Cocks–Pinch curves with efficient ate pairing

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Joint work with A. Guillevic, E. Thomé

Journées C2

October 9, 2018

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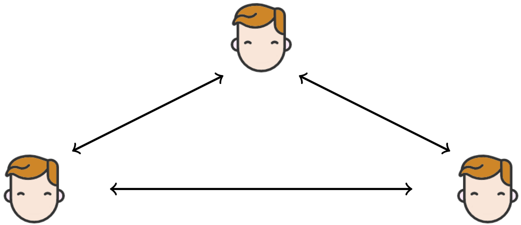

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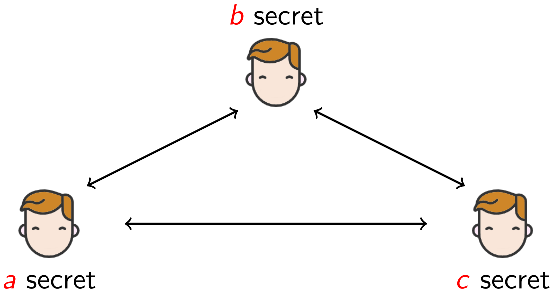

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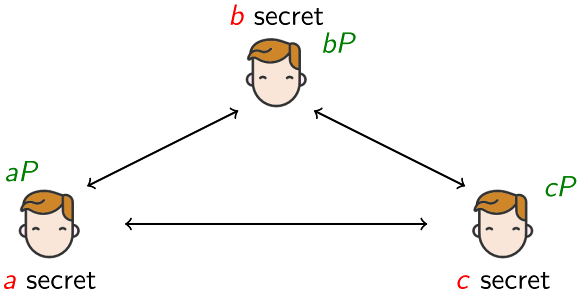

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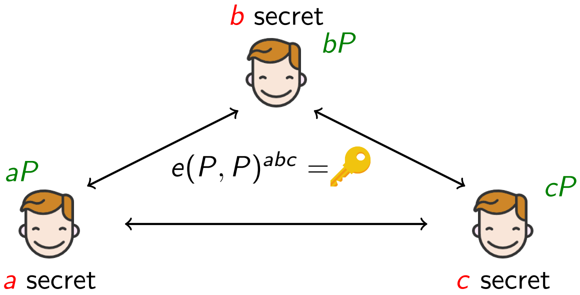

Tate and ate pairing

Definition

The Miller loop computes the function $f_{s,Q}$ such that Q is a zero of order s, and $[s]Q$ is a pole of order 1, i.e

$$\mathrm{div}(f_{s,Q}) = s(Q) - ([s]Q) - (s-1)\mathcal{O}$$

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For $P, Q \in E[r]$ such that $\pi_p(P) = P$, $\pi_p(Q) = [p]Q$,

$$\mathrm{Tate}(P,Q) := f_{r,P}(Q)^{(p^k-1)/r} \qquad \mathrm{ate}(P,Q) := f_{t-1,Q}(P)^{(p^k-1)/r}$$

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2. Compute $x^{(p^k-1)/r}$ (final exponentiation)

Algorithm: $\textsc{MillerLoop}(s, P, Q)$ – Compute $f_{s,Q}(P)$.

$f \leftarrow 1$
$S \leftarrow Q$
for b bit of s from second MSB to LSB **do**
  $f \leftarrow f^2 \cdot \ell_{S,S}(P)/v_{2S}(P)$
  $S \leftarrow [2]S$
  if $b = 1$ **then**
    $f \leftarrow f \cdot \ell_{S,Q}(P)/v_{S+Q}(P)$
    $S \leftarrow S + Q$
  end if
end for
return f such that $\mathrm{div}(f_{s,Q}) = s(Q) - ([s]Q) - (s-1)\mathcal{O}$

Example: $f_{5,Q}(P)$.

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$$\mathrm{div}(f) = 5(Q) - (5Q) - 4(\mathcal{O})$$

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- When k is even, say $\mathbb{F}_{p^k} = \mathbb{F}_{p^{k/2}}(u)$.
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Last part $\frac{\Phi_k(p)}{r}$: more expensive, decompose into polynomials and compute efficiently with Horner rule.

Pairing-friendly curves for 128 bits of security

An elliptic curve E defined over $\mathbb{F}_p$, of trace t and discriminant D is pairing-friendly of embedding degree k if

- p, r are primes and t is relatively prime to p
- r divides $p+1-t$ and p^k-1 but does not divide p^i-1 for $1 \leq i < k$
- $4p-t^2 = Dy^2$ for a sufficiently small positive integer D and an integer y.

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Example.

Barreto-Naehrig curves are elliptic curves of embedding degree $k = 12$, parametrized by

$$p(x) = 36x^4 + 36x^3 + 24x^2 + 6x + 1$$

$$r(x) = 36x^4 + 36x^3 + 18x^2 + 6x + 1$$

$$t(x) = 6x^2 + 1$$

For some integer x_0, $(p(x_0), r(x_0), t(x_0))$ parametrizes a pairing-friendly elliptic curve.

Miller loop.

k is even $\Longrightarrow$ no vertical lines.

$6 \mid k$ and $D = 3 \Longrightarrow$ twist of order 6: $E(\mathbb{F}_{p^{12}})[r] \simeq E'(\mathbb{F}_{p^2})[r]$.

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Efficient pairing.  But how secure are these curves ?

Security of pairing curves.

$$e : E(\mathbb{F}_p) \times E(\mathbb{F}_{p^k}) \longrightarrow \mathbb{F}_{p^k}$$

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- Security against DHP in $\mathbb{F}_{p^k}$: Number Field Sieve attacks in progress.

special prime p	$\Longrightarrow$ 1993: Special NFS attack
$k > 1$	$\Longrightarrow$ 2015: Tower NFS attack
composite k and special p	$\Longrightarrow$ 2016: STNFS attack

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BN curves are threatened by STNFS... 
Need a 5500 bits field $\mathbb{F}_{p^{12}}$ to get 128 bits of security.

Generation of curves with given prime k, square-free D and no structure on p.

Algorithm: $\textsc{Cocks-Pinch}(k, D)$ – Compute a pairing-friendly curve $E/\mathbb{F}_p$ of trace t with a subgroup of order r, such that $t^2 - Dy^2 = 4p$.

Set a prime r such that $k \mid r - 1$ and $\sqrt{-D} \in \mathbb{F}_r$
Set T such that $r \mid \Phi_k(T)$
$t \leftarrow T + 1$
$y \leftarrow (t - 2)/\sqrt{-D}$
Lift $t, y \in \mathbb{Z}$ such that $t^2 + Dy^2 \equiv 0 \mod 4$
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Fix: first fix a small T and then choose r. $t = T + 1$ is small 

Generation of curves with given prime k, square-free D and no structure on p.

Algorithm: $\textsc{Cocks-Pinch}(k, D)$ – Compute a pairing-friendly curve $E/\mathbb{F}_p$ of trace t with a subgroup of order r, such that $t^2 - Dy^2 = 4p$.

Set a small T
Set a prime r such that $k \mid r - 1$, $\sqrt{-D} \in \mathbb{F}_r$ and $r \mid \varphi_k(T)$
$t \leftarrow T + 1$
$y \leftarrow (t-2)/\sqrt{-D}$
Lift $t, y \in \mathbb{Z}$ such that $t^2 + Dy^2 \equiv 0 \mod 4$
$p \leftarrow (t^2 + Dy^2)/4$
if p is prime **then return** $[p, t, y, r]$ **else** Repeat with another r.

Large trace $t \implies$ the ate pairing is not very efficient

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Field	DL attack	Field size needed for 128-bit security	$\log_2(p)$ induced
$\mathbb{F}_{p^5}$	TNFS	3320	664
$\mathbb{F}_{p^6}$	exTNFS	4032	672
$\mathbb{F}_{p^7}$	TNFS	3584	512
$\mathbb{F}_{p^8}$	exTNFS	4352	544

Timings and comparisons

New curves for 128 bits of security.

We generate curves of embedding degree $5, 6, 7$ and 8 with the previous algorithm.

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k	5	6	7	8	12	12	1
$\mathbb{F}_{p^k}$ *size*	3320	4032	3584	4352	5544	5532	3072
$\log_2(p)$	664	672	512	544	462	461	3072
$\mathbb{F}_p$ mul.	230ns	230ns	130ns	154ns	130ns	130ns	4882ns
Miller length	64-bit	128-bit	43-bit	64-bit	117-bit	77-bit	256-bit
Mill. field	3320	672	3584	1088	924	922	3072
Miller step	3.4ms	1.1ms	2.1ms	0.7ms	1.6ms	1.0ms	22.7ms
Expo. step	2.5ms	0.9ms	1.9ms	1.0ms	0.7ms	0.8ms	20.0ms
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Curve	this work				BN	BLS	KSS	–
k	5	6	7	8	12	12	16	1
$\mathbb{F}_{p^k}$ *size*	3320	4032	3584	4352	5544	5532	5424	3072
$\log_2(p)$	664	672	512	544	462	461	339	3072
$\mathbb{F}_p$ mul.	230ns	230ns	130ns	154ns	130ns	130ns	69ns	4882ns
Miller length	64-bit	128-bit	43-bit	64-bit	117-bit	77-bit	35-bit	256-bit
Mill. field	3320	672	3584	1088	924	922	1356	3072
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