Distance between piecewise linear functions

Eric, Olivier , Christian, Guillaume

Barbados 2012

1 Introduction

The problem is given two triangulated terrains (red and blue) of size n to compute the (vertical) volume in between in subquadratic time.

Previous results: Chazelle et al. minimize the distance between red triangulation and a linear transfrom of the blue triangulation when they are constrained to have the same 2d vertical projection

Boris and Guillaume solve the problem for the L_2 norm in $O(n \log^4)$

2 Blue triangles classification

2.1 With respect to one red triangle

We consider one blue triangle as a point of $\mathbb{R}^9$. Given one red triangle it subdivides the space of blue triangles in regions where the intersection of the red and the blue triangles are combinatorially equivalent.

More precisely, these surfaces corresponds to the 15 three dimensional orientation predicates build on 4 of the 6 triangles vertices and the 18 two dimensional orientation predicates build with 3 of the 6 horizontal projection of the 6 vertices so that there are not all of the same color.

The number of blue points in the predicate define the degree of the corresponding surfaces, thus we have 15 linear functions, 15 quadratic functions, and 3 cubic functions.

For a blue triangle that lie in a cell of that subdivision of $\mathbb{R}^9$ the vertical volume between the two triangles is combinatorially fixed as a union of at most 16 tetrahedra whose vertices are either original vertices either computed with formula like $p_1 + \frac{\gamma}{\delta}(p_2 - p_1)$ where p_1 and p_2 are original vertices and γ and δ are determinant build on the original vertices (either 2d or 3d orientation tests). The volume of one of the tetrahedra is a rational function of degree less than 16. Thus the sum over the different tetrahedra is a function of degree less than 256.

2.2 With respect to a group of red triangles

If we consider k red triangles we can subdivide $\mathbb{R}^9$ according to these triangles. That is an arrangement of $O(k)$ surfaces in $\mathbb{R}^9$. A logarithmic time location structure can be computed in time $O(k^{14+\epsilon})$ [Koltun 2004]. For each of the $O(k^9)$ cell of the arrangement, we can sum the $O(k)$ rational functions of constant degree and obtain a rational function of degree $O(k)$.

3 Processing the blue triangles

Given the structure induced by a group of red triangles, we locate all the n blue triangles in that structure. Then for each cell C we perform parallel evaluation of the rational function on the n_C blue triangles that fall in that cell in time $O((k+n_C)\log^2 k)$ [Guillaume].

4 Complexity

There are $\frac{n}{k}$ groups of red triangles.

Computing the data structure takes $O(k^{14+\epsilon})$ per group.

Computing the rational functions takes $O(k\log^2 k)$ for each of the $O(k^9)$ cells of each arrangement.

Locating the n blue triangles in one arrangement takes $O(n\log k)$

Evaluating the rational functions is done in $\sum_C O((k+n_C)\log^2 k) = O((k^{10}+n)\log^2 k)$

The overall cost is $\frac{n}{k}O(k^{14+\epsilon}+n\log^2 k$. Choosing $k = n^{\frac{1}{14}}$ we get total complexity of $O(n^{2+\epsilon-\frac{1}{14}})$.

5 Trade off

If we want to compute the distance from one red surface to several blue terrains, we can study separately the time of preprocessing the red triangles that is $\frac{n}{k}O(k^{14+\epsilon})$ and the time of processing a blue terrain $O(\frac{n}{k}(k^{10}+n)\log^2 k$. $k = n^{\frac{1}{10}}$ yields a preprocessing of $O(n^{\frac{23}{10}+\epsilon})$ and a query time of $O(n^{\frac{19}{10}}\log^2 n)$.

6 Minimization

Find s and t to minimize volume between f and $sg+t$.

The function is convex, try parametric search.