Fast and Exact Geometric Analysis of Real Algebraic Plane Curves

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7th February 2007 / signature franco allemande

Outline

Exact Geometric Computation

Geometric algorithms often

- described in REAL-RAM
- assume non-degeneracy

Real implementations

- must work with real computers
- must handle degenerate inputs

The exact geometric computation paradigm

Commit to return the mathematical true result

- Model REAL-RAM, if necessary
- But: Use numerical calculations whenever possible (controlled approximation)
- (Mainly) solved for straight line objects (LEDA,CGAL)

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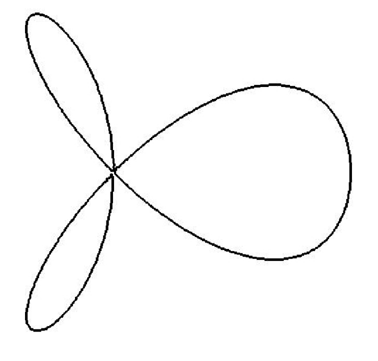
Algebraic curve given as zero locus of polynomial $f \in \mathbb{R}[x, y]$

Example

$f = 2x^4 + y^4 - x^3 + xy^2$



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- Detect event points
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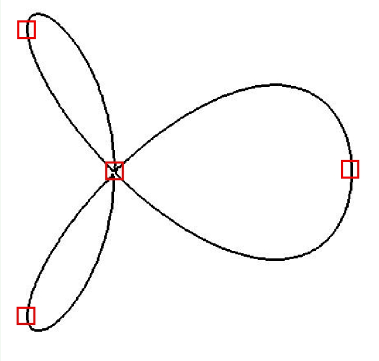

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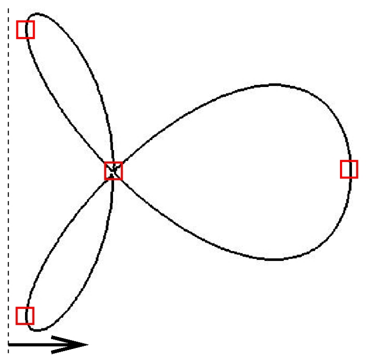

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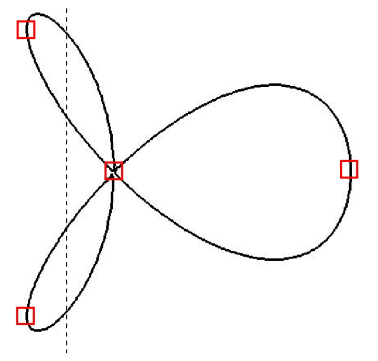

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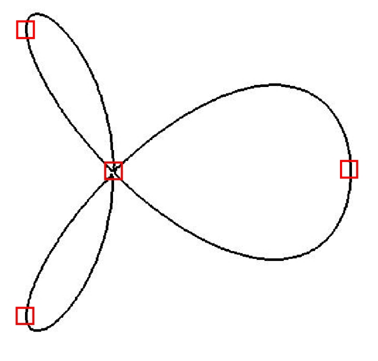

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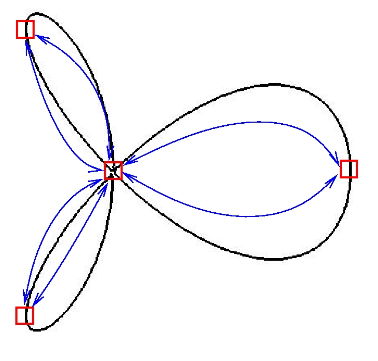

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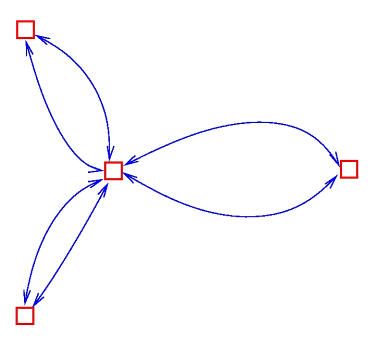

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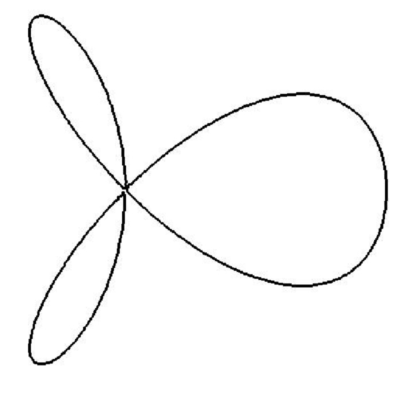
The Projection Approach

Example



1. Find (finitely many) x-values α where event points can occur. α is a root of $R := \mathrm{res}(\mathrm{f}, \frac{\partial \mathrm{f}}{\partial \mathrm{y}}, \mathrm{y})$
2. Analyse the curve along each such value, i.e. we have to analyse $f_\alpha(y) := f(\alpha, y)$ and $f_{\alpha\pm\varepsilon}(y)$.

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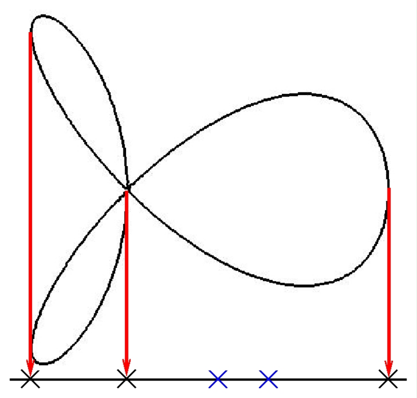

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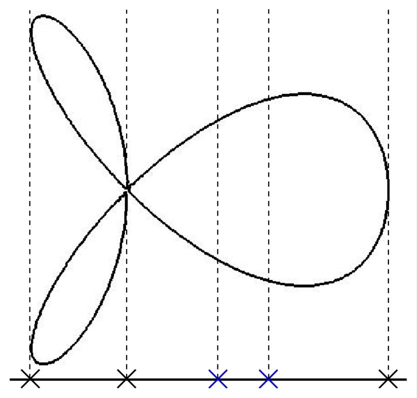

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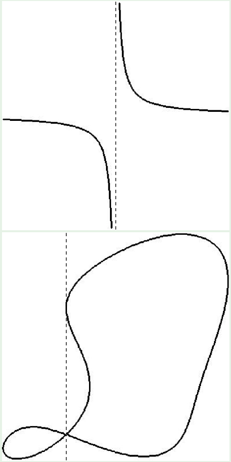
Definition of Genericity

Definition

A curve is in generic position, if it has

- no vertical asymptote and
- no covertical event points

Non-generic:

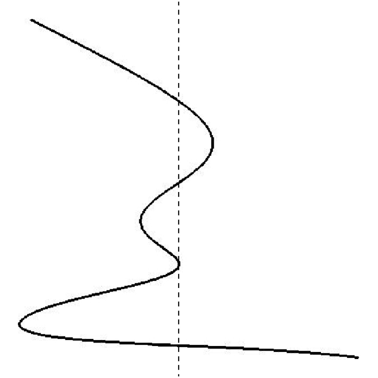


Analysis on a Certain X-Value

For a curve f and some $\alpha \in \mathbb{R}$

1. Count # points on the curve with x-value α
2. Ensure genericity on α and compute a candidate index i, all points except the ith are non-event points

Example



A Closer Look on the Algorithm

Count # points on the curve with x-value α (# distinct real roots of $f_\alpha(y)$) and compute $\deg(\gcd(f_\alpha(y), f_\alpha(y)'))$

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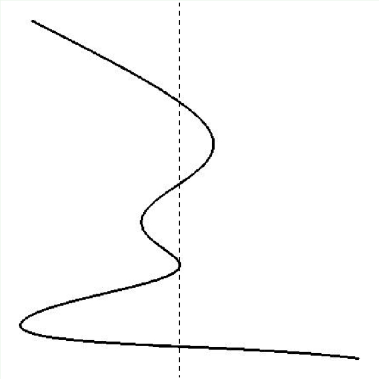

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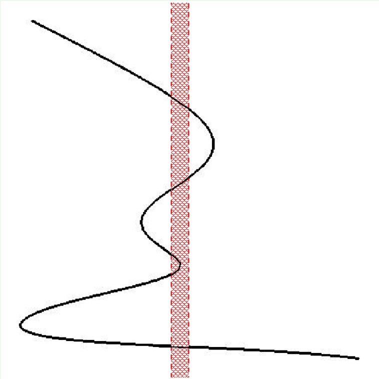

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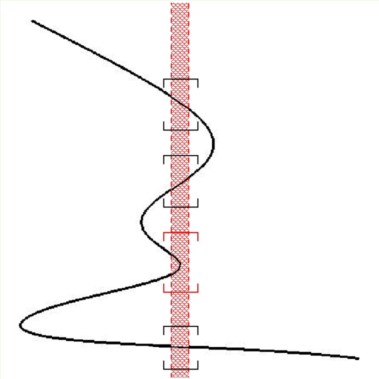
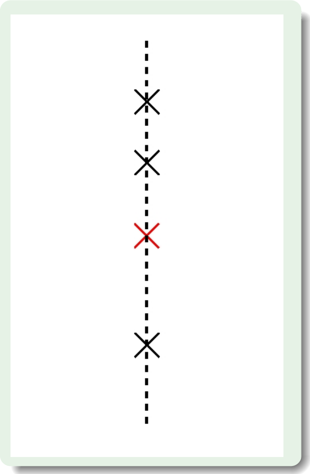


Completing the Analysis

Count the incident arcs for any point at $\alpha \in \mathbb{R}$:



- # of points and index of the candidate
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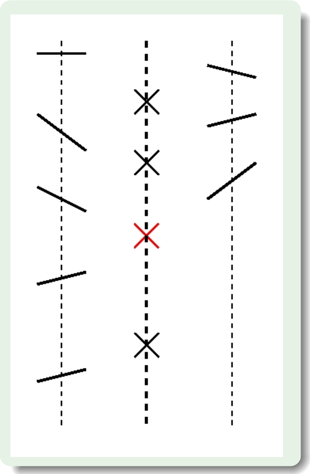

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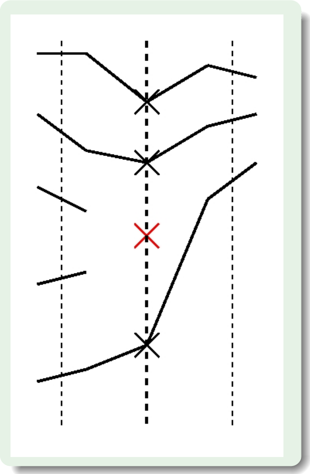

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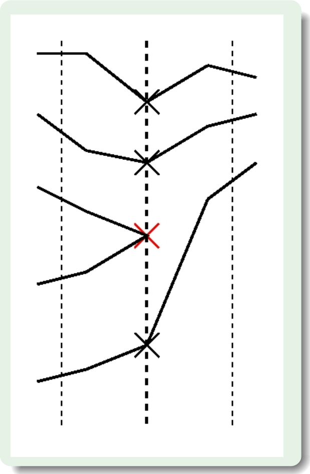

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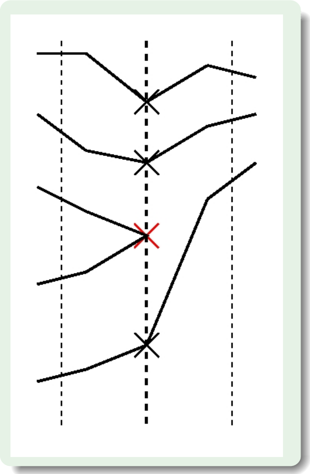

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If the Curve is Not Generic...

Shear of a curve

Transform the curve such that covertical points become non-covertical

Example

Formally, the transformation is

$$f \mapsto f(x + sy, y)$$

with $s \in \mathbb{R}$.

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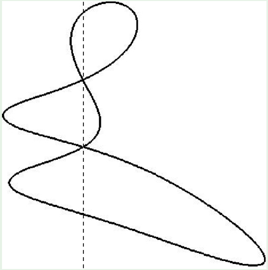

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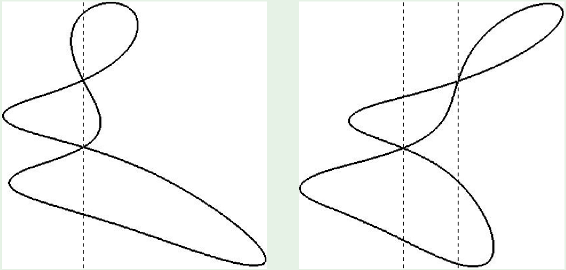

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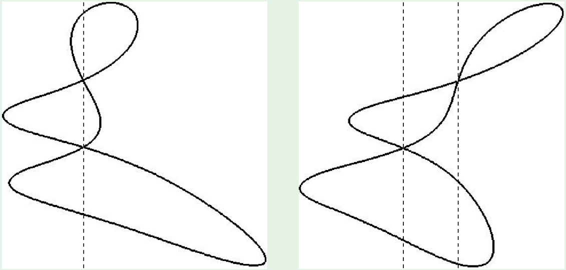

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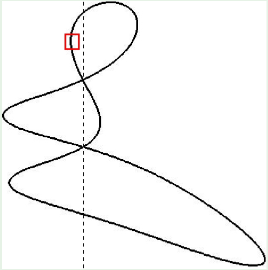

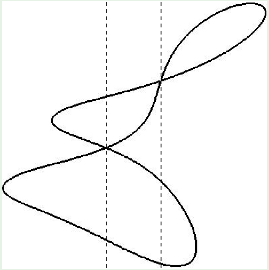

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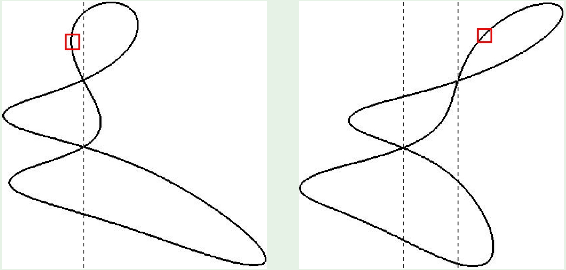

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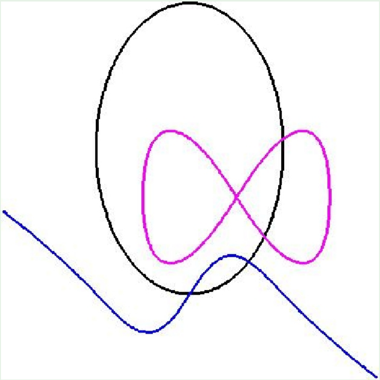

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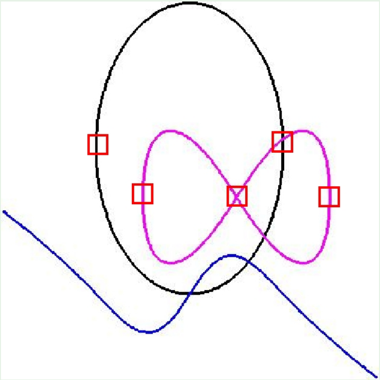

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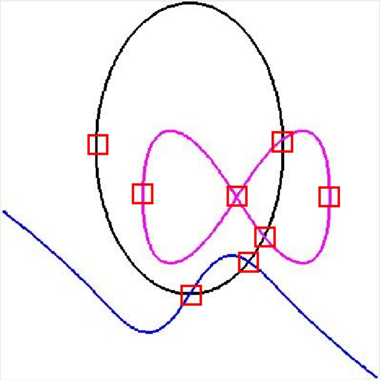

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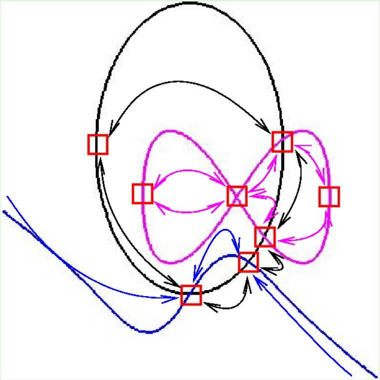

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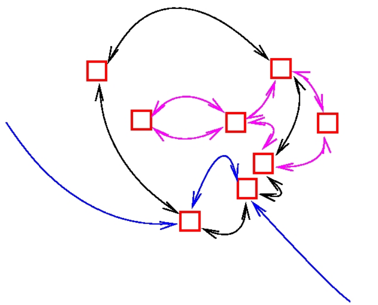

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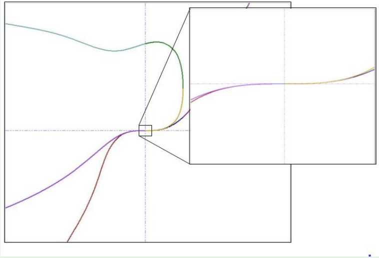
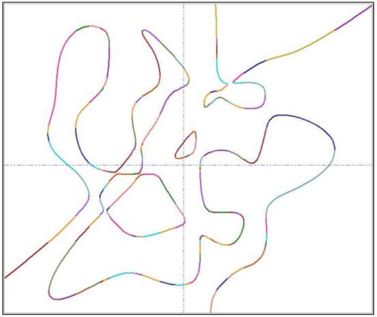
Visualisation of Algebraic Curves

Exact Visualisation of single arcs

- user is provided with topological information (the incidence graph)
- selected arcs are drawn exactly with respect to a fixed resolution

Examples





An implicit curve of degree 10

Summary

Our solution for curve analysis

- improves previous approaches by a combination of exact and approximated computations (as shown).
- does not impose any genericity condition on the input curve.

Details in:

Arno Eigenwillig, Michael Kerber and Nicola Wolpert: Fast and Exact Geometric Analysis of Real Algebraic Plane Curves. submitted, Saarbrücken, January 2007

Projection phase

The algorithm

- Compute $R := \mathrm{res}(\mathrm{f}, \frac{\partial \mathrm{f}}{\partial \mathrm{y}}, \mathrm{y}) \in \mathbb{Z}[\mathrm{x}]$.
- Make R square free, i.e. divide through $\gcd(R, R')$.
- Isolate the real roots of R, using the Descartes method.

Isolating interval representation

Each root is given as $\alpha = (R, I)$, where $R \in \mathbb{Z}[x]$ with $R(\alpha) = 0$ and I some interval containing α and no other root of R.

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Extension phase: Symbolic precomputation

Fix some x-coordinate α. Set $f_\alpha(y) := f(\alpha, y)$

Compute the following two integers:

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Extension phase: Root isolation over x-coordinates

First idea

Use Descartes method for f_α.

Problem

Involves calculation with algebraic numbers in each substep

Second idea

Use the Bitstream Descartes method

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f_α is not square free in general

Third (and final) idea

Modify the (Bitstream) Descartes method s.t. it can handle one multiple root, and detect "worse situations"

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The m-k-Descartes method

Known:

- m, the number of curve points over α.
- $k = \deg(\gcd(f_\alpha, f'_\alpha))$

Interrupt the Descartes method in two situations

"The m-case" (success)

Stop, if $m-1$ simple roots plus one interval that has more than one sign variation are detected.

"The k-case" (failure)

Stop, if no interval counts more than k.

Theorem

The m-k-Descartes algorithm terminates, and if the curve is generic, it isolates the roots over each α successfully.

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Known: Isolating intervals for the roots, plus one "distinguished" root.

- Compute the number of roots at $\alpha - \varepsilon$ and $\alpha + \varepsilon$
- Connect the arcs in the only possible way.

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- Compute the number of roots at $\alpha - \varepsilon$ and $\alpha + \varepsilon$
- Connect the arcs in the only possible way.

For that argument, we need *both* genericity assumptions.

Counting incident arcs

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Real Root Isolation (Descartes method)

Theorem (Descartes' rule of sign in Bernstein basis)

Let $g \in \mathbb{R}[t]$, $\deg g = n$ and $g = \sum b_i B_i[c,d]$, where $B_i[c,d]$ are the Bernstein polynomials of degree n for the interval $[c,d]$. The number of sign variations of the sequence b_i exceeds the number of real roots of g inside $[c,d]$, counted with multiplicities, by an even number.

We have a function $\mathrm{Desc} : \mathbb{R}[x] \times (\mathbb{R} \times \mathbb{R}) \to \mathbb{Z}$ where $\mathrm{Desc}(g, I)$ returns the number of real roots of g in I, possibly overestimated by an even number.

- $\mathrm{Desc}(g, I) = 0 \Rightarrow$ no root in I
- $\mathrm{Desc}(g, I) = 1 \Rightarrow$ exactly one (simple) root in I
- $\mathrm{Desc}(g, I) \geq 2 \Rightarrow$ nothing

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Sturm-Habicht sequences

Subresultants

- $\mathrm{Sres}_0(f,g),\ldots,\mathrm{Sres}_n(f,g)$ *Subresultant sequence* of f and g.
- Definition over minors of the Sylvester matrix
- $\mathrm{res}(f,g) = \mathrm{Sres}_0(f,g)$
- Contain polynomials of the Euclidean remainder sequence of f and g (up to scalar)

Sturm-Habicht-sequences

- $\mathrm{StHa}_i(f) := (-1)^{\delta}\mathrm{Sres}_i(f,f')$
- Negations to create a "Sturm-like" sequence
- Allows to compute the total number of real roots
- Good specialisation properties

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