Verifiable Delay Functions from Supersingular Isogenies and Pairings

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Definition and examples

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VDF based on isogenies and pairings

Implementation and comparison

Definition

A *verifiable delay function* (VDF) is a function $f : X \longrightarrow Y$ such that

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2. the output can be verified efficiently.

- Setup$(\lambda, T) \longrightarrow$ public parameters pp
- Eval$(pp, x) \longrightarrow$ output y such that $y = f(x)$, and a proof π (requires T steps)
- Verify$(pp, x, y, \pi) \longrightarrow$ yes or no.

Application. Distributed randomness

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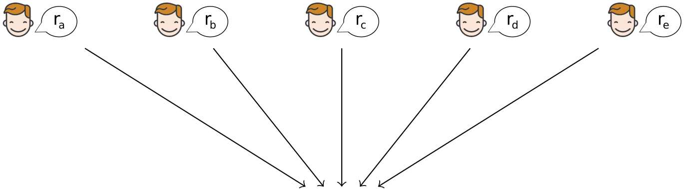

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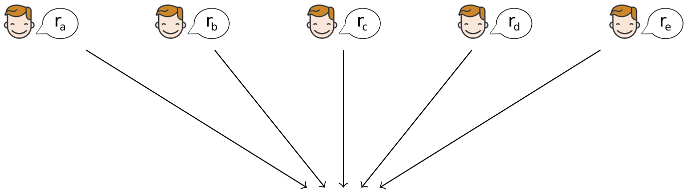

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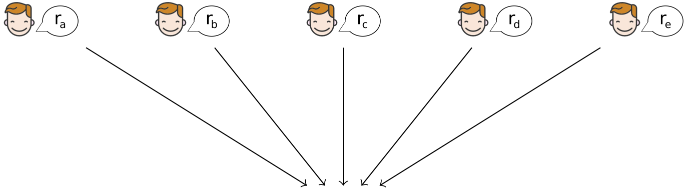

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Idea: slow things down by *adding delay*.

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$$R = r_a \oplus r_b \oplus r_c \oplus r_d \oplus r_e$$

$$\downarrow$$

$$h(R) \in X$$

$$\Big\downarrow \text{VDF}$$

$$f(h(R))$$

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- **Wesolowski verification.** [Eurocrypt '19]
  π is short
  Verification is fast.
- **Pietrzak verification.** [ITCS '19]
  π computation is more efficent
  Verification is slower.

Different security assumptions.

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If one knows the factorization of N, the evaluation can be computed using

$$h(x)^{2^T} \equiv h(x)^{2^T \bmod \varphi(N)} \mod N$$

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VDF based on class group. Let $K = \mathbb{Q}(\sqrt{-D})$ and O_K its ring of integers.

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VDF	**pro**	**con**
RSA	fast verification	trusted setup not post-quantum
Class group	small parameters	slow verification not post-quantum

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- What is an isogeny ?
- What is a pairing ?

Pairing-friendly elliptic curves.

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A pairing is a bilinear non-degenerate application

$$e : \mathbb{G}_1 \times \mathbb{G}_2 \longrightarrow \mathbb{G}_3$$

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For an elliptic curve, we can choose $\mathbb{G}_1 = \langle P \rangle$ and $\mathbb{G}_2 = \langle Q \rangle$ with P, Q points of the curve of order r.
A curve is *pairing-friendly* if P and Q are efficiently computable.

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Applications. BLS signature, identity-based encryption, etc.

Isogenies of elliptic curves.

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Example. $E : y^2 = x^3 - x$ and $K = (1, 0)$ of order 2.

$$\begin{array}{rrcl} \varphi : & E & \longrightarrow & E/\langle K \rangle \\ & (x, y) & \longmapsto & \left(\frac{x^2 - x + 2}{x - 1}, y \frac{x^2 - 2x - 1}{x^2 - 2x + 1} \right) \end{array}$$

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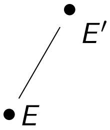
$$e(\varphi(P), Q) = e(P, \hat{\varphi}(Q))$$

Two types of elliptic curves:



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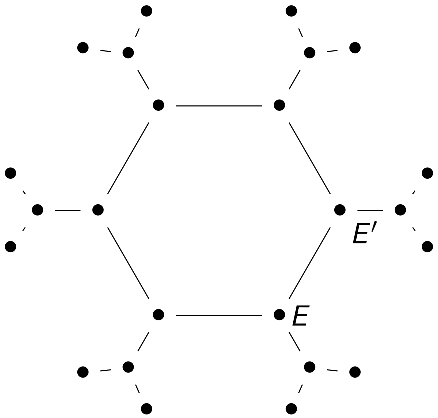
Ordinary curves $End(E)$ is an order in $\mathbb{Q}(\sqrt{-D})$.



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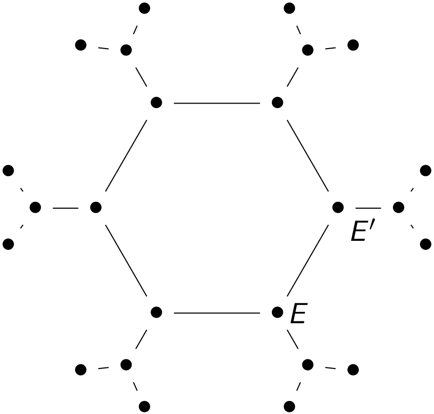

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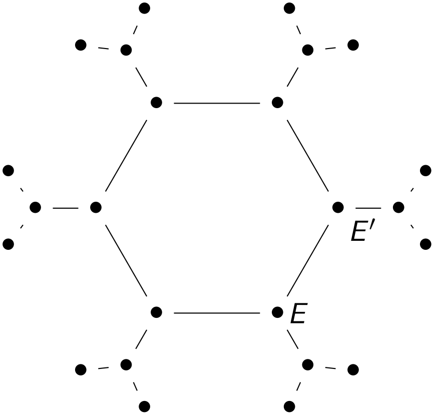

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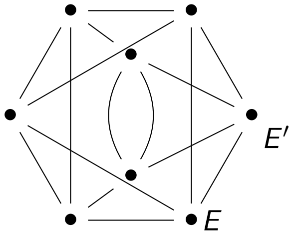

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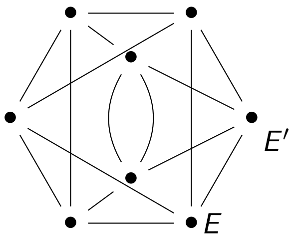

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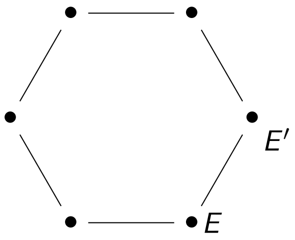
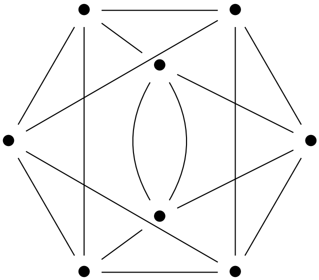
Looking only curves defined over $\mathbb{F}_p$, $\mathrm{End}_p(E)$ is an order in $\mathbb{Q}(\sqrt{-p})$.



VDF over $\mathbb{F}_{p^2}$ supersingular curves.



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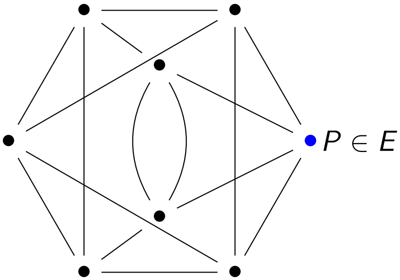
Setup A **public** walk in the isogeny graph.



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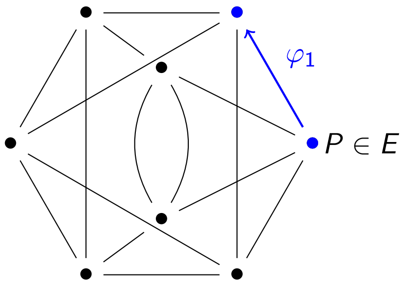

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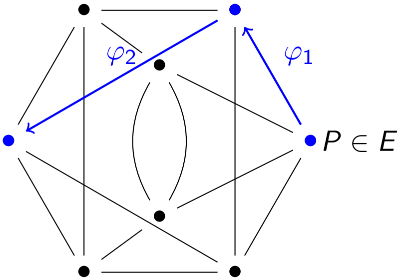

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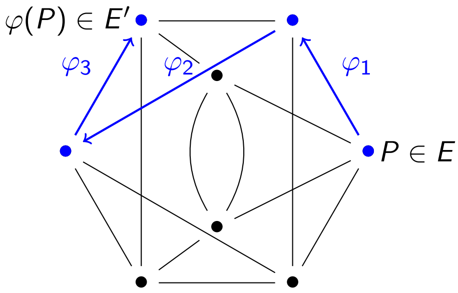

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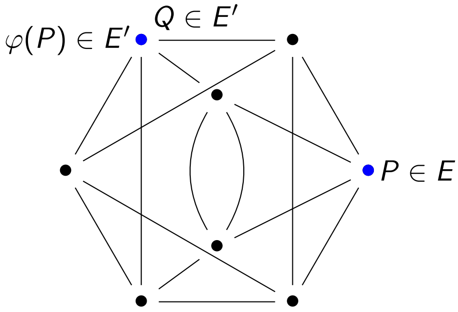

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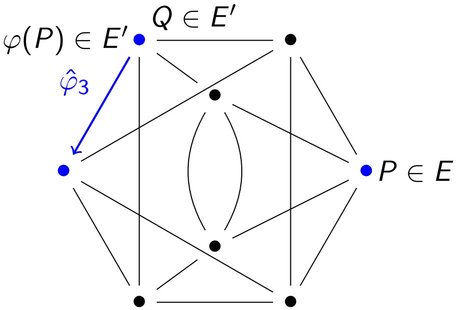

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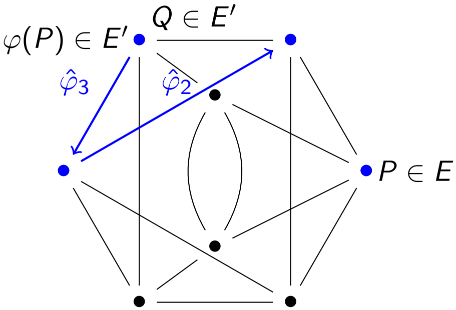

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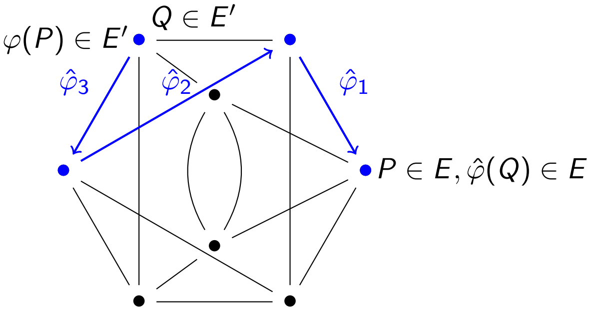

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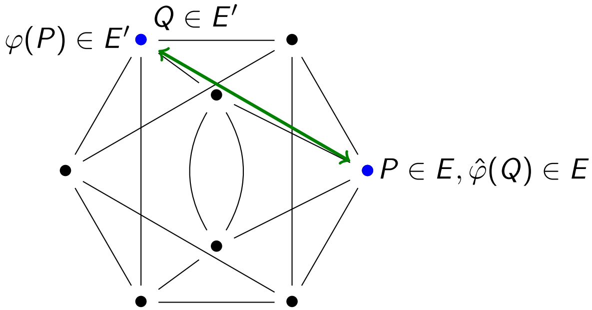

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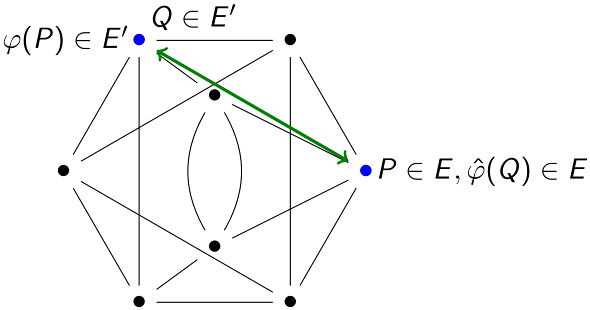

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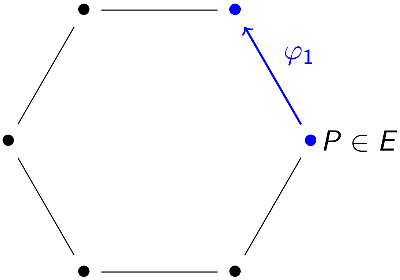

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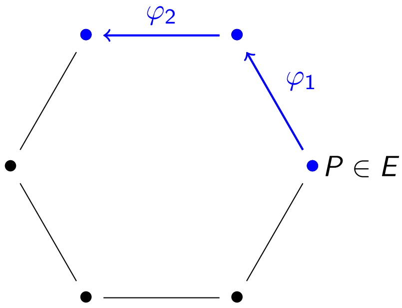

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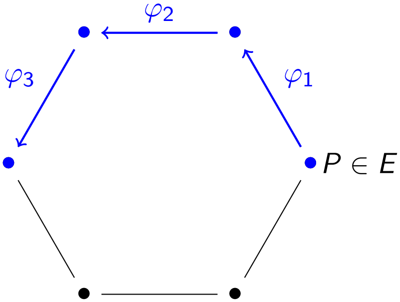

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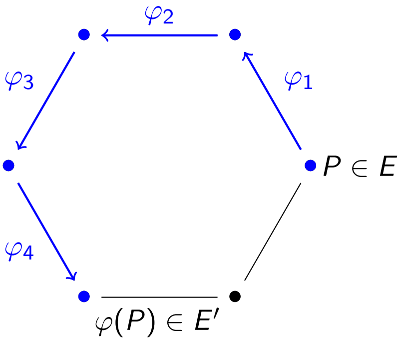

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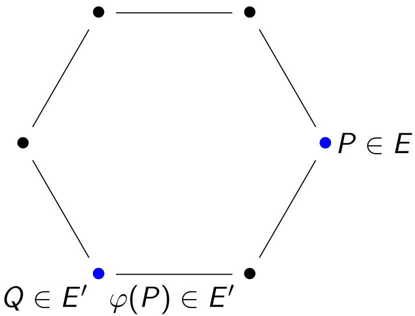

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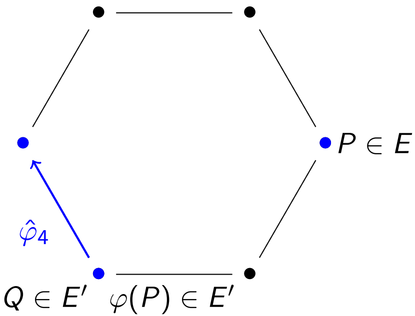

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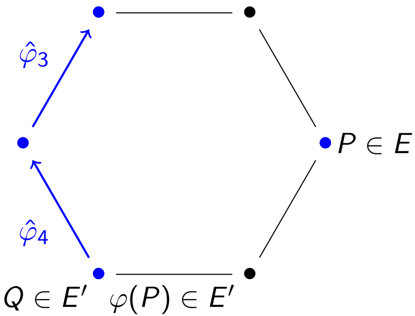

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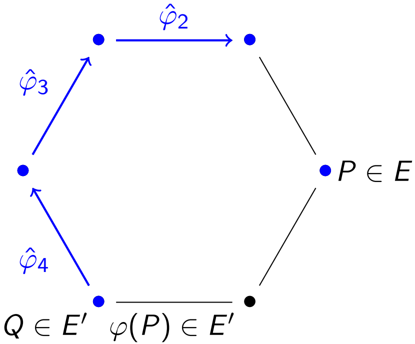

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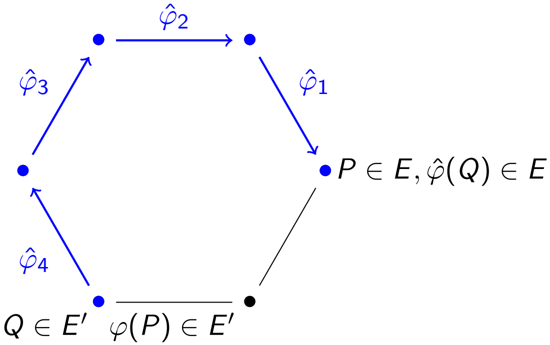

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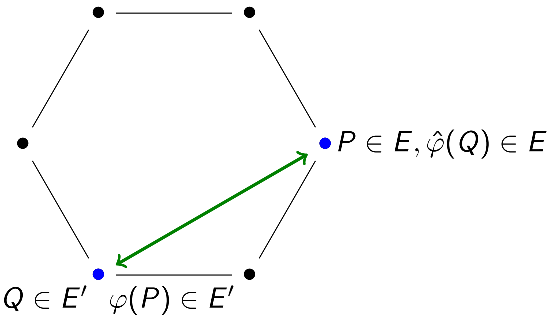

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Isogeny shortcut.

$$y^2 = x^3 + x$$

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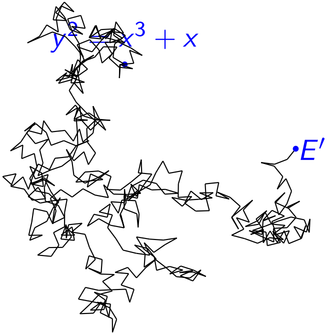
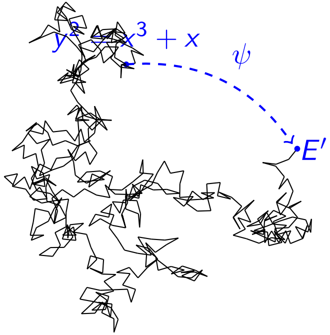


Isogeny shortcut.

If E has a known endomorphism ring, a shortcut can be found!



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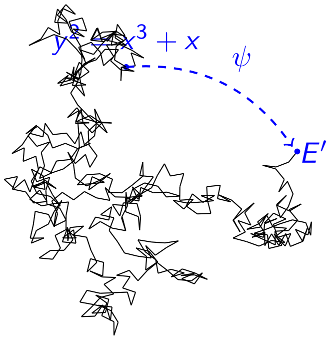
Unknown endomorphism ring.

- Ordinary curves. Pairing friendly $\rightarrow$ small discriminant $\rightarrow$ known End(E).



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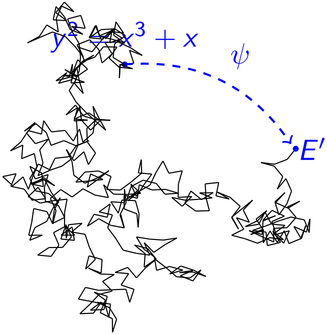
- Ordinary curves. Pairing friendly $\rightarrow$ small discriminant $\rightarrow$ known $\mathrm{End}(E)$.
- Supersingular curves.
  Open problem: compute a supersingular elliptic curve of unknown endomorphism ring.



Isogeny shortcut.
If E has a known endomorphism ring, a shortcut can be found!

Unknown endomorphism ring.

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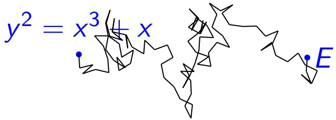

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Isogenies over $\mathbb{F}_p$ and class group.

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$$\begin{tikzcd}
E_1 \arrow[r, "\varphi_1"] \arrow[d] & E_2 \arrow[d] \\
\mathrm{End}_p(E_1) \arrow[r, "I_1"] & \mathrm{End}_p(E_2)
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Security of our VDF.

- Our VDFs are secure on a classical computer.
- The $\mathbb{F}_p$ VDF is insecure on a quantum computer:
  - Once the setup is done, compute $\#\mathrm{Cl}(D)$.
  - Evaluate the $\mathbb{F}_p$ VDF with ideal multiplications faster than isogenies.
- The $\mathbb{F}_{p^2}$ VDF is insecure on a quantum computer.
  It is *quantum-annoying* in the sense that you need to run Shor's algorithm for each evaluation of the VDF.

Implementation and comparison

Definition and examples

VDF based on isogenies and pairings

Implementation and comparison

Computing isogenies.

- Method 1. Degree 2 isogenies using 2-torsion points:

$$\underset{E}{\bullet} - \underset{E_1}{\bullet} - \underset{E_2}{\bullet} - \underset{E_3}{\bullet} - \underset{E_4}{\bullet} - \underset{E_5}{\bullet} - \underset{E_6}{\bullet} - \underset{E_7}{\bullet} - \underset{E_8}{\bullet} - \cdots - \bullet - \bullet - \bullet - \underset{E_{T-1}}{\bullet} - \underset{E'}{\bullet}$$

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$$E \longrightarrow E_4 \longrightarrow E_8 - \cdots - \bullet \longrightarrow \bullet$$

  Time complexity: T/n isogenies of degree $2^n \approx T \log_2(n)$ degree 2 isogenies.
  Storage complexity: $O(T/n)$.
  In practice (for large T), $log_2(n)$ is small and it can be useful to reduce the storage.

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bits of p :

$$p = r \cdot 2^{1244} \cdot f - 1$$

Implementation.

- Proof of concept in SageMath : https://github.com/isogenies-vdf.
- Parameters chosen for 128 bits of security
- Arithmetic of Montgomery curves
- Isogeny computation with recursive strategy
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Protocol	Step	e_k size	Time	Throughput
$\mathbb{F}_p$ graph	Setup	238 kb	–	0.75isog/ms
	Evaluation	–	–	0.75isog/ms
	Verification	–	0.3 s	–
$\mathbb{F}_{p^2}$ graph	Setup	491 kb	–	0.35isog/ms
	Evaluation	–	–	0.23isog/ms
	Verification	–	4 s	–

Table: Benchmarks for our VDFs, on a Intel Core i7-8700 @ 3.20GHz, $T \approx 2^{16}$

VDF	**pro**	**con**
RSA	fast verification	trusted setup
Class group	no trusted setup small parameters	slow verification
Isogenies over $\mathbb{F}_p$	Fast verification	trusted setup
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- Hash to the supersingular set (in order to remove the trusted setup)
- Find a fully post-quantum VDF

Thank you for your attention.