# Computing an ordinary line in linear time*

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**Abstract**

Given a set $P$ of $n$ points in the plane, an ordinary line is a line which passes through exactly two points of $P$. We propose a simple algorithm to compute such a line in randomized expected linear time which improve previous $O(n \log n)$ result.

## 1 Introduction

Warning : Linear randomized solution is false.

[?]

## 2 Basic definition and previous results

## 3 Randomized algorithm

**Lemma 1 ([1])** *If at most $n-k$ points are colinear among $n$ points and*

$$n \geq \frac{1}{2}(3(3k-2)^2 + 3k - 1)$$

*then the number of lines (non necessary ordinary) is at least*

$$kn - \frac{1}{2}(3k+2)(k-1)$$

.

IMPORTANT REMARK : Unfortunately, this lemma bounds the total number of lines and not the number of ordinary lines. Thus the randomized approach I proposed last week and described below do not work.

**Algorithm**

$L' = \emptyset$

choose $p$, $q$ random in $P$

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while true
$L$ = points of $P$ colinear to $p$ and $q$.
if $|L| = 2$ return $(pq)$
if $|L| > |P|/2$ $L' = L$
choose $p$ random in $P$
if $p \in L$ choose $q$ random in $P \setminus L'$
else choose $q$ random in $P$
continue

**Lemma 2** *Above algorithm compute an ordinary line in a set of $n$ points in expected randomized linear time.*

**Proof:** The analysis is as follows: assume that the line containing the most points of $P$ contains $n-k$ points.

If $k > \frac{n}{2}$ then there is at least $O(n^2)$ ordinary lines [1], thus the probability than a random picking of $p$ and $q$ will work is lower bounded by a constant and thus the expected running time of the algorithm is $O(n)$.

If $k < \frac{n}{2}$ then there is at least $O(nk)$ ordinary lines [1]. the probability that we pick the line $L_0$ containing the $n-k$ points is $\frac{\begin{pmatrix} 2 \\ n-k \end{pmatrix}}{\begin{pmatrix} 2 \\ n \end{pmatrix}}$, once this line is picked, the number of possible trials goes down to $k^2 + (n-k)n < nk + k^2 < 2nk$ thus the probability that the second trial succeeds is constant and the complexity is also $O(n)$. ■

# 4 Remarks

Finding a deterministic algorithm is difficult

**Lemma 3** *There exists three points which none of them support an ordinary line.*

**Proof:** Consider a set of points $\mathcal{S}$ and $p_1, p_2, p_3 \in \mathcal{S}$ such that $(p_1p_2)$, $(p_2p_3)$ and $(p_3p_1)$ are ordinary. For each ordinary line through $p_i$ add the intersection point of that line and line $(p_jp_k)$ $(i \neq j \neq k \neq i)$, if they are parallel add another point $q$ on the ordinary line such that $(p_jq)$ non parallel to $(p_ip_k)$ and $(p_kq)$ non parallel to $(p_ip_j)$ (see Figure 1). ■

**Lemma 4** *There exists three colinear points which none of them support an ordinary line.*

**Proof:** There is a set of 11 points as described on the left of Figure 2 such that the three grey points do not support any ordinary line. The size of the set

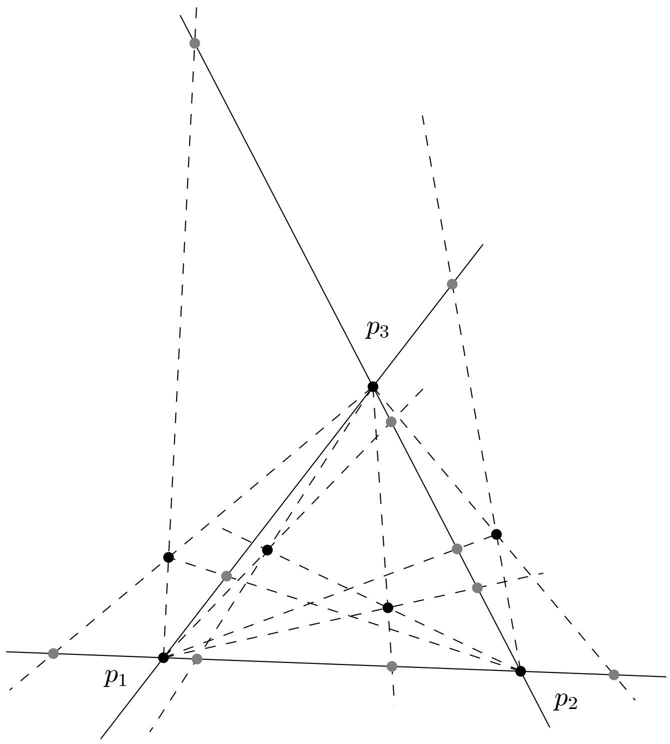


Figure 1: 3 points non supporting an ordinary line

can reached a bigger size by creating copies of the original set through affine transformation which let the three relevant points invariant as shown on the right of Figure 2. ■

**Lemma 5** *If the line trough two points $p$ and $q$ of $\mathcal{S}$ contains half of the points or more, then an ordinary line can be found in $O(n)$ time*

**Proof:** Let $L$ be the line $(pq)$ and $\mathcal{S}'$ the points on $L$. A projective transformation, sending a point close enough to $L$ at infinity sends $L$ on a cupporting line of the convex hull, thus we assume without lost of generality than $\mathcal{S}'$ is on an horizontal edge of the convex hull. Now we determine $r$ the closest point to $L$ in $\mathcal{S} \setminus \mathcal{S}'$. Since $\mathcal{S}'' = \mathcal{S} \setminus \mathcal{S}' \setminus \{r\}$ as less points than $\mathcal{S}'$ at least one of line through a point of $\mathcal{S}'$ and $r$ is ordinary. We find the median $m$ of $\mathcal{S}'$ which split $\mathcal{S}'$ in $\{m\}$, $\mathcal{S}'_1$ and $\mathcal{S}'_2$, we then compare all points of $\mathcal{S}''$ with line $(mr)$ which splits $\mathcal{S}''$ in points on $(mr)$, $\mathcal{S}''_1$ and $\mathcal{S}''_2$ then either
– $(mr)$ is ordinary and we are done
– $|\mathcal{S}''_1| < |\mathcal{S}'_1|$ and we recurse on $\mathcal{S}''_1, \mathcal{S}'_1$.
– $|\mathcal{S}''_2| < |\mathcal{S}'_2|$ and we recurse on $\mathcal{S}''_2, \mathcal{S}'_2$.

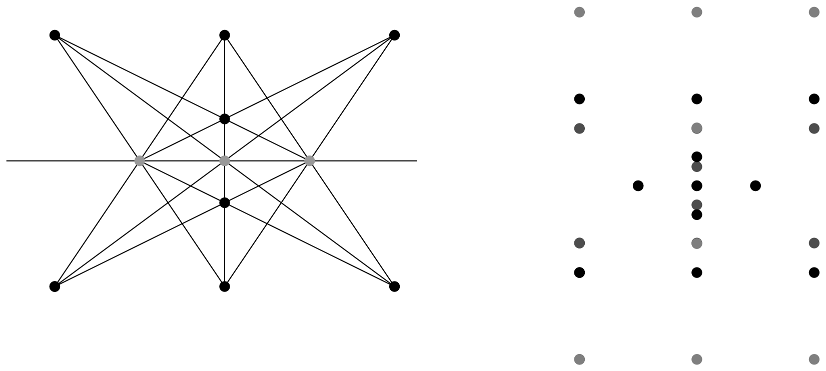

Figure 2: 3 colinear points non supporting an ordinary line

The complete algorithm ends in linear time. ■

**Lemma 6** *Given a set of points, determining if there exists a line containing half of the points or more and finding this line can be done in $O(n)$ time*

**Proof:** Using $\epsilon$ cuttings. Herv is responsible of that proof. ■

# References

[1] L. Kelly and W. Moser. On the number of ordinary lines determined by $n$ points. *Canad. J. Math.*, 10:210–219, 1958.

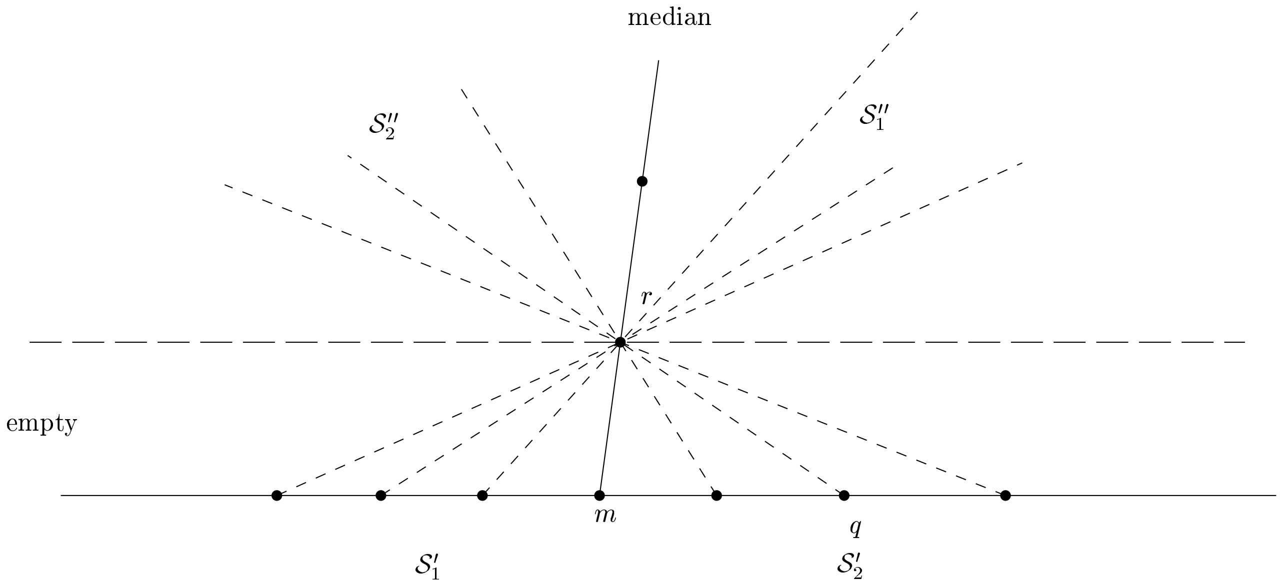


Figure 3: Finding an ordinary line when half of the points are colinear