Fast and Exact Geometric Analysis of Real Algebraic Plane Curves

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Outline

1 Introduction to Curve Analysis
2 Curve Analysis Details
3 Application and Further Work
Exact Geometric Computation

Geometric algorithms often
- described in REAL-RAM
- assume non-degeneracy

Real implementations
- must work with real computers
- must handle degenerate inputs

The exact geometric computation paradigm

Commit to return the mathematical true result
- Model REAL-RAM, if necessary
- But: Use numerical calculations whenever possible (controlled approximation)
- (Mainly) solved for straight line objects (LEDA, CGAL)
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What is Curve Analysis?

Algebraic curve given as zero locus of polynomial \( f \in \mathbb{R}[x, y] \)

Example

\[
f = 2x^4 + y^4 - x^3 + xy^2
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Curve analysis:
- Detect event points
- Count incident arcs to the left and to the right
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The Projection Approach

Example

1. Find (finitely many) $x$-values $\alpha$ where event points can occur. $\alpha$ is a root of $R := \text{res}(f, \frac{\partial f}{\partial y}, y)$

2. Analyse the curve along each such value, i.e. we have to analyse $f_\alpha(y) := f(\alpha, y)$ and $f_{\alpha \pm \varepsilon}(y)$.
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Definition of Genericity

Definition

A curve is in generic position, if it has

- no vertical asymptote and
- no covertical event points
Analysis on a Certain X-Value

For a curve \( f \) and some \( \alpha \in \mathbb{R} \)

1. Count \# points on the curve with \( x \)-value \( \alpha \)

2. Ensure genericity on \( \alpha \) and compute a candidate index \( i \), all points except the \( i \)th are non-event points
A Closer Look on the Algorithm

Count \# points on the curve with x-value α (\# distinct real roots of \( f_α(y) \)) and compute \( \deg(\gcd(f_α(y), f_α(y)')) \)

- Exact calculation on the real number
- Uses Sturm-Habicht sequences

Ensure genericity on α and compute a candidate index \( i \)

- Uses controlled approximation of α
- Bitstream Descartes method

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Completing the Analysis

Count the incident arcs for any point at $\alpha \in \mathbb{R}$:

- Count the number of points at $\alpha - \varepsilon$ and $\alpha + \varepsilon$
- Connect simple arcs
- Connect remaining arcs with the candidate
- Decide whether the candidate is an event point or not
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If the Curve is Not Generic...

Shear of a curve
Transform the curve such that covertical points become non-covertical

Example

Formally, the transformation is

\[ f \mapsto f(x + sy, y) \]

with \( s \in \mathbb{R} \).
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Arrangements of Algebraic Curves

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- curve analysis (Event points)
- curve-pair analysis (Intersection points)
- Transform into combinatorial object
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Visualisation of Algebraic Curves

Exact Visualisation of single arcs

- user is provided with topological information (the incidence graph)
- selected arcs are drawn exactly with respect to a fixed resolution

Examples

An implicit curve of degree 10
Our solution for curve analysis

- improves previous approaches by a combination of exact and approximated computations (as shown).
- does not impose any genericity condition on the input curve.

Details in:

Projection phase

The algorithm

- Compute $R := \text{res}(f, \frac{df}{dy}, y) \in \mathbb{Z}[x]$.
- Make $R$ square free, i.e. divide through $\gcd(R, R')$.
- Isolate the real roots of $R$, using the Descartes method.

Isolating interval representation

Each root is given as $\alpha = (R, I)$, where $R \in \mathbb{Z}[x]$ with $R(\alpha) = 0$ and $I$ some interval containing $\alpha$ and no other root of $R$. 

Michael Sagraloff (MPI AG1)  
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Extension phase: Symbolic precomputation

Fix some $x$-coordinate $\alpha$. Set $f_\alpha(y) := f(\alpha, y)$

Compute the following two integers:

- $m = \# \{ \beta \in \mathbb{R} \mid f_\alpha(\beta) = 0 \}$, the number of curve points over $\alpha$.
- $k = \deg(\gcd(f_\alpha, f'_\alpha))$

Both numbers are computed with Sturm-Habicht sequences.
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Extension phase: Root isolation over x-coordinates

First idea
Use Descartes method for $f_\alpha$.

Problem
Involves calculation with algebraic numbers in each substep

Second idea
Use the Bitstream Descartes method

Problem
$f_\alpha$ is not square free in general

Third (and final) idea
Modify the (Bitstream) Descartes method s.t. it can handle one multiple root, and detect “worse situations”
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The m-k-Descartes method

Known:
- $m$, the number of curve points over $\alpha$.
- $k = \deg(\gcd(f_\alpha, f'_\alpha))$

Interrupt the Descartes method in two situations

“The $m$-case” (success)
Stop, if $m - 1$ simple roots plus one interval that has more than one sign variation are detected.

“The $k$-case” (failure)
Stop, if no interval counts more than $k$.

Theorem
The m-k-Descartes algorithm terminates, and if the curve is generic, it isolates the roots over each $\alpha$ successfully.
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Real Root Isolation (Descartes method)

Theorem (Descartes’ rule of sign in Bernstein basis)

Let \( g \in \mathbb{R}[t] \), \( \deg g = n \) and \( g = \sum b_i B_i[c, d] \), where \( B_i[c, d] \) are the Bernstein polynomials of degree \( n \) for the interval \( [c, d] \). The number of sign variations of the sequence \( b_i \) exceeds the number of real roots of \( g \) inside \( [c, d] \), counted with multiplicities, by an even number.

We have a function \( \text{Desc} : \mathbb{R}[x] \times (\mathbb{R} \times \mathbb{R}) \to \mathbb{Z} \) where \( \text{Desc}(g, I) \) returns the number of real roots of \( g \) in \( I \), possibly overestimated by an even number.

- \( \text{Desc}(g, I) = 0 \Rightarrow \) no root in \( I \)
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Sturm-Habicht sequences

Subresultants

- $\text{Sres}_0(f, g), \ldots, \text{Sres}_n(f, g)$ \textit{Subresultant sequence} of $f$ and $g$.
- Definition over minors of the Sylvester matrix
- $\text{res}(f, g) = \text{Sres}_0(f, g)$
- Contain polynomials of the Euclidean remainder sequence of $f$ and $g$ (up to scalar)

Sturm-Habicht-sequences

- $\text{StHa}_i(f) := (-1)^\delta \text{Sres}_i(f, f')$
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- Allows to compute the total number of real roots
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