Cocks–Pinch curves with efficient ate pairing

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Thales – LORIA

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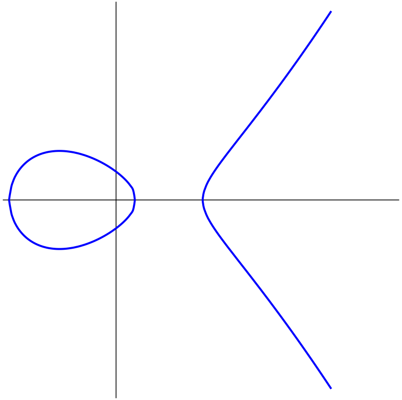
Elliptic curve. $y^2 = x^3 + Ax + B$



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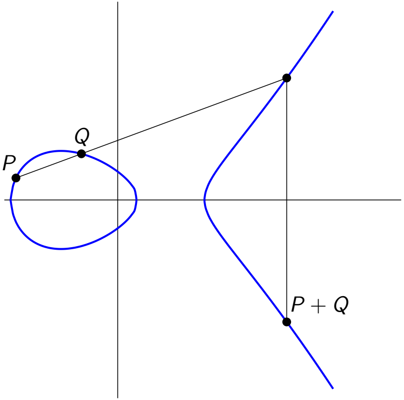

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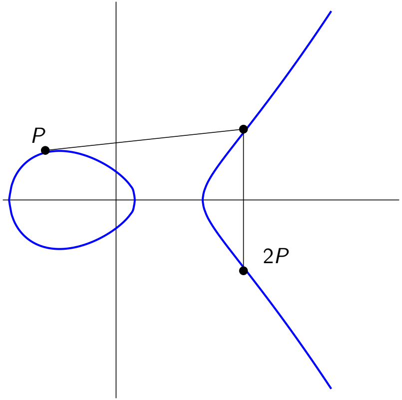

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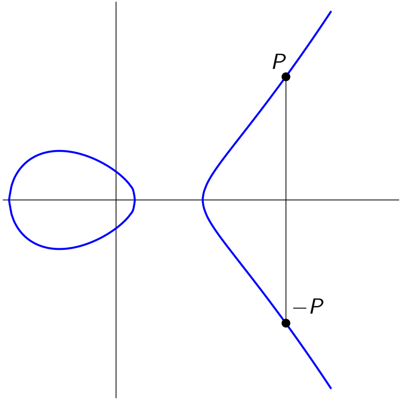

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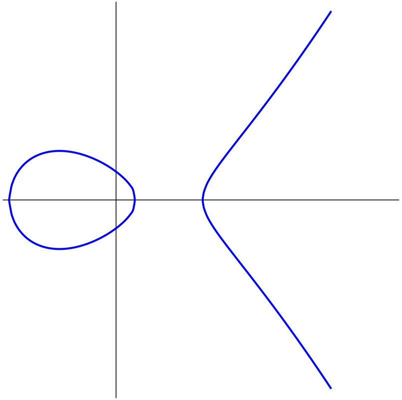

Points on an elliptic curve form a group (with group law $+$).

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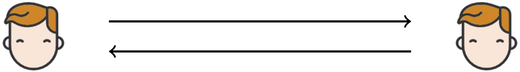

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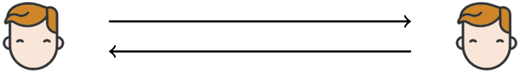

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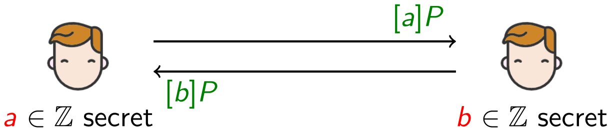

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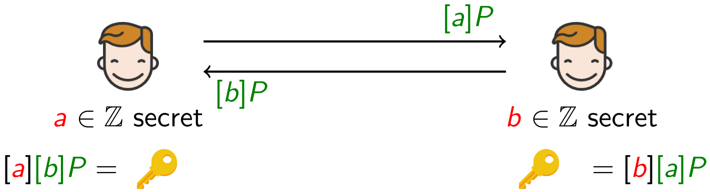

WhatsApp

Terms

Public Key Types

- `Identity Key Pair` – A long-term Curve25519 key pair, generated at install time.
- `Signed Pre Key` – A medium-term Curve25519 key pair, generated at install time, signed by the `Identity Key`, and rotated on a periodic timed basis.
- `One-Time Pre Keys` – A queue of Curve25519 key pairs for one time use, generated at install time, and replenished as needed.

Session Key Types

- `Root Key` – A 32-byte value that is used to create `Chain Keys`.
- `Chain Key` – A 32-byte value that is used to create `Message Keys`.
- `Message Key` – An 80-byte value that is used to encrypt message contents. 32 bytes are used for an AES-256 key, 32 bytes for a HMAC-SHA256 key, and 16 bytes for an IV.

Initiating Session Setup

To communicate with another WhatsApp user, a WhatsApp client first needs to establish an encrypted session. Once the session is established, clients do not need to rebuild a new session with each other until the existing session state is lost through an external event such as an app reinstall or device change.

To establish a session:

1. The initiating client ("initiator") requests the public `Identity Key`, public `Signed Pre Key`, and a single public `One-Time Pre Key` for the recipient.
2. The server returns the requested public key values. A `One-Time Pre Key` is only used once, so it is removed from server storage after being requested. If the recipient's latest batch of `One-Time Pre Keys` has been consumed and the recipient has not replenished them, no `One-Time Pre Key` will be returned.
3. The initiator saves the recipient's `Identity Key` as $I_{recipient}$, the `Signed Pre Key` as $S_{recipient}$, and the `One-Time Pre Key` as $O_{recipient}$.
4. The initiator generates an ephemeral Curve25519 key pair, $E_{initiator}$.
5. The initiator loads its own `Identity Key` as $I_{initiator}$.
6. The initiator calculates a master secret as $master\_secret = ECDH(I_{initiator}, S_{recipient}) \,||\, ECDH(E_{initiator}, I_{recipient}) \,||\, ECDH(E_{initiator}, S_{recipient}) \,||\, ECDH(E_{initiator}, O_{recipient})$. If there is no `One Time Pre Key`, the final ECDH is omitted.
7. The initiator uses HKDF to create a `Root Key` and `Chain Keys` from the `master_secret`.

Discrete logarithm problem (DLP).

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Subgroup attack.

$\#E(\mathbb{F}_p) =$

$2^2 \times 5^5 \times 13 \times 37 \times 18575429 \times 505818037 \times 1089749937157876379177809361515176882436093600552189158080830008040550806174507 3$,
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Counter the attack.

- Checking that P is of order 108...073:
  $[108...073]P = 0$, $[2^2 \times 5^5 \times 13 \times 37 \times 18575429 \times 505818037]P \neq 0$ and $P \neq 0$.
- Choosing a curve with $\#E(\mathbb{F}_p)$ with no small factor.

If $\gcd(r, p) = 1$ (think $r = 1089749937157876379177809361515176882436093600552189158080830008040550806174507 3$),

$$E(\overline{\mathbb{F}}_p)[r] \simeq \underbrace{\mathbb{Z}/r\mathbb{Z}}_{\mathbb{G}_1} \times \underbrace{\mathbb{Z}/r\mathbb{Z}}_{\mathbb{G}_2}$$

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Definition (embedding degree)

The embedding degree of E w.r.t. r (coprime to p) is the smallest integer k such that $E[r]$ is defined over $\mathbb{F}_{p^k}$.

Pairings on elliptic curves

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Secure pairing-friendly elliptic curve with an efficient pairing

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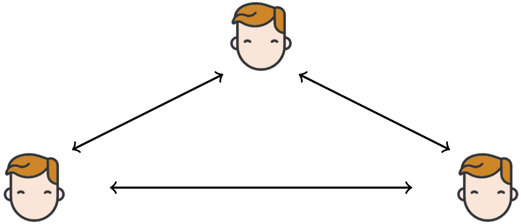

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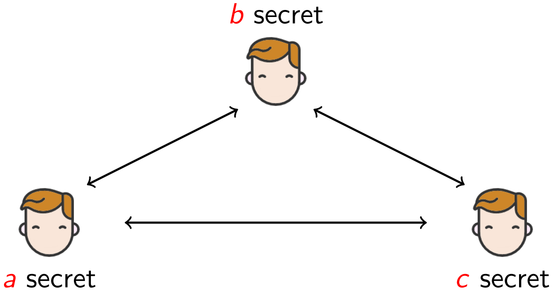

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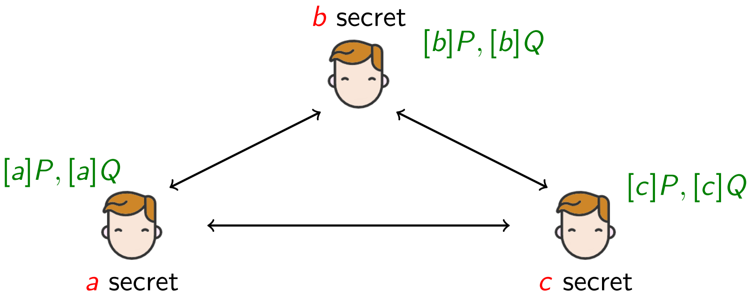

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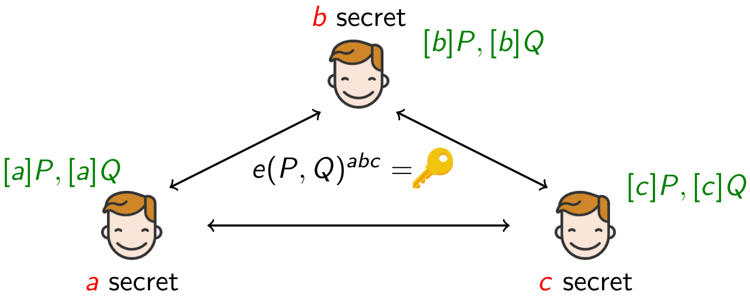

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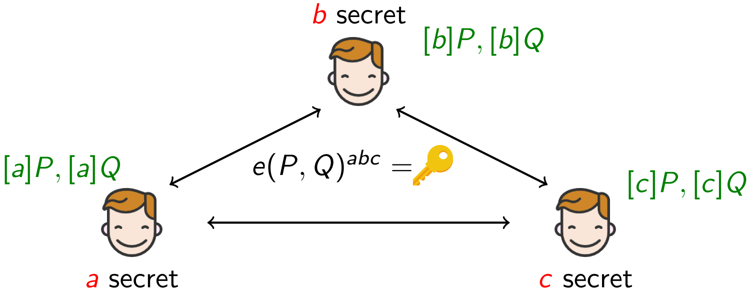

$$e([b]P, [c]Q)^{a} = e(P, Q)^{bca}$$

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$$e(P_k, H(M)) = e([s_k]P, H(M)) = e(P, [s_k]H(M)) = e(P, \sigma)$$

Many other applications:

- Blind signature
- Identity-based encryption
- Post-quantum cryptography compressions (eprint 2017/1143)
- Short group signature (eprint 2018/1115)
- Verifiable delay functions (eprint 2019/166)
- etc.

Tate and ate pairing

1 Tate and ate pairing

2 Pairing-friendly curves for 128 bits of security

3 Timings and comparisons

The Tate and ate pairings are computed in two steps:

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Definition

For $P \in \mathbb{G}_1 = E(\mathbb{F}_p)[r]$, $Q \in \mathbb{G}_2 = E(\mathbb{F}_{p^k})[r]$,

$$\mathrm{Tate}(P, Q) := f_{r,P}(Q)^{(p^k-1)/r} \qquad \mathrm{ate}(P, Q) := f_{t-1,Q}(P)^{(p^k-1)/r}$$

Miller loop step.

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Definition

The Miller loop computes the function $f_{s,Q}$ such that Q is a zero of order s, and $[s]Q$ is a pole of order 1, i.e

$$\mathrm{div}(f_{s,Q}) = s(Q) - ([s]Q) - (s-1)\mathcal{O}$$

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Compute $x = f_{r,P}(Q)$ with $P \in E(\mathbb{F}_p)[r]$ and $Q \in E(\mathbb{F}_{p^k})[r]$.

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Miller loop for ate.
For ate: compute $x = f_{t-1,Q}(P)$ with $P \in E(\mathbb{F}_p)[r]$ and $Q \in E(\mathbb{F}_{p^k})[r]$.

Algorithm: $\textsc{MillerLoop}(s, P, Q)$ – Compute $f_{s,Q}(P)$.

$f \leftarrow 1$
$S \leftarrow Q$
for b bit of s from second MSB to LSB **do**
  $f \leftarrow f^2 \cdot \ell_{S,S}(P)/v_{2S}(P)$
  $S \leftarrow [2]S$
  if $b = 1$ **then**
    $f \leftarrow f \cdot \ell_{S,Q}(P)/v_{S+Q}(P)$
    $S \leftarrow S + Q$
  end if
end for
return f such that $\mathrm{div}(f_{s,Q}) = s(Q) - ([s]Q) - (s-1)\mathcal{O}$

Example: $f_{5,Q}(P)$.

$$s = 5 = \overline{101}^2$$

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$$s = 5 = \overline{\boxed{1}01}^{2}$$

$$f = 1$$

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$$f = \left(1^2 \cdot \ell_{Q,Q}(P)/v_{2Q}(P)\right)^2$$

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$$s = 5 = \overline{10\boxed{1}}^{2}$$

$$f = \left(1^2 \cdot \ell_{Q,Q}(P)/v_{2Q}(P)\right)^2 \cdot \ell_{2Q,2Q}(P)/v_{4Q}(P)$$

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$$\mathrm{div}(f) = 5(Q) - (5Q) - 4(\mathcal{O})$$

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- First exponentiation: $\frac{p^k-1}{\Phi_k(p)}$.Polynomial in p with very small coefficients.
  Very efficent with Frobenius: if $\mathbb{F}_{p^k} = \mathbb{F}_p[x]/(x^k - \alpha)$,
  $a^p = \left(\sum_{i=0}^{k-1} a_i x^i\right)^p = \sum_{i=0}^{k-1} a_i x^{ip}$ and x^{ip} can be precomputed.

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  Very efficent with Frobenius: if $\mathbb{F}_{p^k} = \mathbb{F}_p[x]/(x^k-\alpha)$,
  $a^p = \left(\sum_{i=0}^{k-1} a_i x^i\right)^p = \sum_{i=0}^{k-1} a_i x^{ip}$ and x^{ip} can be precomputed.
  A Frobenius costs $k-1$ multiplications over $\mathbb{F}_p$.

Final exponentiation step.
$f_{r,P}(Q)$ and $f_{t-1,Q}(P)$ are r-th roots of unity.
We obtain a unique coset representative by elevating to the power $(p^k - 1)/r$.

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- Second exponentiation: more expensive. Possible optimizations.

Pairing-friendly curves for 128 bits of security

1 Tate and ate pairing

2 Pairing-friendly curves for 128 bits of security

3 Timings and comparisons

Parameters of pairing-friendly curves.

An elliptic curve E defined over $\mathbb{F}_p$, of trace t and discriminant D is pairing-friendly of embedding degree k if

- p, r are primes and t is relatively prime to p
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Example (BN curves). Barreto-Naehrig curves are elliptic curves of embedding degree $k = 12$ with

$$p = 36x_0^4 + 36x_0^3 + 24x_0^2 + 6x_0 + 1$$

$$r = 36x_0^4 + 36x_0^3 + 18x_0^2 + 6x_0 + 1$$

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BN Miller loop.

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Efficient pairing. But how secure are these curves ?

Security of pairing curves.

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BN curves are threatened by STNFS... 
Need a 5500 bits field $\mathbb{F}_{p^{12}}$ to get 128 bits of security.

Curves of embedding degree 1. [eprint 2016/403]

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p is very large, only the Tate pairing on these curves: not efficient

Generation of curves with given prime k, square-free D and no structure on p.

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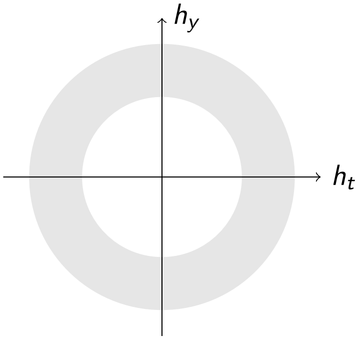

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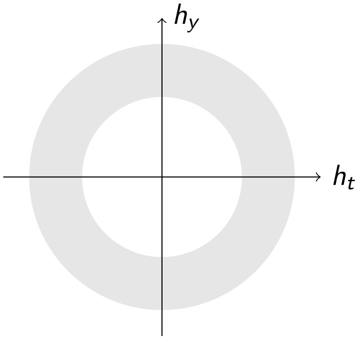

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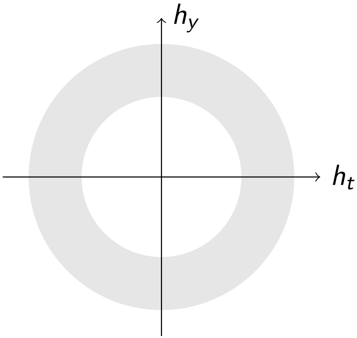

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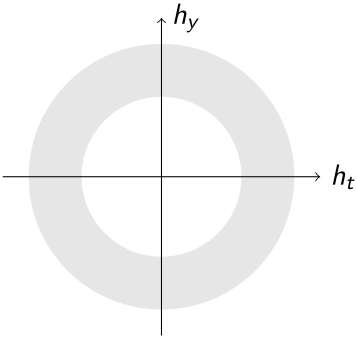

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Large p, no compression for $\mathbb{G}_2$

Parameter choices for 128 bits of security.

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Example of generation for $k = 8$.

Code is available at `https://gitlab.inria.fr/smasson/cocks-pinch-variant`.

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Check subgroup-security and twist-subgroup-security.

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```
CocksPinchVariantResult(
    k=8,D=4,T=0xffffffffeff7c200,i=5,ht=5,hy=-0xd700,
    allowed_cofactor=420,allowed_size_cofactor=10,
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Subgroup- and twist-subgroup- secure curves found for $k = 5, 6, 7$ and 8 !

Timings and comparisons

1. Tate and ate pairing
2. Pairing-friendly curves for 128 bits of security
3. Timings and comparisons

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2. Count the number of $\mathbb{F}_p$ multiplications for a pairing computation on each curve
3. Compare the estimated costs between the different curves.

Curve	64-bit words for p	$\mathbb{F}_p$ mult. timing
BN-462	□□□□□□□□	
BLS12-461	□□□□□□□□	
$k = 5$	□□□□□□□□□□□	
$k = 6$	□□□□□□□□□□□	
$k = 7$	□□□□□□□□	
$k = 8$	□□□□□□□□□	
$k = 1$	□ $\times$ 48	

Curve	64-bit words for p	$\mathbb{F}_p$ mult. timing
BN-462	□□□□□□□□	120ns
BLS12-461	□□□□□□□□	120ns
$k = 5$	□□□□□□□□□□□	
$k = 6$	□□□□□□□□□□□	
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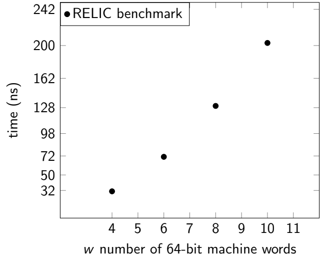
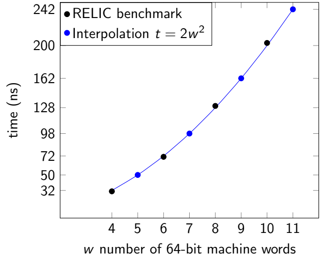


Curve	64-bit words for p	$\mathbb{F}_p$ mult. timing
BN-462	□□□□□□□□	120ns
BLS12-461	□□□□□□□□	120ns
$k = 5$	□□□□□□□□□□□	230ns*
$k = 6$	□□□□□□□□□□□	230ns*
$k = 7$	□□□□□□□□	120ns
$k = 8$	□□□□□□□□□	154ns*
$k = 1$	$\square \times 48$	



* Interpolation from the graph

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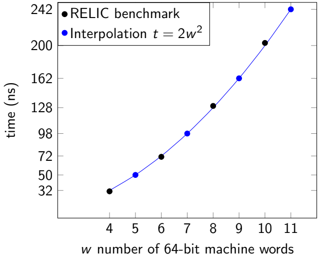
BN-462	□□□□□□□□	120ns
BLS12-461	□□□□□□□□	120ns
$k = 5$	□□□□□□□□□□□	230ns*
$k = 6$	□□□□□□□□□□□	230ns*
$k = 7$	□□□□□□□□	120ns
$k = 8$	□□□□□□□□□	154ns*
$k = 1$	□ $\times$ 48	4882ns**



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**Benchmark with GMP.

Pairing computation on BLS and BN curves.

- Automorphism of degree 6: $\mathbb{G}_2 \simeq {}^{t6}E(\mathbb{F}_{p^2})$
- Miller length: 117-bit for BN, 77-bit for BLS
- Efficient final exponentiation using $p = p(x_0)$ and $r = r(x_0)$
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Benchmarks with RELIC: $\approx$ 10% of error 

Pairing computation on $k = 5$ and $k = 7$ curves.

- No compression: very large $\mathbb{G}_2 \simeq E(\mathbb{F}_{p^k})$
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Pairing estimations: very expensive 

Pairing computation on $k = 6$ curves.

- Automorphism of degree 6: $\mathbb{G}_2$ defined over $\mathbb{F}_p$
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- Final exponentiation faster than $k = 5$ and 7 (structure on p, cyclotomic squaring). See gitlab.inria.fr.

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Pairing computation on $k = 8$ curves.

- Automorphism of degree 4: $\mathbb{G}_2$ defined over $\mathbb{F}_{p^2}$
- Structure on p: $p = \frac{1}{4}((t + h_t \cdot r)^2 + 4(y + h_y \cdot r)^2) = p(T, h_t, h_y)$
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Pairing estimations: competitive 

Curve	**Miller loop** time estimation	**Exponentiation** time estimation	time estimation
$k = 5$	3.3ms	2.3ms	5.6ms
$k = 6$	1.1ms	0.9ms	2.0ms
$k = 7$	2.2ms	1.6ms	3.8ms
$k = 8$	0.7ms	1.1ms	1.8ms
BN	1.5ms	0.7ms	2.2ms
BLS12	0.9ms	0.7ms	1.6ms
$k = 1$	22.7ms	20.0ms	42.7ms

Curve	**Miller loop** time estimation	**Exponentiation** time estimation	time estimation
$k = 5$	3.3ms	2.3ms	5.6ms
$k = 6$	1.1ms	0.9ms	2.0ms
$k = 7$	2.2ms	1.6ms	3.8ms
$k = 8$	0.7ms	1.1ms	1.8ms
BN	1.5ms	0.7ms	2.2ms
BLS12	0.9ms	0.7ms	1.6ms
$k = 1$	22.7ms	20.0ms	42.7ms

Thank you for your attention.