Drawing Maximal Outer Planar Graphs as Unit Distance Graphs

Barbados Workshop

March 17, 2005

Abstract

A fundamental problem for I/O efficient bio-informatics and homeland security.

1 results

Lemma 1. *Any maximal outer planar graph with maximum degree* 4 *can be realized as a unit distance graph.*

Proof. There is a unique maximal outer planar graph with n vertices and maximum degree at most 4. It can be embedded as represented in Figure 1. □

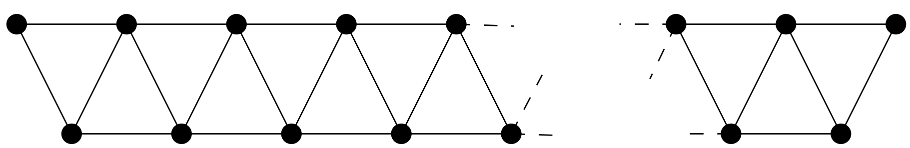

Figure 1: The maximal outer planar graph on n vertices with maximum degree 4.

Theorem 2. *There exist maximal outer planar graph with maximum degree* 9 *that cannot be realized as a unit distance graph.*

2 Basic properties

Lemma 3. *The angle between two edges* (a, b) *and* (a, c) *is at least* $\frac{\pi}{3}$ *if* b *and* c *are not connected.*

Lemma 4. *A graph containing* $K_{1,6}$ *as an induced subgraph cannot be drawn as a unit distance graph.*

Corrolary 5. *A maximal outer planar graph graph with a vertex of degree* 11 *or more cannot be drawn as a unit distance graph.*

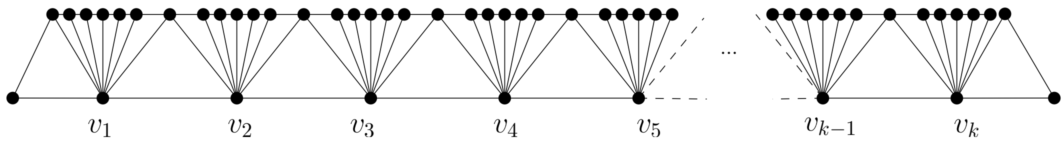

Figure 2: The graph $mop_9(k)$.

3 Construction

Let us denote the graph represented on Figure 2 by $mop_9(k)$. This graph is maximal outer planar, has maximum degree 9. The k vertices of degree 9 form a path and are denoted $v_1, \ldots, v_k$ in the order given by that path.

The following lemma proves Theorem 2.

Lemma 6. *The graph $mop_9(k)$ cannot be realized as a unit disc graph for k arbitrary large.*

Proof. Assume the graph $mop_9(k)$ can be realized by a unit disc graph and consider the path $v_1, \ldots, v_k$. These vertices must be embedded as vertices of a poly-line Γ satisfying the following conditions:

(i) the angle $\angle(v_{i-1}v_i, v_iv_{i+1})$ is in the interval $]\frac{\pi}{3}, \frac{2\pi}{3}[$

(ii) the distance between two non-consecutive vertices is strictly greater than 1

(iii) every edge has length at most 1

If the Γ has a self-intersection, the 4 endpoints of the 2 intersecting edges form a convex quadrilateral. At most one side of this quadrilateral corresponds to an edge of the Γ so there must be two consecutive sides whose length add up to at least 2. Since the diagonal between the endpoints of those two edges is an edge of the Γ, it has length at most 1, which contradicts the triangular inequality. Thus, Γ is a simple curve. Furthermore, condition (i) ensures that it is locally convex. Therefore Γ is a spiral.

Let h denote the number of vertices on the convex hull of Γ. The sum of the internal angles of a convex h-gone is $\pi(h-2)$. Let α denote the internal angle in this polygon that is not between consecutive edges of Γ. From condition (i) we get:

$$\pi(h-2) - (h-1)\frac{2\pi}{3} \leq \alpha \qquad \textit{i.e.} \qquad h \leq 3\frac{\alpha}{\pi} + 4.$$

Since $\alpha \leq 2\pi$, there are at most 10 vertices on $Conv(\Gamma)$. Since all but one edge of this polygon has length at most one, its area is bounded. Now, let us place a disc of radius $1/2$ at every v_i with odd index. By condition (ii) all those discs are disjoint. By condition (i), a constant fraction of the area of each of these discs must be contained in $Conv(\Gamma)$. Since the area of $Conv(\Gamma)$ is bounded, k cannot be arbitrarily large. $\square$

4 Questions and observations

1. The obvious question is whether one can narrow the gap between 4 and 9. The proof of Lemma 6 collapses for a maximum degree 8 (the spiral can have an arbitrary number of vertices on its convex hull). Actually, the similar class of graphs with maximum degree 8 (i.e for each v_i remove one of the vertices that is incident to only him and connect its neighbors) seems to be realizable for any k.