Verifiable Delay Functions from Supersingular Isogenies and Pairings

Luca De Feo[1] **Simon Masson[2]** Christophe Petit[3] Antonio Sanso[4]

[1] IBM Research Zürich

[2] Thales and Université de Lorraine, Nancy

[3] University of Birmingham

[4] Adobe Inc. and Ruhr Universität Bochum

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Definition and examples

Definition

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- Setup$(\lambda, T) \longrightarrow$ public parameters pp
- Eval$(pp, x) \longrightarrow$ output y such that $y = f(x)$, and a proof π (requires T steps)
- Verify$(pp, x, y, \pi) \longrightarrow$ yes or no.

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- **Wesolowski verification.** [Eurocrypt '19]
  π is short
  Verification is fast.
- **Pietrzak verification.** [ITCS '19]
  π computation is more efficent
  Verification is slower.

Different security assumptions.

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VDF based on class group. Let $K = \mathbb{Q}(\sqrt{-D})$ and O_K its ring of integers.

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This group is finite and it is hard to compute $\#\text{ClassGroup}(D)$.

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VDF	**pro**	**con**
RSA	fast verification	trusted setup not post-quantum
Class group	small parameters	slow verification not post-quantum

VDF based on isogenies and pairings

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- What is an isogeny ?
- What is a pairing ?

Pairing-friendly elliptic curves.

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For an elliptic curve E, we can choose $\mathbb{G}_1$ and $\mathbb{G}_2$ two groups of points of E, and $\mathbb{G}_3$ a multiplicative subgroup of a finite field.
A curve is *pairing-friendly* if the $\mathbb{G}_i$ are efficiently computable.

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An isogeny between two elliptic curves E and E' is an algebraic map

$$\varphi : E \longrightarrow E'$$

such that $\varphi(0_E) = 0_{E'}$.

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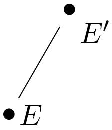
$$e(\varphi(P), Q) = e(P, \hat{\varphi}(Q))$$

Two types of elliptic curves:



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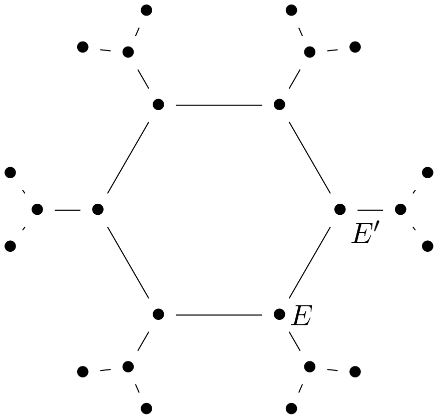
Ordinary curves $End(E)$ is an order in $\mathbb{Q}(\sqrt{-D})$.



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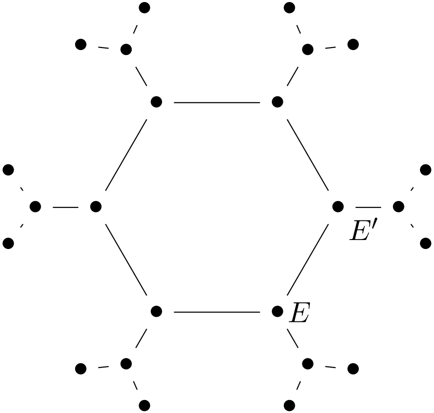

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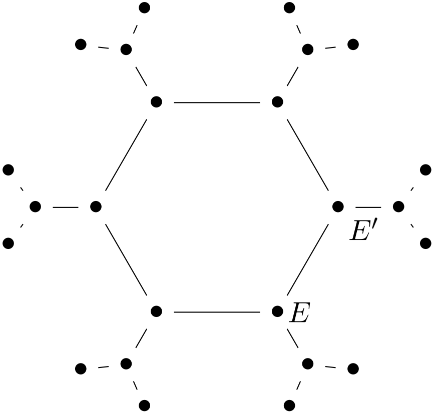

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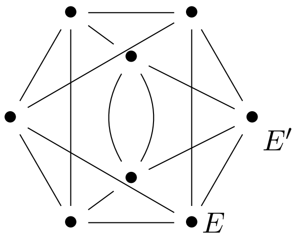

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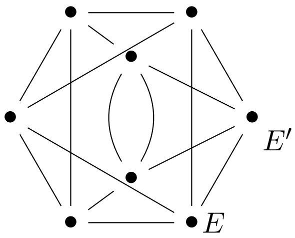
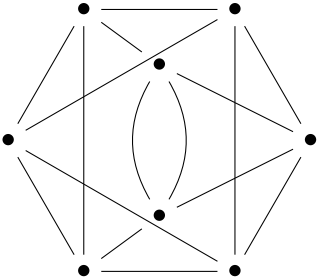
Supersingular curves can be defined over $\mathbb{F}_{p^2}$.



VDF over $\mathbb{F}_{p^2}$ supersingular curves.



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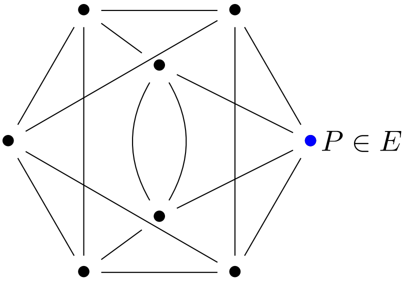
Setup A **public** walk in the isogeny graph.



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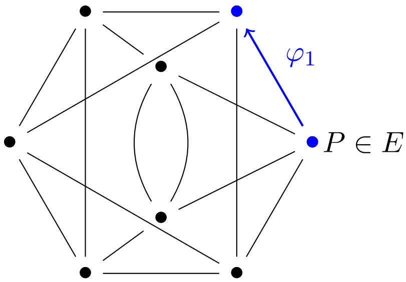

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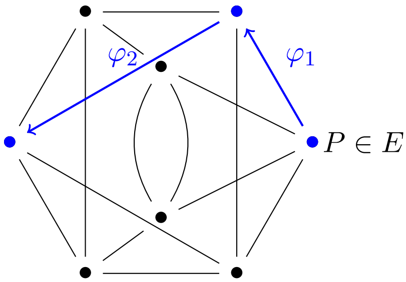

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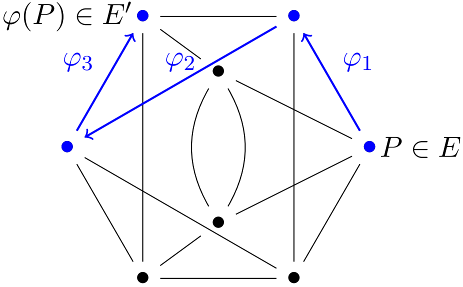

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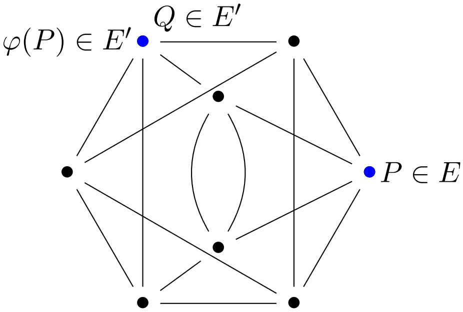

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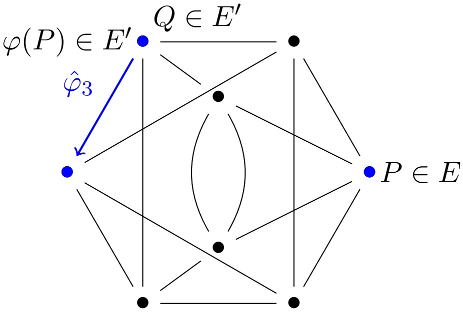

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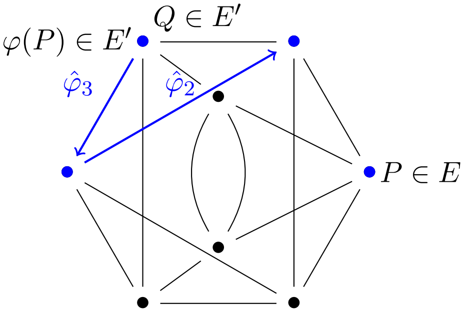

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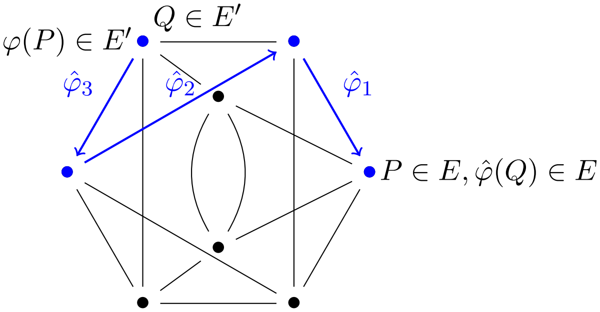

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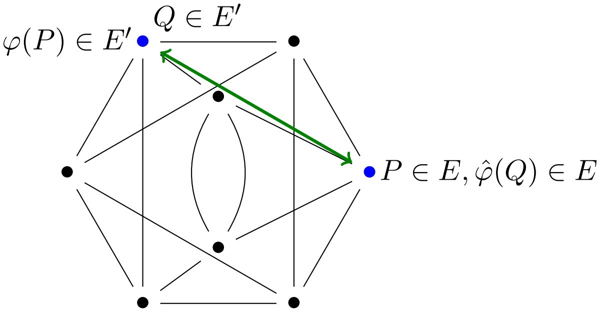

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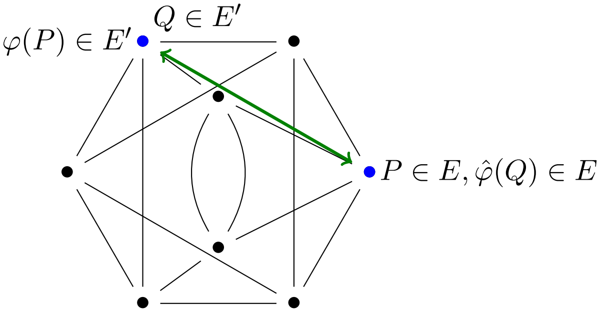

Another version with isogenies defined over $\mathbb{F}_p$ in the paper.

Security considerations

Definition and examples

VDF based on isogenies and pairings

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Implementation and comparison

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One cannot evaluate in less than T steps.

- Attacking the DLP in $\mathbb{F}_{p^2}$.
  Writing $\mathbb{G}_2 = \langle G \rangle$, find x such that $e(P, G)^x = e(\varphi(P), Q)$.
  Solution: choose a large prime p (1500 bits) such that DLP is hard in $\mathbb{F}_{p^2}$.

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- Find a shortcut.
  Find a way to compute the isogeny in less than T steps.

Isogeny shortcut.

$$y^2 = x^3 + x$$

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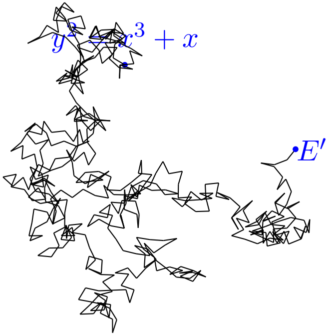

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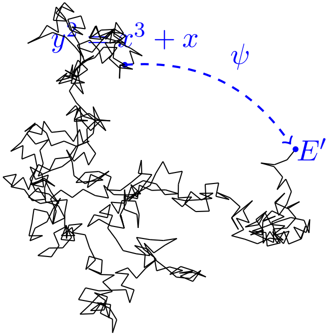
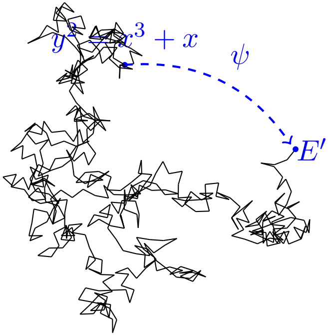


Isogeny shortcut.
If E has a *known* endomorphism ring, a shortcut can be found.

- Convert the isogeny into an ideal of $\mathrm{End}(E)$;
- Find an equivalent ideal J of different (smaller) norm;
- Convert J into another isogeny ψ of smaller degree.



Example. $p = 3 \bmod 4$.

$$\begin{tikzcd} y^2 = x^3 + x : \ E_0 \arrow[rr, bend left, "{\ker\varphi = \langle P \rangle, \deg\varphi = 2}"] & \varphi : (x,y) \mapsto \left(\frac{x^2-1}{x}, y\frac{x^2+1}{x^2}\right) & E \end{tikzcd}$$

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$$n_1\mathbf{1}(P) + n_2\mathbf{i}(P) + n_3\left(\frac{\mathbf{1}+\mathbf{j}}{2}\right)(P) + n_4\left(\frac{\mathbf{i}+\mathbf{k}}{2}\right)(P) = 0_{E_0}.$$

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$\alpha = u_1 + u_3 + u_4 = \frac{3+\mathbf{i}+\mathbf{j}+\mathbf{k}}{2}$ and

$$\mathcal{I} = \mathcal{O}_0 \cdot 2 + \mathcal{O}_0 \cdot \alpha = \mathbb{Z}\left\langle \frac{\mathbf{1}+\mathbf{i}+\mathbf{j}+3\mathbf{k}}{2}, \mathbf{i}+\mathbf{k}, \mathbf{j}+\mathbf{k}, 2\mathbf{k} \right\rangle.$$

$$\begin{tikzcd}
E_0 \arrow[rr, bend left, "\varphi"'] & & E \\
\mathcal{O}_0 \arrow[rr, leftrightarrow, "\mathcal{I}"'] & & \mathcal{O}
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Algorithms based on quaternion computations provide an equivalent ideal $J = I\beta$ of norm coprime to 2.

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Algorithms based on quaternion computations provide an equivalent ideal $J = I\beta$ of norm coprime to 2.
It corresponds to an isogeny of degree $N(J)$.

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Implementation in Magma
https://gitlab.inria.fr/smasson/endomorphismsthroughisogenies.

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Conclusion: **do not use a curve with a known endomorphism ring!**

Do we known curves with an unknown endomorphism ring?

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Pairing-friendly
ordinary curves
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Do we known curves with an unknown endomorphism ring?

Pairing-friendly ordinary curves	Special supersingular curves
no	**no**

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Trusted setup (supersingular case).

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$E_0 \bullet$

Trusted setup (supersingular case)**.**

- Start from a well known supersingular curve,

Do we known curves with an unknown endomorphism ring?

Pairing-friendly ordinary curves	Special supersingular curves
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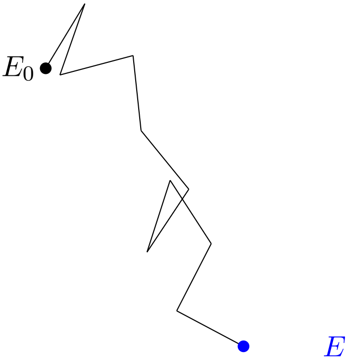
Trusted setup (supersingular case)**.**

- Start from a well known supersingular curve,
- Do a random walk,



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Trusted setup (supersingular case).

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E has an unknown endomorphism ring.

Implementation and comparison

Definition and examples

VDF based on isogenies and pairings

Security considerations

Implementation and comparison

Implementation of the VDF.

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	Step	e_k size	Time	Throughput
$\mathbb{F}_p$ graph	Setup	238 kb	–	0.75isog/ms
	Evaluation	–	–	0.75isog/ms
	Verification	–	0.3 s	–
$\mathbb{F}_{p^2}$ VDF	Setup	491 kb	–	0.35isog/ms
	Evaluation	–	–	0.23isog/ms
	Verification	–	4 s	–

Table: Benchmarks for our VDFs, on a Intel Core i7-8700 @ 3.20GHz, $T \approx 2^{16}$

Comparison.

VDF	pro	con
RSA	fast verification	trusted setup
Class group	no trusted setup small parameters	slow verification
Isogenies over $\mathbb{F}_p$	Fast verification	trusted setup long setup
Isogenies over $\mathbb{F}_{p^2}$	Quantum-annoying Fast verification	trusted setup long setup

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Thank you for your attention.