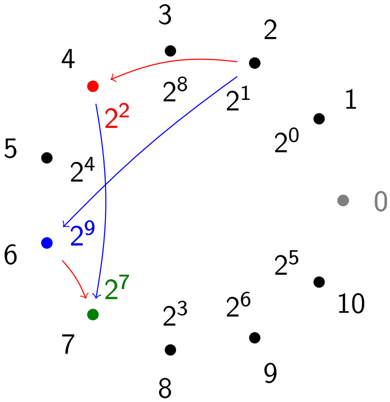

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$\mathrm{DL}_2(6)$? $2^9 = 6 \bmod 11$ ✓

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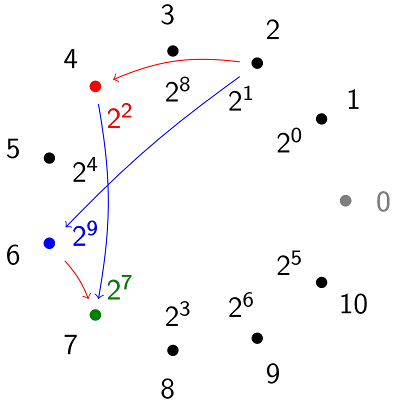

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$\mathrm{DL}_2(6)$? $2^9 = 6 \bmod 11$ ✓
Complexity $O(\#G)$ operations in G.
Exponential in the size of G.

Index calculus method

We set a factor basis S of *small* elements of G.

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3. *Target discrete logarithm.* Find a relation between h and the elements of S, and recover $\log h$ from solutions of Step 2.

The Number Field Sieve

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$$\mathbb{Z}[x]$$

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$$\begin{tikzcd} & \mathbb{Z}[x] \arrow[dl] \\ \mathbb{Z}[x]/(f_1(x)) & \end{tikzcd}$$

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The Number Field Sieve

$$\begin{tikzcd}
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NFS variant	f_1	f_2	Complexity
Original	$\deg(f_1) = 1$	$\deg(f_2) \approx (3 \log p / \log \log p)^{1/3}$	$L_{p^k}(1/3, \sqrt[3]{64/9} + o(1))$

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SNFS	small coeffs, chosen with the structure of p		$L_{p^k}(1/3, \sqrt[3]{32/9} + o(1))$
TNFS	defined over $\mathbb{Z}[x]/(h(x))$		$L_{p^k}(1/3, \sqrt[3]{48/9} + o(1))$

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Original	$\deg(f_1) = 1$	$\deg(f_2) \approx (3 \log p / \log\log p)^{1/3}$	$L_{p^k}(1/3, \sqrt[3]{64/9} + o(1))$
SNFS	small coeffs, chosen with the structure of p		$L_{p^k}(1/3, \sqrt[3]{32/9} + o(1))$
TNFS	defined over $\mathbb{Z}[x]/(h(x))$		$L_{p^k}(1/3, \sqrt[3]{48/9} + o(1))$
STNFS	conditions of SNFS on $\mathbb{Z}[x]/(h(x))$		$L_{p^k}(1/3, \sqrt[3]{32/9} + o(1))$

Finite fields for a 128-bit security level

Estimation of $\log_2(p)$ so that the best NFS variant has complexity $\approx 2^{128}$ operations.

Field	$\mathbb{F}_p$	$\mathbb{F}_{p^5}$	$\mathbb{F}_{p^6}$	$\mathbb{F}_{p^7}$	$\mathbb{F}_{p^8}$	$\mathbb{F}_{p(x_0)^{12}}$	$\mathbb{F}_{p(x_0)^{16}}$
Efficent variants	NFS	NFS	NFS TNFS	NFS	NFS TNFS	NFS TNFS SNFS STNFS	NFS TNFS SNFS STNFS
Field size	3072	3315	4032	3584	4352	5352	5424
$\log_2(p)$	3072	663	672	512	544	446	339

Benchmarks of multiplications in finite fields

Field prop.	**64-bit words for p**	$\mathbb{F}_p$ **mult. timing**
special p, $k = 12$	$\square\square\square\square\square\square\square\square$	
$k = 5$	$\square\square\square\square\square\square\square\square\square\square\square$	
$k = 6$	$\square\square\square\square\square\square\square\square\square\square\square$	
$k = 7$	$\square\square\square\square\square\square\square\square$	
$k = 8$	$\square\square\square\square\square\square\square\square\square$	
$k = 1$	$\square \times 48$	

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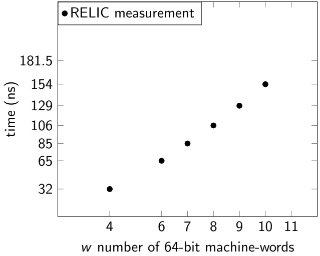
$k = 6$	□□□□□□□□□□□	
$k = 7$	□□□□□□□□	106ns
$k = 8$	□□□□□□□□□	129ns
$k = 1$	□ $\times$ 48	



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special p, $k=12$	□□□□□□□□	106ns
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$k=7$	□□□□□□□□	106ns
$k=8$	□□□□□□□□□	129ns
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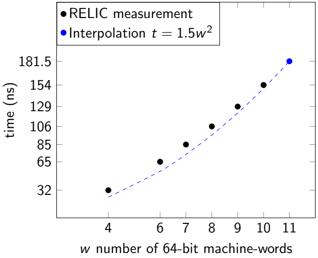

* Interpolation from the graph

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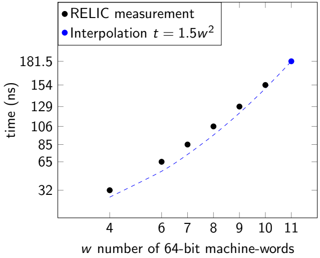
$k = 8$	□□□□□□□□□	129ns
$k = 1$	□ $\times$ 48	3800ns**



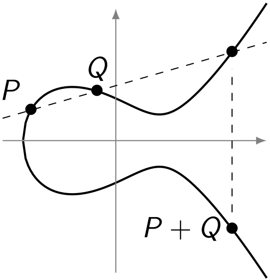
* Interpolation from the graph

**Benchmark with GMP.

Elliptic curves

Elliptic curves group law

$E_{a,b} : y^2 = x^3 + ax + b$



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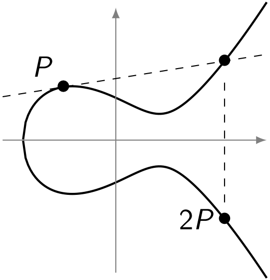

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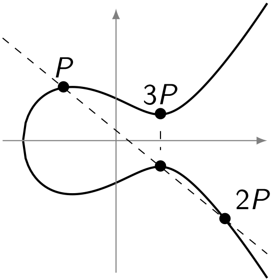

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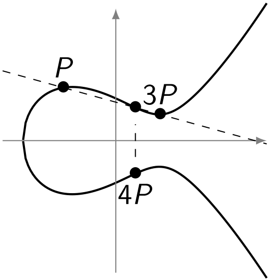

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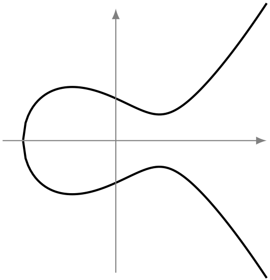

Points on an elliptic curve form a group G (with group law $+$).
Attacking the discrete log costs $O(\sqrt{\#G})$.

Torsion

Let $E_{a,b}$ be an elliptic curve defined over $\mathbb{F}_p$.

$$\begin{array}{rrcl} \pi: & E_{a,b} & \longrightarrow & E_{a,b} \\ & (x,y) & \longmapsto & (x^p, y^p) \end{array}$$

has characteristic polynomial $X^2 - tX + p$ and the order of the curve satisfies

$$\#E(\mathbb{F}_p) = p + 1 - t.$$

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For an integer ℓ, the ℓ-torsion is

$$\begin{aligned} E[\ell] &:= \left\{P \in E(\bar{\mathbb{F}}_p), \ell P = 0_E\right\} \\ &\simeq \mathbb{Z}/\ell\mathbb{Z} \times \mathbb{Z}/\ell\mathbb{Z} \text{ if } \gcd(\ell, p) = 1. \end{aligned}$$

Torsion

Let $E : y^2 = x^3 + 6$ defined over $\mathbb{F}_p$ with $p = 27631$.
$\#E(\mathbb{F}_p) = r$ prime, we denote $\mathbb{G}_1 = E[r](\mathbb{F}_p)$.

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$$\#E(\mathbb{F}_p) = 27481$$

$$\#E(\mathbb{F}_{p^{12}}) = 2^6 3^6 5^2 7^4 13^2 \cdot 73 \cdot 97 \cdot 109 \cdot 127 \cdot 283 \cdot 853 \cdot 2053 \cdot 2137 \cdot 6991 \cdot 27481^2 \cdot 7634397$$

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- $\mathbb{G}_2$, often defined over $\mathbb{F}_{p^k}$ where k is the embedding degree.

$$\mathbb{F}_{p^2} = \mathbb{F}_p(i) = \mathbb{F}_p[x]/(x^2+1) \qquad \mathbb{F}_{p^{12}} = \mathbb{F}_{p^2}(u) = \mathbb{F}_{p^2}[y]/(y^6 - (1121i + 404))$$

$$x_P = (20678i + 23625)u^5 + (1861i + 10882)u^4 + (16355i + 5810)u^3 + (20962i + 7790)u^2 + (13621i + 26347)u + 19587i + 23498,$$
$$y_P = (11673i + 12944)u^5 + (5902i + 22858)u^4 + (11246i + 24609)u^3 + (802i + 13087)u^2 + (3722i + 15960)u + 8881i + 13552.$$

Twists of curves

- If E has equation $y^2 = x^3 + ax$ for $a \in \mathbb{F}_p$, E has four quartic twists.
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Using the sextic twist, we get a point P of order r with *sparse* coordinates:

$$x_P = (0i + 0)u^5 + (17983i + 9957)u^4 + (0i + 0)u^3 + (0i + 0)u^2 + (0i + 0)u + 0i + 0,$$
$$y_P = (0i + 0)u^5 + (0i + 0)u^4 + (12752i + 19494)u^3 + (0i + 0)u^2 + (0i + 0)u + 0i + 0.$$

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Given an order $\mathcal{O}$ of discriminant $-D$, $H_D = \texttt{HilbertClassPolynomial}(D)$.
Roots of H_D are invariants leading to curves of endomorphism ring $\mathcal{O}$.

Pairings

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Secure pairing-friendly elliptic curve with an efficient pairing

Tate and ate pairings

The Tate and ate pairings are computed in two steps:

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Definition

For $P \in \mathbb{G}_1 = E(\mathbb{F}_p)[r]$, $Q \in \mathbb{G}_2 = E(\mathbb{F}_{p^k})[r]$,

$$\mathrm{Tate}(P, Q) := f_{r,P}(Q)^{(p^k-1)/r} \qquad \mathrm{ate}(P, Q) := f_{t-1,Q}(P)^{(p^k-1)/r}$$

Miller function

Definition

The Miller loop computes the function $f_{s,Q}$ such that Q is a zero of order s, and $[s]Q$ is a pole of order 1, i.e $\mathrm{div}(f_{s,Q}) = s(Q) - ([s]Q) - (s-1)\mathcal{O}$.

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Example of $f_{5,Q}(P)$.

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Example of $f_{5,Q}(P)$. $s = 5 = \overline{1\boxed{0}1}^{2}$

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Example of $f_{5,Q}(P)$. $s = 5 = \overline{10\boxed{1}}^2$

$$f = \left(1^2 \cdot \ell_{Q,Q}(P)/v_{2Q}(P)\right)^2$$

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Example of $f_{5,Q}(P)$. $s = 5 = \overline{10\boxed{1}}^{2}$

$$f = \left(1^2 \cdot \ell_{Q,Q}(P)/v_{2Q}(P)\right)^2 \cdot \ell_{2Q,2Q}(P)/v_{4Q}(P) \cdot \ell_{4Q,Q}(P)/v_{5Q}(P)$$

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Example of $f_{5,Q}(P)$. $s = 5 = \overline{101}^2$

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Example of $f_{5,Q}(P)$. $s = 5 = \overline{101}^2$

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Example of $f_{5,Q}(P)$. $s = 5 = \overline{101}^2$

$$f = \left(1^2 \cdot \ell_{Q,Q}(P)/v_{2Q}(P)\right)^2 \cdot \ell_{2Q,2Q}(P)/v_{4Q}(P) \cdot \color{red}{\ell_{4Q,Q}(P)}/v_{5Q}(P)$$

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$$\mathrm{div}(f) = 5(Q) - (5Q) - 4(\mathcal{O})$$

Final exponentiation

$f_{r,P}(Q)$ and $f_{t-1,Q}(P)$ are cosets modulo r-th powers.
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  Polynomial in p with very small coefficients.
  Very efficient with Frobenius: if $\mathbb{F}_{p^k} = \mathbb{F}_p[x]/(x^k - \alpha)$,
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  More expensive. Possible optimizations.

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3. Choose a curve with p non-special.
   We investigated this solution for $5 \leq k \leq 8$.
   A. Guillevic, S. Masson, and E. Thomé, 2020.
   $\Longrightarrow$ Competitive pairing?

Barreto-Lynn-Scott family

Curves of embedding degree $k = 12$: TNFS applies!
Discriminant $-D = -3$, twists of degree 6: $\mathbb{G}_2 \simeq {}^{t6}E(\mathbb{F}_{p^2})$.

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	Estimation of the pairing cost.	**Measurements.**	
Miller loop:	$7805\text{m} \approx 0.7\text{ms}$	0.7ms	
Final expo.:	$7723\text{m} \approx 0.7\text{ms}$	0.7ms	
Total:	$15528\text{m} \approx 1.3\text{ms}$	1.4ms	(error$<$ 10%)

The Cocks–Pinch construction

Given an integer k, and a discriminant $-D$.

Algorithm: $\textsc{Cocks-Pinch}(k, -D)$ – Compute a pairing-friendly curve $E/\mathbb{F}_p$ of trace t with a subgroup of order r, such that $t^2 - 4p = -Dy^2$.

Set a prime r such that $k \mid r - 1$ and $\sqrt{-D} \in \mathbb{F}_r$
Set T such that $r \mid \Phi_k(T)$
$t \leftarrow T + 1$
$y \leftarrow (t - 2)/\sqrt{-D}$
Lift $t, y \in \mathbb{Z}$ such that $t^2 + Dy^2 \equiv 0 \mod 4$
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if p is prime **then return** $[p, t, y, r]$ **else** Repeat with another r.

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Large trace $t \implies$ the ate pairing is not very efficient 

Our Cocks–Pinch variant

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$p \leftarrow (t^2 + Dy^2)/4$
if p is prime and $p = 1 \bmod k$ **then return** $[p, t, y, r]$ **else** Repeat with another r.

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$\mathbb{F}_{p^k} = \mathbb{F}_p[u]/(u^k - \alpha)$

Properties of our modified Cocks–Pinch curves

Example of generation for $k = 8$.

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Lift t and y with 16-bit h_t and h_y, and restrict on $\log_2(p) = 544$
Check subgroup-security and twist-subgroup-security.

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```
CocksPinchVariantResult(k=8,D=4,T=0xffffffffeff7c200,i=5,ht=5,
hy=-0xd700,allowed_cofactor=420,allowed_size_cofactor=10,max_B1=600)
```

k	$-D$	NFS	TNFS	$\log(p^k)$	$\log(p)$	Twist	$\mathbb{G}_2$ size
5	34-bit	yes	slower	3315	663	no	3315
6	-3	yes	faster	4032	672	yes	672
7	-20	yes	slower	3584	512	no	3584
8	-4	yes	faster	4352	544	yes	1088

Pairing time Comparison

At the 128-bit security level

Curve	**Miller loop** time estimation	**Exponentiation** time estimation	time estimation	RELIC **Measurement**
$k = 5$				
$k = 6$				
$k = 7$				
$k = 8$				
BN				
BLS12				
KSS16				
$k = 1$				

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$k = 5$	2.6ms	1.8ms	4.4ms	
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Isogeny of elliptic curve

Definition

An isogeny is a morphism $\varphi : E \to E'$ between elliptic curves such that $\varphi(0_E) = 0_{E'}$.

- We focus here on cyclic separable isogenies.
- A generator of $\ker(\varphi)$ defines the isogeny.
- $\deg(\varphi) \approx \# \ker(\varphi)$. Efficient for small degrees.
- An isogeny $\varphi : E \to E'$ has a dual $\hat{\varphi} : E' \to E$ s.t. $\varphi \circ \hat{\varphi} = \hat{\varphi} \circ \varphi = [\deg \varphi]$.

The SIDH key exchange

E supersingular curve defined over $\mathbb{F}_{p^2}$.
$E[2^n] = \langle P_2, Q_2 \rangle$, $E[3^m] = \langle P_3, Q_3 \rangle$.

$$E$$

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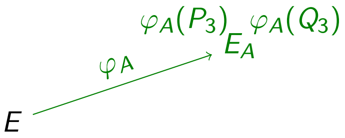
She also computes the image of P_3 and Q_3 by her isogeny.



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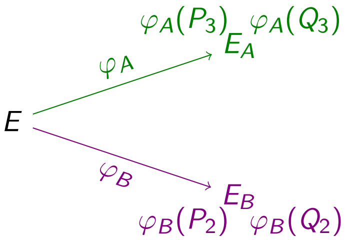

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Bob computes the isogeny of kernel $\varphi_A(P_3) + s_B \varphi_A(Q_3)$.
They arrive at the same curve E_{AB}.

$$\begin{tikzcd}
 & \overset{\varphi_A(P_3)\quad\varphi_A(Q_3)}{E_A} \arrow[dr, "\varphi_B"] & \\
E \arrow[ur, "\varphi_A"] \arrow[dr, "\varphi_B"'] & & E_{AB} \\
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Security assumption.

- It is hard to compute φ_A given E, E_A, $\varphi_A(P_3)$ and $\varphi_A(Q_3)$

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  .
- (stronger) Given E, E', it is hard to find an isogeny $E \to E'$ of given degree

  .

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- It is hard to compute φ_A given E, E_A, $\varphi_A(P_3)$ and $\varphi_A(Q_3)$
- (stronger) Given E, E', it is hard to find an isogeny $E \to E'$ of given degree
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The SIDH key exchange

E supersingular curve defined over $\mathbb{F}_{p^2}$.
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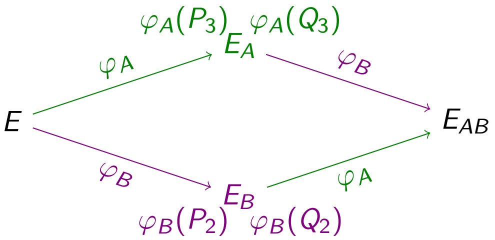

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  Here, we consider for primes p and q the quaternion algebra
  $H_{-q,-p} = \mathbb{Q}1 + \mathbb{Q}\mathrm{i} + \mathbb{Q}\mathrm{j} + \mathbb{Q}\mathrm{ij}$ where $\mathrm{i}^2 = -q$ and $\mathrm{j}^2 = -p$.

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- Maximal order: no order contain this order.
  Maximal orders are not unique!

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Let $p = 3 \bmod 4$.
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Endomorphism ring is easy to compute because it is a particular curve: reduction of a $\mathbb{Q}$-curve of discriminant $-D = -4$.

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Here, we focus on computing endomorphism rings given the knowledge of an isogeny (in a practical point of view).

Endomorphism ring through isogenies

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y^2 = x^3 + x : \; E_0 \arrow[rr, bend left, "\ker\varphi = \langle P \rangle,\, \deg\varphi = 2"] & \varphi : (x,y) \mapsto \left(\frac{x^2-1}{x}, y\frac{x^2+1}{x^2}\right) & E
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https://gitlab.inria.fr/smasson/endomorphismsthroughisogenies.

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2. the output can be verified efficiently.

- $\text{Setup}(\lambda, T) \longrightarrow$ public parameters pp
- $\text{Eval}(pp, x) \longrightarrow$ output y such that $y = f(x)$, and a proof π (requires T steps)
- $\text{Verify}(pp, x, y, \pi) \longrightarrow$ yes or no.

VDF based on RSA

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- **Wesolowski verification.** [Eurocrypt '19]
  π is short
  Verification is fast.
- **Pietrzak verification.** [ITCS '19]
  π computation is more efficient
  Verification is slower.

Different security assumptions.

Generalization of the RSA VDF

If one knows the factorization of N, the evaluation can be computed using

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VDF	**pro**	**con**
RSA	fast verification	trusted setup not post-quantum
Class group	small parameters	slow verification not post-quantum

VDF constructions with isogenies and pairings

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Not post-quantum because of pairings.

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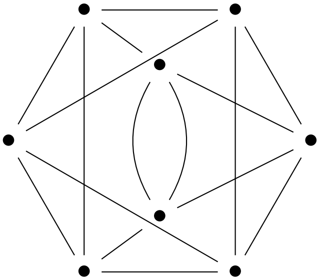

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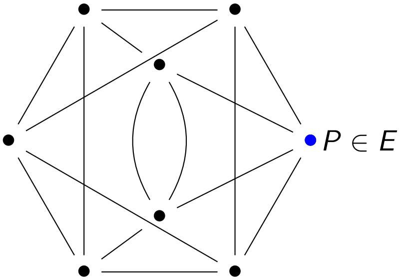

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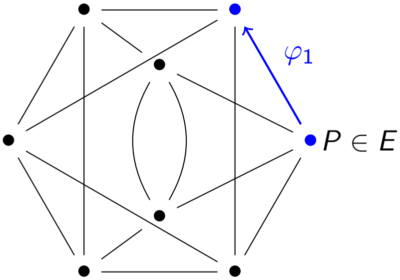

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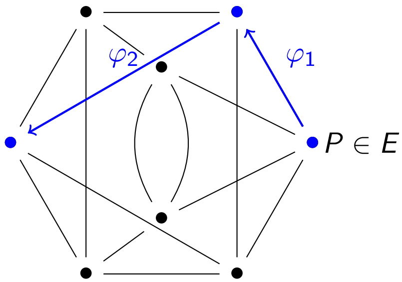

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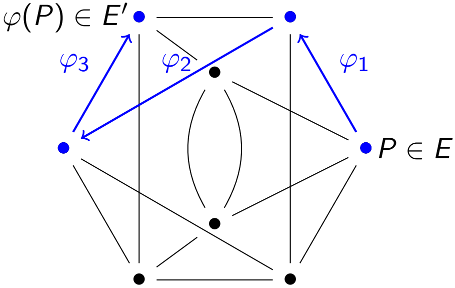

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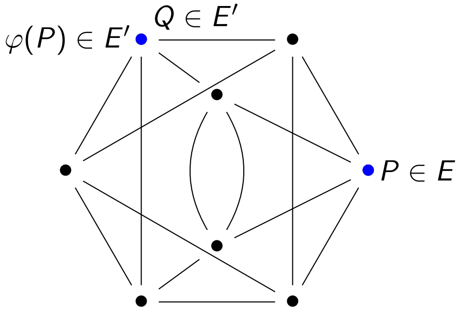

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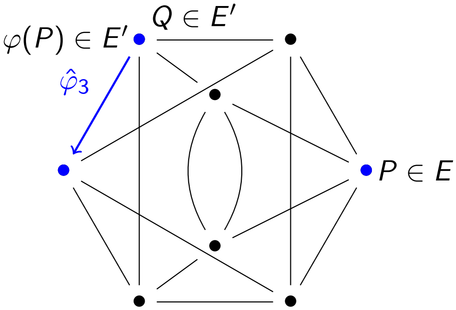

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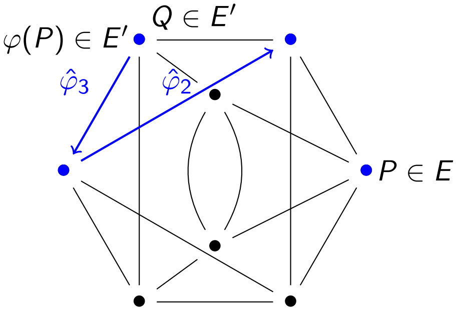

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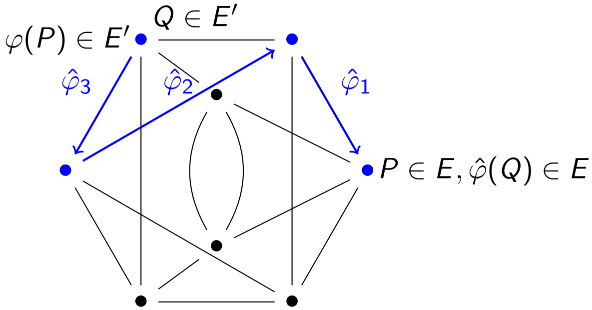

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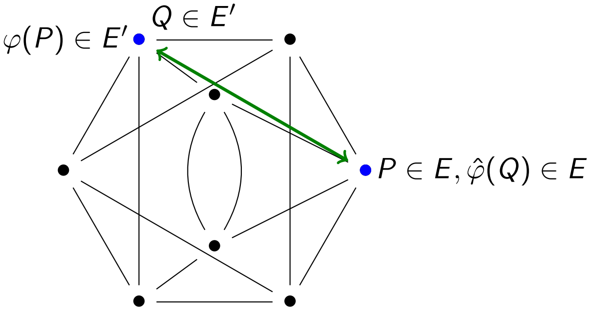

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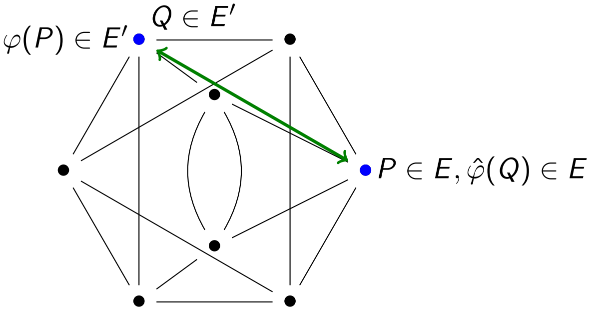

Another version with isogenies defined over $\mathbb{F}_p$ in the paper.

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  Writing $\mathbb{G}_2 = \langle G \rangle$, find x such that $e(P, G)^x = e(\varphi(P), Q)$.
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- Find a shortcut.
  Find a way to compute the isogeny in less than T steps.

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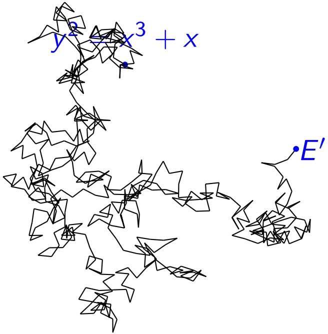

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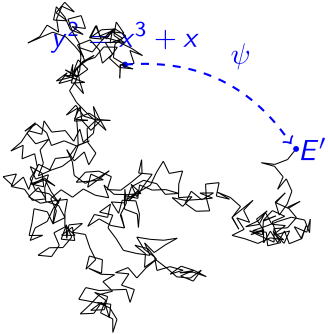
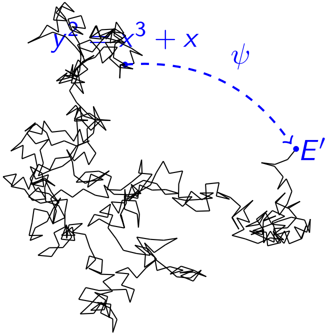


Isogeny shortcut

If E has a *known* endomorphism ring, a shortcut can be found.

- Convert the isogeny into an ideal of $\mathrm{End}(E)$;
- Find an equivalent ideal J of different (smaller) norm;
- Convert J into another isogeny ψ of smaller degree.



Isogeny shortcut

If E has a *known* endomorphism ring, a shortcut can be found.

- Convert the isogeny into an ideal of $\mathrm{End}(E)$;
- Find an equivalent ideal J of different (smaller) norm;
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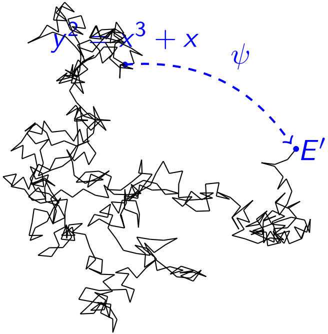

Conclusion: **do not use a curve with a known endomorphism ring!**

The need of trusted setup

Do we know curves with an unknown endomorphism ring?

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Pairing-friendly ordinary curves
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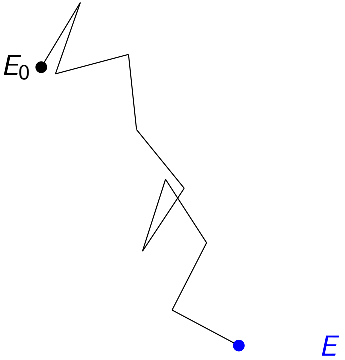

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E has an unknown endomorphism ring.

E

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	Step	**e_k size**	**Time**	**Throughput**
$\mathbb{F}_p$ graph	Setup	238 kb	90s	0.75isog/ms
	Evaluation	–	89s	0.75isog/ms
	Verification	–	0.3s	–
$\mathbb{F}_{p^2}$ VDF	Setup	491 kb	193s	0.35isog/ms
	Evaluation	–	297s	0.23isog/ms
	Verification	–	4s	–

Table: Benchmarks for our VDFs, on a Intel Core i7-8700 @ 3.20GHz, $T \approx 2^{16}$

Comparison of the VDFs

VDF	**pro**	**con**
RSA	fast verification	trusted setup
Class group	no trusted setup small parameters	slow verification
Isogenies over $\mathbb{F}_p$	Fast verification	trusted setup long setup
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- Find a fully post-quantum VDF.

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