Cocks–Pinch curves with efficient ate pairing

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Pairings on elliptic curves

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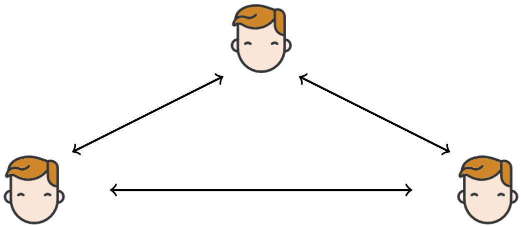

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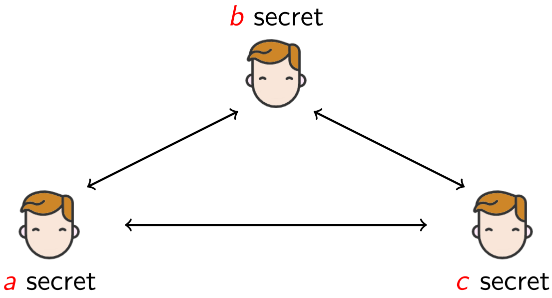

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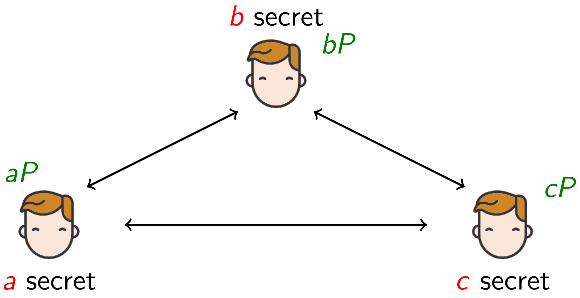

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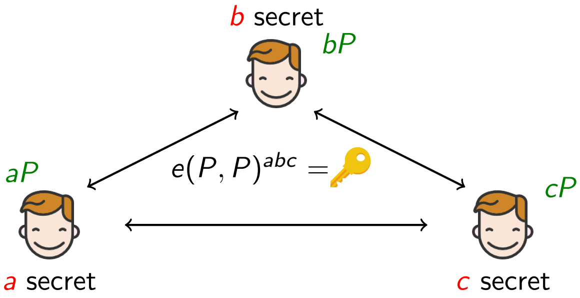

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Verification: same as for BLS.

Application 4. Identity based encryption
$H_1 : \{0,1\}^* \to E$ and $H_2 : \mathbb{F}_{p^k} \to \{0,1\}^n$ are hash functions.
The PKG has a secret key s and a public key $P_k = [s]P$.
$Q_{\text{id}} = H_1(id)$ and $S_{\text{id}} = [s]Q_{\text{id}}$ is obtained from the PKG.

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Send $(u, v) = ([r]P, m \oplus H_2(g_{\text{id}}^r))$.

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$$e(S_{\mathrm{id}}, u) = e([s]Q_{\mathrm{id}}, [r]P) = e(Q_{\mathrm{id}}, P)^{rs} = e(Q_{\mathrm{id}}, P_k)^r$$

Tate and ate pairing

1. Tate and ate pairing
2. Pairing-friendly curves for 128 bits of security
3. Timings and comparisons

The Tate and ate pairings are computed in two steps:

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Definition

For $P, Q \in E[r]$ such that $\pi_p(P) = P$, $\pi_p(Q) = [p]Q$,

$$\mathrm{Tate}(P, Q) := f_{r,P}(Q)^{(p^k-1)/r} \qquad \mathrm{ate}(P, Q) := f_{t-1,Q}(P)^{(p^k-1)/r}$$

Miller loop step.

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Definition

The Miller loop computes the function $f_{s,Q}$ such that Q is a zero of order s, and $[s]Q$ is a pole of order 1, i.e

$$\operatorname{div}(f_{s,Q}) = s(Q) - ([s]Q) - (s-1)\mathcal{O}$$

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Compute $x = f_{r,P}(Q)$ with $P \in E(\mathbb{F}_p)[r]$ and $Q \in E(\mathbb{F}_{p^k})[r]$.

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Miller loop for ate.
For ate: compute $x = f_{t-1,Q}(P)$ with $P \in E(\mathbb{F}_p)[r]$ and $Q \in E(\mathbb{F}_{p^k})[r]$.

Algorithm: $\textsc{MillerLoop}(s, P, Q)$ – Compute $f_{s,Q}(P)$.

$f \leftarrow 1$
$S \leftarrow Q$
for b bit of s from second MSB to LSB **do**
  $f \leftarrow f^2 \cdot \ell_{S,S}(P)/v_{2S}(P)$
  $S \leftarrow [2]S$
  if $b = 1$ **then**
    $f \leftarrow f \cdot \ell_{S,Q}(P)/v_{S+Q}(P)$
    $S \leftarrow S + Q$
  end if
end for
return f such that $\mathrm{div}(f_{s,Q}) = s(Q) - ([s]Q) - (s-1)\mathcal{O}$

Example: $f_{5,Q}(P)$.

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$$\mathrm{div}(f) = 5(Q) - (5Q) - 4(\mathcal{O})$$

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Factors in subfields do not need to be computed !

- When k is even, say $\mathbb{F}_{p^k} = \mathbb{F}_{p^{k/2}}(\sqrt{\alpha})$.
  Quadratic twist :

$$\begin{array}{ccc} E' & \xrightarrow{\sim} & E \\ (x, y) & \longmapsto & (\alpha x, \sqrt{\alpha}^3 y) \end{array}$$

  The isomorphism is defined over $\mathbb{F}_{p^k}$ and E has full r-torsion defined over $\mathbb{F}_{p^k}$.
  $Q \in E(\mathbb{F}_{p^k})[r]$ is seen as twist$(\tilde{Q})$ with $\tilde{Q}$ with two coordinates in $\mathbb{F}_{p^{k/2}}$.

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  The isomorphism is defined over $\mathbb{F}_{p^k}$ and E has full r-torsion defined over $\mathbb{F}_{p^k}$.
  $Q \in E(\mathbb{F}_{p^k})[r]$ is seen as twist$(\tilde{Q})$ with $\tilde{Q}$ with two coordinates in $\mathbb{F}_{p^{k/2}}$.
  Vertical lines $v_S(P) = x_S - x_P \in \mathbb{F}_{p^{k/2}}$ because $x_S \in \mathbb{F}_{p^{k/2}}$ and $P \in E(\mathbb{F}_p)$.
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$$f = (1^2 \cdot \ell_{Q,Q}(P)/v_{2Q}(P))^2 \cdot \ell_{2Q,2Q}(P)/v_{4Q}(P) \cdot \ell_{4Q,Q}(P)/v_{5Q}(P)$$

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Pairing-friendly curves for 128 bits of security

An elliptic curve E defined over $\mathbb{F}_p$, of trace t and discriminant D is pairing-friendly of embedding degree k if

- p, r are primes and t is relatively prime to p
- r divides $p+1-t$ and p^k-1 but does not divide p^i-1 for $1 \leq i < k$
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Cyclotomic families.

1. Find $r(x) \in \mathbb{Z}[x]$ such that $K := \mathbb{Q}[x]/(r(x))$ is a number field containing $\sqrt{-D}$ and $\mathbb{Q}(\zeta_k)$ for a chosen primitive k-th root ζ_k.

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If $p(x)$ represents primes, choosing $x_0 \in \mathbb{Z}$ such that $y(x_0) \in \mathbb{Z}$ gives a pairing-friendly elliptic curve of embedding degree k, defined over $\mathbb{F}_{p(x_0)}$, of trace $t(x_0)$, with a subgroup of order $r(x_0)$ and discriminant D.

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`sage: EllipticCurve_from_j(j0).`

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Barreto-Naehrig curves are elliptic curves of embedding degree $k = 12$, parametrized by

$$p(x) = 36x^4 + 36x^3 + 24x^2 + 6x + 1$$

$$r(x) = 36x^4 + 36x^3 + 18x^2 + 6x + 1$$

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What about efficiency of the pairing computation ?

Miller loop.

k is even $\implies$ no vertical lines.

$6 \mid k$ and $D = 3 \implies$ twist of degree 6: $E(\mathbb{F}_{p^{12}})[r] \simeq E'(\mathbb{F}_{p^2})[r]$.

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Efficient pairing. But how secure are these curves ?

Security of pairing curves.

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  BN curves are threatened by STNFS... 
  Need a 5500 bits field $\mathbb{F}_{p^{12}}$ to get 128 bits of security.

Generation of curves with given prime k, square-free D and no structure on p.

Algorithm: $\textsc{Cocks-Pinch}(k, D)$ – Compute a pairing-friendly curve $E/\mathbb{F}_p$ of trace t with a subgroup of order r, such that $t^2 - Dy^2 = 4p$.

Set a prime r such that $k \mid r - 1$ and $\sqrt{-D} \in \mathbb{F}_r$
Set T such that $r \mid \Phi_k(T)$
$t \leftarrow T + 1$
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Lift $t, y \in \mathbb{Z}$ such that $t^2 + Dy^2 \equiv 0 \mod 4$
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- Low hamming weight of T (Miller loop).
- When lifting in $\mathbb{Z}$, add a multiple of r in y

$$y = y + h_y \cdot r$$

such that p is large enough to resist NFS attacks.

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Sometimes (for instance $k = 8$) p is parametrized by a *two-variables* polynomial: $p \in \mathbb{Z}[T, h_y]$, but today NFS-variants do not use this property.

Field	DL attack	Field size needed for 128-bit security	$\log_2(p)$ induced
$\mathbb{F}_{p^5}$	TNFS	3320	664
$\mathbb{F}_{p^6}$	exTNFS	4032	672
$\mathbb{F}_{p^7}$	TNFS	3584	512
$\mathbb{F}_{p^8}$	exTNFS	4352	544

Timings and comparisons

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1. Bench $\mathbb{F}_p$ arithmetic and pairing computation for new BN and BLS primes.
2. Bench $\mathbb{F}_p$ arithmetic for our non-special primes of different sizes.
3. Count the number of $\mathbb{F}_p$ multiplications to get an estimation of the cost.

New curves for 128 bits of security.

We generate curves of embedding degree $5, 6, 7$ and 8 with the previous algorithm.

Curve	this work				BN	BLS	–
k	5	6	7	8	12	12	1
$\mathbb{F}_{p^k}$ *size*	3320	4032	3584	4352	5544	5532	3072
$\log_2(p)$	664	672	512	544	462	461	3072
$\mathbb{F}_p$ mul.	230ns	230ns	130ns	154ns	130ns	130ns	4882ns
Miller length	64-bit	128-bit	43-bit	64-bit	117-bit	77-bit	256-bit
Mill. field	3320	672	3584	1088	924	922	3072
Miller step	3.4ms	1.1ms	2.1ms	0.7ms	1.6ms	1.0ms	22.7ms
Expo. step	2.5ms	0.9ms	1.9ms	1.0ms	0.7ms	0.8ms	20.0ms
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Curve	this work				BN	BLS	KSS	–
k	5	6	7	8	12	12	16	1
$\mathbb{F}_{p^k}$ *size*	3320	4032	3584	4352	5544	5532	5424	3072
$\log_2(p)$	664	672	512	544	462	461	339	3072
$\mathbb{F}_p$ mul.	230ns	230ns	130ns	154ns	130ns	130ns	69ns	4882ns
Miller length	64-bit	128-bit	43-bit	64-bit	117-bit	77-bit	35-bit	256-bit
Mill. field	3320	672	3584	1088	924	922	1356	3072
Miller step	3.4ms	1.1ms	2.1ms	0.7ms	1.6ms	1.0ms	0.5ms	22.7ms
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