Disjoint Unit Spheres Admit At Most Two Line Transversals

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Abstract

We show that a set of n disjoint unit spheres in $\mathbb{R}^d$ admits at most *two* distinct geometric permutations, or line transversals, if n is large enough. This bound is optimal.

1 Introduction

A line ℓ is a *line transversal* for a set $\mathcal{S}$ of pairwise disjoint convex bodies in $\mathbb{R}^d$ if it intersects every element of $\mathcal{S}$. A line transversal defines two linear orders on $\mathcal{S}$, namely the order in which ℓ intersects the bodies, where we can choose to orient ℓ in two directions. Since the two orders are essentially the same (one is the reverse of the other), we consider them as a single *geometric permutation*.

Bounds on the maximum number of geometric permutations were established about a decade ago: a tight bound of $2n-2$ is known for $d = 2$ [2], for higher dimension the number is in $\Omega(n^{d-1})$ [5] and in $O(n^{2d-2}$ [10]. The gap was closed for the special case of spheres by Smorodinsky et al. [8], who showed that n spheres in $\mathbb{R}^d$ admit $\Theta(n^{d-1})$ geometric permutations. This result can be generalized to "fat" convex objects [7].

The even more specialized case of congruent spheres was treated by Smorodinsky et al. [8] and independently by Asinowski [1]. They proved that n unit circles in $\mathbb{R}^2$ admit at most two geometric permutations if n is large enough (the proof by Asinowski holds for all $n \geq 4$). Zhou and Suri established an upper bound of 16 for all d and n sufficiently large, a result quickly improved by Katchalski, Suri, and Zhou [6] and independently by Huang, Xu, and Chen [4] to 4.

Katchalski et al. show that for n large enough, two line transversals can make an angle of at most $O(1/n)$ with each other, so all line transversals are "essentially" parallel. They define a *switched pair* to be pair of spheres (A, B) such that there are two line transversals ℓ and ℓ' (for all n spheres) where ℓ visits A before B, while ℓ' visits B before A. Katchalski et al. prove that any sphere can participate in at most one switched pair, and that the two spheres forming a switched pair must appear consecutively in any geometric perturbation of the set. It follows that any two geometric perturbations differ only in that the elements of some switched pair may have been exchanged. Katchalski et al.'s main result is that there are at most two switched pairs in a set of n disjoint unit spheres, implying the bound of four geometric permutations.

We show that in fact there cannot be more than one switched pair. This implies that a set of n disjoint unit spheres admits at most two geometric perturbations, which differ only by the swapping of two adjacent elements. Since there are arbitrarily large sets of unit spheres in $\mathbb{R}^d$ with one switched pair, this bound is optimal.

The study of geometric permutations is motivated mostly by theoretical interest, yet there is an important application to computer graphics. In various applications, space is decomposed into boxes, and light can pass only through *portals*, that is openings between the boxes (such as doors,

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windows, and stairwell openings in building models) [9]. For a box to be visible from another box, light has to pass through all the portals in-between, and if no line transversals for these portals exists, then the two boxes are mutually invisible.

Surveys of geometric transversal theory are Goodman et al. [3] and Wenger [11].

2 The proof

A *unit sphere* is a sphere of radius 1. We say that two unit spheres are *disjoint* if their interiors are (in other words, we allow the spheres to touch). A line *stabs* a sphere if it intersects the closed sphere (and so a tangent to a sphere stabs it). A *line transversal* for a set of disjoint unit spheres is a line that stabs all the spheres, with the restriction that it is not allowed to be tangent to two spheres in a common point.

Given two disjoint unit spheres A and B, let $g(A, B)$ be their center of gravity and $\Pi(A, B)$ be their bisecting hyperplane. If the centers of A and B are a and b, then $g(A, B)$ is the mid-point of a and b, and $\Pi(A, B)$ is the hyperplane through $g(A, B)$ orthogonal to the line ab.

The first repeat two lemmas that were proven by Katchalski et al. [6].

Lemma 1 ([6] Lemma 2.3) *Let ℓ and ℓ' be two different line transversals of a set $\mathcal{S}$ of n disjoint unit spheres in $\mathbb{R}^d$. Then the angle between the direction vectors of ℓ and ℓ' is $O(1/n)$.*

Lemma 2 ([6] Lemmas 3.1 and 3.2) *Let $\mathcal{S}$ be a set of n disjoint unit spheres in $\mathbb{R}^d$, for n large enough. Assume there are three line transversals for $\mathcal{S}$ that realize three different geometric permutations. Then there exist two line transversals ℓ and ℓ' and four spheres A, B, C, D in $\mathcal{S}$ such that ℓ stabs them in the order $ABCD$ and ℓ' stabs them in the order $BADC$, and*

$$d(g(A, B), g(C, D)) \geq \sqrt{2} - \varepsilon,$$

for arbitrarily small $\varepsilon > 0$.

We start with a warm-up lemma in two dimensions.

Lemma 3 *Let S and T be two unit-radius disks in $\mathbb{R}^2$ with centers $(-\lambda, 0)$ and $(\lambda, 0)$, where $\lambda \geq \cos\beta$ for some angle β with $0 < \beta \leq \pi/2$. Then $S \cap T$ is contained in the ellipse*

$$\left(\frac{x}{\sin^2\beta}\right)^2 + \left(\frac{y}{\sin\beta}\right)^2 \leq 1.$$

Proof. Let $(\mu, 0)$ and $(0, \nu)$ be the rightmost and topmost point of $S \cap T$ (see Figure 1). Consider the ellipse E defined as

$$(\frac{x}{\mu})^2 + (\frac{y}{\nu})^2 \leq 1.$$

E intersects the boundary of S in $(0, \nu)$ and $(0, -\nu)$, and is tangent to it in $(\mu, 0)$. An ellipse can intersect a circle in at most four points and the tangency counts as two intersections, and so there is no further intersection between the two curves. Since $(-\mu, 0)$ lies on E and inside S, the part of E between $(0, -\nu)$ and $(0, \nu)$ lies outside S, and $S \cap T$ is contained in E. It remains to observe that

$$\nu^2 = 1 - \lambda^2 \leq 1 - \cos^2\beta = \sin^2\beta,$$

so $\nu \leq \sin\beta$, and

$$\mu = 1 - \lambda \leq 1 - \cos\beta \leq 1 - \cos^2\beta = \sin^2\beta,$$

which proves the lemma. ◻

We now show that a transversal for two spheres cannot pass too far from their common center of gravity.

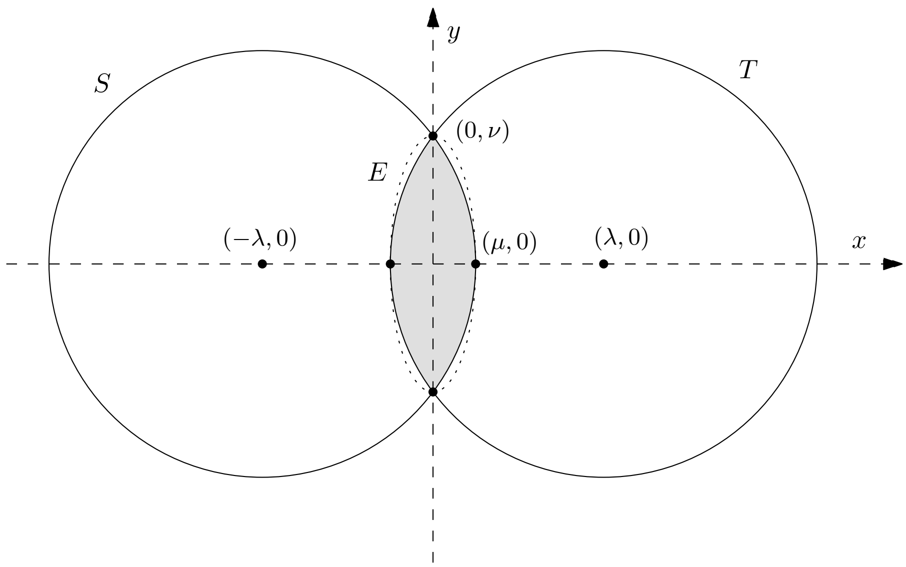

Figure 1: The intersection of two disks is contained in an ellipse.

Lemma 4 *Given two disjoint unit spheres A and B in $\mathbb{R}^d$ and a line ℓ stabbing both spheres, let p be the point of intersection of ℓ and $\Pi(A,B)$, and let β be the angle between ℓ and $\Pi(A,B)$. Then*

$$d(p, g(A,B)) \leq \sin\beta.$$

Proof. Let a and b be the centers of A and B and let v be the direction vector of ℓ, that is, ℓ can be written as $\{p + \lambda v \mid \lambda \in \mathbb{R}\}$. We first argue that proving the lemma for $d = 3$ is sufficient. Indeed, assume $d > 3$ and consider the 3-dimensional subspace Γ containing ℓ, a, and b. Since we have $d(a,\ell) \leq 1$ and $d(b,\ell) \leq 1$, the line ℓ stabs the 3-dimensional unit spheres $A \cap \Gamma$ and $B \cap \Gamma$. And since $\pi/2 - \beta$ is the angle between two vectors in Γ, namely v and $b - a$, β is also the angle between ℓ and the two-dimensional plane $\Pi(A \cap \Gamma, B \cap \Gamma)$. So if the lemma holds in Γ, then it also holds in $\mathbb{R}^d$.

In the rest of the proof we can therefore assume that $d = 3$. We choose a coordinate system where $a = (0,0,-\rho)$, $b = (0,0,\rho)$ with $\rho \geq 1$, and $v = (\cos\beta, 0, \sin\beta)$. Then $\Pi := \Pi(A,B)$ is the xy-plane and $g := g(A,B) = (0,0,0)$. Consider the cylinders $\mathrm{cyl}(A) := \{u + \lambda v \mid u \in A, \lambda \in \mathbb{R}\}$ and $\mathrm{cyl}(B)$. Since ℓ stabs A and B, we have $p \in \mathrm{cyl}(A) \cap \mathrm{cyl}(B) \cap \Pi$.

The intersection $B' := \mathrm{cyl}(B) \cap \Pi$ is the ellipse (see Figure 2)

$$\sin^2\beta(x + \frac{\rho}{\tan\beta})^2 + y^2 \leq 1,$$

and symmetrically $A' := \mathrm{cyl}(A) \cap \Pi$ is

$$\sin^2\beta(x - \frac{\rho}{\tan\beta})^2 + y^2 \leq 1.$$

If we let τ be the linear transformation

$$\tau(x,y) = (x\sin\beta, y),$$

then $\tau(A')$ and $\tau(B')$ are unit-radius disks with centers $(\rho\cos\beta, 0)$ and $(-\rho\cos\beta, 0)$. By Lemma 3, the intersection $\tau(A' \cap B')$ is contained in the ellipse

$$\Big(\frac{x}{\sin^2\beta}\Big)^2 + \Big(\frac{y}{\sin\beta}\Big)^2 \leq 1.$$

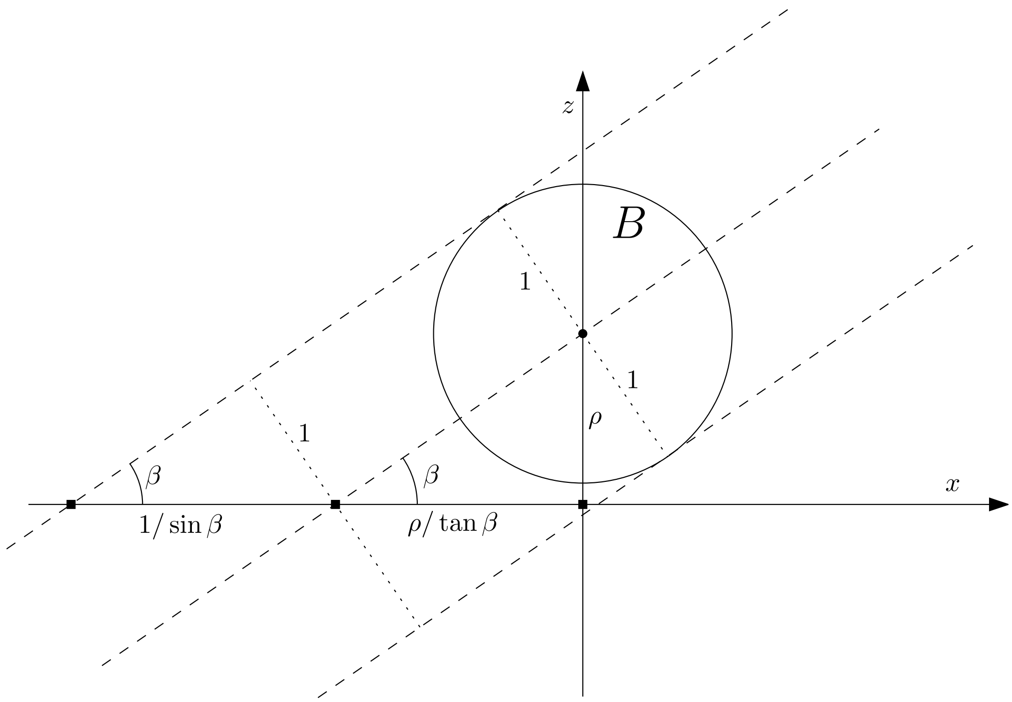

Figure 2: The intersection of the cylinder with the xy-plane is an ellipse.

Applying τ^{-1} we find that $A' \cap B'$ is contained in the circle with radius $\sin\beta$ around g. Since $p \in A' \cap B'$, the lemma follows. ◻

We now prove our key lemma.

Lemma 5 *Let A, B, C, D be four spheres from a set $\mathcal{S}$ of n disjoint unit spheres in $\mathbb{R}^d$, for n large enough. Assume there are two line transversal ℓ and ℓ' for $\mathcal{S}$, such that ℓ stabs the four spheres in the order $ABCD$, and ℓ' stabs them in the order $BADC$. Then*

$$d(g(A,B), g(C,D)) < 1 + \varepsilon,$$

for arbitrarily small $\varepsilon > 0$.

Proof. Let $\Pi_1 := \Pi(A,B)$, $\Pi_2 = \Pi(C,D)$, $g_1 := g(A,B)$, and $g_2 := g(C,D)$. We choose a coordinate system where Π_1 is the hyperplane $x_1 = 0$, and the intersection $\Pi_1 \cap \Pi_2$ is the subspace $x_1 = x_2 = 0$. We can make this choice such that the x_1-coordinate of the center of A is < 0, and that the x_2-coordinate of the center of C is less than the x_2-coordinate of the center of D. We can also assume that the x_2-coordinate of g_1 is ≥ 0 (otherwise we swap A with B, C with D, and ℓ with ℓ'). Figure 3 shows the projection of the situation on the x_1x_2-plane.

Let $p_i := \ell \cap \Pi_i$, $p'_i := \ell' \cap \Pi_i$, let β_i be the angle between ℓ and Π_i, and let β'_i be the angle between ℓ' and Π_i. By Lemma 1 we have $\beta_i, \beta'_i \in O(1/n)$.

Since ℓ stabs A before B and C before D, it intersects Π_1 from bottom to top, and Π_2 from left to right. The segment p_1p_2 therefore lies in the top-left quadrant of Figure 3. On the other hand, ℓ' stabs B before A and D before C, so it intersects Π_1 from top to bottom, and Π_2 from right to left, and the segment $p'_1p'_2$ lies in the bottom-right quadrant of the figure.

Let now $t := d(p_1, p_2)$ and $t' := d(p'_1, p'_2)$. Lemma 4 implies

$$d(g_1, g_2) \leq d(g_1, p_1) + d(p_1, p_2) + d(p_2, g_2) \leq \sin\beta_1 + t + \sin\beta_2 \leq t + O(1/n),$$

and similarly

$$d(g_1, g_2) \leq d(g_1, p'_1) + d(p'_1, p'_2) + d(p'_2, g_2) \leq \sin\beta'_1 + t' + \sin\beta'_2 \leq t' + O(1/n),$$

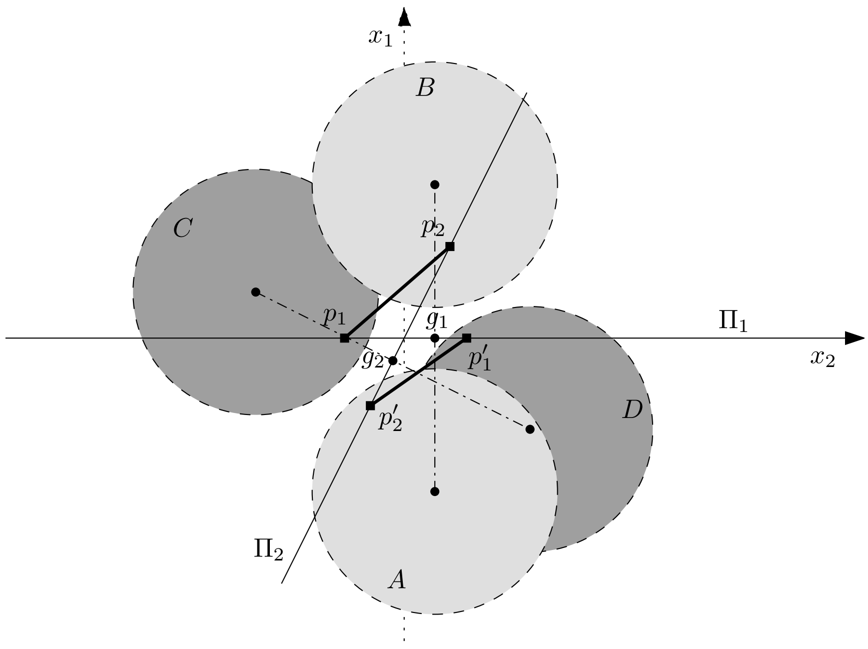

Figure 3: The two hyperplanes define four quadrants

and so

$$d(g_1, g_2) \leq O(1/n) + \min\{t, t'\}.$$

It remains to prove that $\min\{t, t'\} \leq 1$. Let u_1 (u'_1) be the orthogonal projection of p_1 (p'_1) on Π_2, u_2 (u'_2) the orthogonal projection of p_2 (p'_2) on Π_1. Consider the rectangular triangle $p_1u_2p_2$. We have $\angle u_2p_1p_2 = \beta_1$, and so

$$t \sin\beta_1 = d(p_2, u_2) = d(p_2, \Pi_1). \tag{1}$$

Similarly, we can consider the rectangular triangles $p_2u_1p_1$, $p'_1u'_2p'_2$, and $p'_2u'_1p'_1$ to obtain

$$\begin{aligned} t \sin\beta_2 &= d(p_1, u_1) = d(p_1, \Pi_2), && (2)\\ t' \sin\beta'_1 &= d(p'_2, u'_2) = d(p'_2, \Pi_1), && (3)\\ t' \sin\beta'_2 &= d(p'_1, u'_1) = d(p'_1, \Pi_2). && (4) \end{aligned}$$

We now distinguish two cases.

The *first case* occurs if, as in the figure, the x_1-coordinate of g_2 is ≤ 0. By Lemma 4 we have $d(p_2, g_2) \leq \sin\beta_2$. Since p_2 and g_2 lie on opposite sides of Π_1, we have $d(p_2, \Pi_1) \leq \sin\beta_2$. Similarly, we have $d(p_1, g_1) \leq \sin\beta_1$, and p_1 and g_1 lie on opposite sides of Π_2, implying $d(p_1, \Pi_2) \leq \sin\beta_1$. Plugging into Eq. (1) and (2), we obtain

$$t \leq \min\left\{\frac{\sin\beta_2}{\sin\beta_1}, \frac{\sin\beta_1}{\sin\beta_2}\right\} \leq 1,$$

which proves the lemma for this case.

The *second case* occurs if the x_1-coordinate of g_2 is > 0. We let $s_1 := d(g_1, \Pi_2)$, and $s_2 := d(g_2, \Pi_1)$. Applying Lemma 4 , we then have

$$\begin{aligned} d(p_2, \Pi_1) &\leq d(p_2, g_2) + s_2 \leq \sin\beta_2 + s_2, && (5)\\ d(p_1, \Pi_2) &\leq d(p_1, g_1) - s_1 \leq \sin\beta_1 - s_1, && (6)\\ d(p'_2, \Pi_1) &\leq d(p'_2, g_2) - s_2 \leq \sin\beta'_2 - s_2, && (7)\\ d(p'_1, \Pi_2) &\leq d(p'_1, g_1) + s_1 \leq \sin\beta'_1 + s_1. && (8) \end{aligned}$$

Plugging Ineqs. (5) to (8) into (1) to (4), we obtain

$$t \leq \frac{\sin\beta_2 + s_2}{\sin\beta_1}, \tag{9}$$

$$t \leq \frac{\sin\beta_1 - s_1}{\sin\beta_2}, \tag{10}$$

$$t' \leq \frac{\sin\beta_2' - s_2}{\sin\beta_1'}, \tag{11}$$

$$t' \leq \frac{\sin\beta_1' + s_1}{\sin\beta_2'}. \tag{12}$$

We want to prove that $\min(t, t') \leq 1$. We assume the contrary. From $t > 1$ and Ineq. (10) we obtain

$$\sin\beta_2 < \sin\beta_1 - s_1,$$

and from $t' > 1$ and Ineq. (11) we get

$$\sin\beta_1' < \sin\beta_2' - s_2.$$

Plugging this into Ineq. (9) and (12) results in

$$\begin{aligned} t &\leq \frac{\sin\beta_2 + s_2}{\sin\beta_1} < \frac{\sin\beta_1 - s_1 + s_2}{\sin\beta_1} = 1 + \frac{s_2 - s_1}{\sin\beta_1}, \\ t' &\leq \frac{\sin\beta_1' + s_1}{\sin\beta_2'} < \frac{\sin\beta_2' - s_2 + s_1}{\sin\beta_2'} = 1 + \frac{s_1 - s_2}{\sin\beta_2'}. \end{aligned}$$

It follows that if $s_2 < s_1$ then $t < 1$, otherwise $t' \leq 1$. In either case the lemma follows. ◻

Theorem 1 *A set $\mathcal{S}$ of n disjoint unit spheres in $\mathbb{R}^d$, for n large enough, admits at most two different geometric permutations that can be realized with line transversals.*

Proof. Assume there are at least three different geometric permutations of $\mathcal{S}$. By Lemma 2, there exist two line transversals ℓ and ℓ' and four spheres A, B, C, D in $\mathcal{S}$ such that ℓ stabs them in the order $ABCD$ and ℓ' stabs them in the order $BADC$, and

$$d(g(A,B), g(C,D)) \geq \sqrt{2} - 1/5.$$

By Lemma 5, we also have

$$d(g(A,B), g(C,D)) < 1 + 1/5 < \sqrt{2} - 1/5,$$

a contradiction. The theorem follows. ◻

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