Coinduction, Equilibrium and Escalation or the Rationality of Madness

Pierre Lescanne

ENS de Lyon

July 22nd, 2011
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   - Madness vs extreme rationality
   - Escalation in the Dollar Auction Game

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   - Coinductives in COQ
   - How does coinduction work?

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   - Finite sequential games in COQ
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   - “Illogic” conflict of escalation revisited

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Many programs for automatic bidding
Auction Auto Bidder
Stop losing auctions today!
I am the creator and copyright owner of this software

The best Auction Sniper software on eBay!
Actual user comments (see more in my eBay feedback):

"Excellent program. Just saved $150 on my first use."

"Best auction software out there (tried them all)"

"I list nine or ten [auction sniper] programs/services, but recommend yours as being the best buy."

"Out of all the money I spent on eBay this is the best thing $ can buy"

"This is the best support I had of any computer program and I thank you very much for your rapid response."

"The best software I have ever bought"

"First reliable sniping software I finally bought. Just sniped a $367.09 auction!"

"Better than the $60.00 stuff, great ease of use"

"Awesome program! Already recovered triple what I paid!"

"Tried others and this is the best hands down."

"VASTLY superior to anything else available"

Available on CD and by download
Free shipping is for the download only!

Available Editions:

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<td>• Leave notes for items</td>
<td>• Bid groups (win only one item in every group of items)</td>
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$9.99 One-Time Cost

$29.99 One-Time Cost
Why use Bidnapper, the eBay Auction Sniper?

Bidnapper's free sniper software delivers bids to eBay auctions at the last moment in a process called "sniping."

Snipe eBay auctions with Bidnapper to:
- Win more auctions, your competition won't have time to respond
- Protect your username from searches by your competition
- Change or delete your bids without the eBay hassle
- Hold prices down by not bidding throughout the auction
- Use the proxy system to advance your bid just enough to win an auction
- Place your bids automatically, in any currency, while you do other things
- eBay Security Key support

Read what University Studies say about sniping eBay auctions.

An eBay bidding sniper beyond compare!

Bidnapper's flat rate auction sniper service prices give you unlimited sniping for one price. No percentage commission or per-snipe charge. No other costs. Pay monthly, quarterly, semi-annually, or annually. Click for reviews of Bidnapper.

Bidnapper IS FREE TO TRY. Snipe for 15 days, no obligation!

Nothing to download or Install! And, you are Guaranteed to Succeed!

JOIN NOW!

The Bidnapper eBay Sniper System

**Unique Sniper Software** - Our software multiplies your chances of winning, and at a lower price.

**Special Sniper Hardware** - Unmatched speed, accuracy & uptime.

**Simple User Interface** - Our interface software offers cluttered convenience and control.

**Group Bidding** - Manages bids on groups of items.

**Snapper** - Easily adds your bid directly from eBay, making it ready to snipe.

**Email Alerts** - Tells you when your bids have been exceeded.

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"I started using eBay the day before to collect coins for my son and I lost 95% of the auctions I was bidding on in last few seconds. Today I used Bidnapper and won 13 of 15 (and I under bid on 2 that I lost). I can't tell you what the difference it made between the 2 days. I would suggest that if anyone wants to buy anything on eBay, even only a few times, Bidnapper will save you money and get you what you want! Well Done!"

-- S.D. 5/9/11

"I'M A BELIEVER! I thought I knew how to win items by placing my bids by holding off till the last 15 seconds...BOY I WAS WRONG...when my friend told me about this site...I had to try it...thank goodness for the trial period...I gave it a try and OMG, my first item won was by 4 SECONDS!!!! 4 SECONDS people!...BIDNAPPER: YOU ARE NOW MY NEW BEST FRIEND!!!"

-- K.T. 4/24/11

More Testimonials
eBay Auction Sniper
Stop losing your eBay Auctions in the final seconds.

Top 4 reasons people use Powersnipe
• Places your bid just before auction ends - Completely automatic.
• Your computer doesn’t need to be on - Set it and forget it.
• Members save an average of $33.75 and 25% per auction!
• 30 Day Money Back Guarantee.

New users get a 25% discount.
*Offer expires 07/08/2011.

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Powersnipe has been featured in the following:

Savings counter
Our members have saved:

$36,687,873.20
Total as of June 29, 2011

Testimonials
*Powersnipe pays for itself after the first use.*
*J. We have great deals for members too!*
*Saves you time and improves your odds of victory.*
*Time Magazine*
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I can calculate the movement of the stars,
but not the madness of men.

claimed to be Newton’s view
on the outcome of the South Sea Bubble.
The Dollar Auction

In 1971, in a paper called
The Dollar Auction game:
A paradox in noncooperative behavior and escalation
Martin Shubik described an infinite game.
The Dollar Auction \textit{(the story revisited)}

For charity, an object is sold on an auction made a special way. There is a piggy bank (or a hat).

To bid, each person puts one euro in the piggy bank which is never returned to him.
The Dollar Auction: a game with costs

Assume that there are two bidders (Alice and Bob) and that the value of the object is \( v \) €.

We count in term of cost for the bidder.
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After $v + n$ turns

- the bidder who has the object has a cost of $n$ and
The Dollar Auction: a game with costs

Assume that there are two bidders (Alice and Bob) and that the value of the object is $v \in \mathbb{R}$.

We count in term of cost for the bidder.

After $v + n$ turns

- the bidder who has the object has a cost of $n$ and
- the bidder who does not get the object has a cost of $v + n$. 
The Dollar Auction game may lead to escalation, i.e., players may play forever.
Escalation

The Dollar Auction game may lead to escalation, i.e., players may play forever.

- The Dollar Auction Game is by definition an infinite game,

  We could add an upper limit to the amount that anyone is allowed to bid. However the analysis is confined to the (possibly infinite) game without a specific termination point, as no particularly interesting general phenomena appear if an upper bound is introduced. \(\text{Shubik (1971), p. 109.}\)
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- It should be studied using tools designed for infiniteness.
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- It should be studied using tools designed for infiniteness. namely coinduction.
• Escalation is irrational.

*Once two bids have been obtained from the crowd, the paradox of escalation is real [...] A total of payments between three and five dollars is not uncommon*  

Is escalation in the Dollar Auction rational?

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- Escalation is rational.

  **Theorem**: The dollar auction game has an escalation.

  There is no paradox.
Why this discrepancy?

- For Osborne et al. the resources are finite.

  *Each person’s wealth is w, which exceeds v; neither player may bid more than her wealth.*

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*In a world of finite resources escalation is irrational.*
Why this discrepancy?

- For Osborne et al. the resources are finite.

  \textit{Each person's wealth is w, which exceeds v; neither player may bid more than her wealth.}

  Osborne \textit{An Introduction to Game Theory},

  Hence the escalation cannot occur, as noticed by Shubik.

- With infinite resources, escalation can happen.

\textit{In a world of \textbf{finite resources} escalation is irrational.}

\textit{In a world of \textbf{infinite resources} escalation is rational.}
The Dollar Auction (pictured)
The Dollar Auction (pictured)

We will focus on a simpler game

The game 0, 1
Coinduction a tool for reasoning on infinite structures

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6. Conclusion
Infinite objects (informally)

- An **infinite list** or a **stream** has infinite many items:
  - $0, 1, 2, ..., n, n + 1, ...$
Infinite objects (informally)

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  - \(0, 1, 2, \ldots, n, n + 1, \ldots\)
  - \(0, 0, 0, 1, \ldots, 1, 1, \ldots\)
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Infinite objects (informally)

- An **infinite list** or a **stream** has infinite many items:
  - $0, 1, 2, \ldots, n, n+1, \ldots$
  - $0, 0, 0, 1, \ldots, 1, 1, \ldots$,

- An **infinite** binary tree may have infinite branches:

  ![Backbone Zigzag Diagram]

  1 eventually always
Coinduction is a way to reason on an infinite set which contains infinite objects.

- The streams or infinite lists
Coinduction is a way to reason on an infinite set which contains infinite objects.

- The streams or infinite lists
  - \([0, 0, 0, \ldots]\),
  - ...,
What is coinduction?

CoInduction is a way to reason on an infinite set which contains infinite objects.

- The streams or infinite lists
  - \([0, 0, 0, \ldots]\),
  - \(\ldots\),
  - \([0, 1, 2, \ldots]\),
  - \(\ldots\),
  - \([2, 3, 5, 7, 11, \ldots]\),
  - \(\ldots\),
The finite and infinite lists
- [ ],
- [0],
- [1],
- ...
- [0; 0],
- [0; 1], ...
- [0; 0; 0],
- [0; 0; 1],
- ...

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The finite and infinite lists

- [ ],
- [0],
- [1],
- ...,
- [0; 0],
- [0; 1], ...
- [0; 0; 0],
- [0; 0; 1],
- ...
- [0, 0, 0, ...],
- ...
- [0, 1, 2, ...],
- ...

infinite lists
• The finite and infinite binary trees
Coinduction a tool for reasoning on infinite structures

Examples of infinite objects

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Let us be formal!
A stream (or an infinite list) on $A$ is

- of the form $a :: s$, where $a$ is an element of $A$ and $s$ is a stream.
A coinductive list (or a lazy list) on $A$ is

- either the empty lazy list $[]$,
- or a lazy list of the form $a :: \ell$, where $a$ is an element of $A$ and $\ell$ is a lazy list.
A **coinductive** binary tree (or lazy binary tree) is

- either the empty lazy binary tree
A **coinductive** binary tree (or lazy binary tree) is

- either the empty lazy binary tree
- or a lazy binary tree made of two lazy binary trees.
Let us be even more formal!
CoInductive Stream (A:Set) : Set :=
| SCons: A -> Stream A -> Stream A.
CoInductive Stream (A:Set) : Set :=
| SCons: A -> Stream A -> Stream A.

CoInductive LList (A:Set) : Set :=
| LNil: LList A
| LCons: A -> LList A -> LList A.
CoInductive **Stream** (A:Set) : Set :=
| SCons: A -> Stream A -> Stream A.

CoInductive **LList** (A:Set) : Set :=
| LNil: LList A
| LCons: A -> LList A -> LList A.

CoInductive **InfFinBintree** : Set :=
| InfFinBtNil: InfFinBintree
| InfFinBtNode: InfFinBintree -> InfFinBintree -> InfFinBintree.
CoInductives in COQ

CoInductive **Stream** \( (A:\text{Set}) : \text{Set} := \)
| SCons: \( A \rightarrow \text{Stream} A \rightarrow \text{Stream} A \).

CoInductive **LList** \( (A:\text{Set}) : \text{Set} := \)
| LNil: \( \text{LList} A \)
| LCons: \( A \rightarrow \text{LList} A \rightarrow \text{LList} A \).

CoInductive **InfFinBintree** : \( \text{Set} := \)
| InfFinBtNil: \( \text{InfFinBintree} \)
| InfFinBtNode: \( \text{InfFinBintree} \rightarrow \text{InfFinBintree} \rightarrow \text{InfFinBintree} \).

CoInductive **InfiniteInfFinBT**: \( \text{InfFinBintree} \rightarrow \text{Prop} := \)
| IBTLeft : \( \forall \, bl \, br, \text{InfiniteInfFinBT} \, bl \rightarrow \text{InfiniteInfFinBT} \, (\text{InfFinBtNode} \, bl \, br) \)
| IBTRight : \( \forall \, bl \, br, \text{InfiniteInfFinBT} \, br \rightarrow \text{InfiniteInfFinBT} \, (\text{InfFinBtNode} \, bl \, br) \).
Defining an infinite object

One defines infinite objects as fixpoints.
Defining an infinite object

One defines infinite objects as fixpoints.

CoFixpoint InfFinBackBone : InfFinBintree := InfFinBtNode InfFinBackBone InfFinBtNil.
Defining an infinite object

One defines infinite objects as fixpoints.

CoFixpoint **InfFinBackBone**: InfFinBintree :=
InfFinBtNode InfFinBackBone InfFinBtNil.

CoFixpoint **Zig**: InfFinBintree := InfFinBtNode Zag InfFinBtNil
with **Zag**: InfFinBintree := InfFinBtNode InfFinBtNil Zig.
Defining an infinite object

One defines infinite objects as fixpoints.

CoFixpoint InfFinBackBone: InfFinBintree :=
InfFinBtNode InfFinBackBone InfFinBtNil.

CoFixpoint Zig: InfFinBintree := InfFinBtNode Zag InfFinBtNil
with Zag: InfFinBintree := InfFinBtNode InfFinBtNil Zig.

One can prove three lemmas:

Lemma InfiniteBackbone: InfiniteInfFinBT InfFinBackBone.

Lemma InfiniteZig: InfiniteInfFinBT Zig.

Lemma InfiniteZag: InfiniteInfFinBT Zag.
The coinduction principle

**Sketch:** Assume one considers a coinductive object.

One proves that

if a property which holds for a sub-object implies
that the property holds for the whole object,
then the property holds for the whole object.
The coinduction principle

**Sketch:** Assume one considers a coinductive object.

One proves that
- if a property which holds for a sub-object implies that the property holds for the whole object,
- then the property holds for the whole object.

**Basically:** one proves that the property is an invariant.
Applications to sequential games

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6. Conclusion
What is a **sequential** game?

A **sequential game** is described by a labeled tree.

![Labeled Tree Diagram]
What is a *strategy profile*?

A *strategy profile* is described by a labeled tree.
What is a Nash equilibrium?

A Nash equilibrium is a strategy profile where if an agent changes alone his action he will get a utility which is not better.
What is a Nash equilibrium?

A **Nash equilibrium** is a strategy profile where if an agent changes alone his action he will get a utility which is not better.
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A **Nash equilibrium** is a strategy profile where if an agent changes alone his action he will get a utility which is not better.

This method for computing a Nash equilibrium is called **backward induction**.
We are interested in infinite games,
We are interested in infinite games,

and to the extension of backward induction to infinite games.
A finite sequential games is described by induction from its subgames.
A finite sequential games is described by induction from its subgames.

Without loss of generality, we restrict to binary sequential games.

A binary finite sequential game is

- either a node, assigned to a player, with two subgames,
- or a leaf.
Finite sequential games as inductive objects

A *finite sequential games* is described *by induction* from its subgames.

Without loss of generality, we restrict to binary sequential games.

A *binary finite sequential game* is

- either a **node**, assigned to a **player**, with **two subgames**, or a **leaf**.

**Inductive** \( \text{FinGame} \) : \( \text{Set} := \)

\[ \text{gLeaf} : \ \text{Utility} \_\text{fun} \ \rightarrow \ \text{FinGame} \]

\[ \text{gNode} : \ \text{Agent} \ \rightarrow \ \text{FinGame} \ \rightarrow \ \text{FinGame} \ \rightarrow \ \text{FinGame}. \]
Utility and utility functions

Utility\_fun is a function which associates a utility (a cost or a payoff) with an agent:

**Definition** \texttt{Utility\_fun} := Agent $\rightarrow$ Utility.
A finite strategy profile is also an inductive

\textbf{Inductive} \texttt{FinStratProf} : \text{Set} := \\
\mid \texttt{sLeaf} : \text{Utility}_\text{fun} \to \text{FinStratProf} \\
\mid \texttt{sNode} : \text{Agent} \to \text{Choice} \to \text{FinStratProf} \to \text{FinStratProf} \to \text{FinStratProf}. 
Applications to sequential games  Finite sequential games in COQ

A finite strategy profile is also an inductive

\textbf{Inductive} \texttt{FinStratProf} : Set :=
\begin{itemize}
  \item \texttt{sLeaf} : Utility\_fun \rightarrow \texttt{FinStratProf}
  \item \texttt{sNode} : Agent \rightarrow \texttt{Choice} \rightarrow \texttt{FinStratProf} \rightarrow \texttt{FinStratProf} \rightarrow \texttt{FinStratProf}.
\end{itemize}

A strategy profil is written \langle f \rangle and \langle a, c, sl, sr \rangle.
Fixpoint \( f2u \) (s:FinStratProf) : Utility_fun :=
match s with
| ≪uf≫ => uf
| ≪a, c, sl, sr≫ => (f2u sl)
| ≪a, c, sl, sr≫ => (f2u sr)
end.
Backward induction

On finite strategy profiles.

**Inductive** $\text{BI}$: $\text{FinStratProf} \to \text{Prop} :=$

| $\text{BILeaf}$: $\forall \text{uf:Utility_fun}, \text{BI} (\text{sLeaf uf})$ |
Backward induction

On finite strategy profiles.

**Inductive BI**: $\text{FinStratProf} \rightarrow \text{Prop} :=$

- **BILeaf**: $\forall \text{uf:Utility\_fun}, \text{BI} (\text{sLeaf uf})$

- **BINode\_left**: $\forall (a:\text{Agent}) (\text{sl sr: FinStratProf})$,
  \[ \text{BI sl} \rightarrow \text{BI sr} \rightarrow (f2u \text{ sr a} \leq f2u \text{ sl a}) \rightarrow \text{BI} (\langle a \text{ left sl sr } \rangle) \]
On finite strategy profiles.

**Inductive** \( BI \): \( FinStratProf \rightarrow Prop := \)

| \( BILeaf \): \( \forall uf:\text{Utility}_\text{fun}, BI (sLeaf uf) \) |
| \( BINode\_left \): \( \forall (a:A) (sl sr: FinStratProf), \)
  \( BI sl \rightarrow BI sr \rightarrow (f2u sr a \preceq f2u sl a) \rightarrow BI (\ll a \text{ left } sl sr \gg) \) |
| \( BINode\_right \): \( \forall (a:A) (sl sr: FinStratProf), \)
  \( BI sl \rightarrow BI sr \rightarrow (f2u sl a \preceq f2u sr a) \rightarrow BI (\ll a \text{ right } sl sr \gg) \).
CoInductive Games:

CoInductive Game : Set :=
| gLeaf : Utility_fun → Game
| gNode : Agent → Game → Game → Game.
CoInductive Games

CoInductive Game : Set :=
| gLeaf : Utility_fun → Game
| gNode : Agent → Game → Game → Game.

Either a leaf or a triple with an agent and two subgames that are infinite.
Coinductive Games

CoInductive Game : Set :=
| gLeaf : Utility_fun → Game
| gNode : Agent → Game → Game → Game.

Either a leaf or a triple with an agent and two subgames that are infinite.

The concept of infinite strategy profile is also defined as a coinductive:

CoInductive StratProf : Set :=
| sLeaf : Utility_fun → StratProf
| sNode : Agent → Choice → StratProf → StratProf → StratProf.
CoInductive Games:

CoInductive Game : Set :=
| gLeaf : Utility_fun → Game
| gNode : Agent → Game → Game → Game.

Either a leaf or a triple with an agent and two subgames that are infinite.

The concept of infinite strategy profile is also defined as a coinductive:

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The utility function $s2u$ is no more a function, but a relation, since it is no more total.
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It returns a value only on strategies which go eventually to a leaf.
The predicate *Leads to a leaf*

This requires to introduce a predicate *LeadsToLeaf* on strategies,
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Roughly speaking, 
*following the path given by the strategy profile one is lead to a leaf.*
Existence and uniqueness

On strategies that leads to a leaf, one gets **existence** and **uniqueness** of the utility associated with each agent.
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**Lemma** *Existence_i2u*: \( \forall (a:\text{Agent}) (s:\text{StratProf}) \), 
\( \text{LeadsToLeaf } s \rightarrow \exists u:\text{Utility}, \ i2u \ a \ u \ s \).

**Lemma** *Uniqueness_i2u*: \( \forall (a:\text{Agent}) (u \ v:\text{Utility}) (s:\text{StratProf}) \), 
\( \text{LeadsToLeaf } s \rightarrow i2u \ a \ u \ s \rightarrow i2u \ a \ v \ s \rightarrow u=v \).
A strategy profile always leads to leaf, if all strategy sub profiles lead to a leaf.
CoInductive $SGPE$ : $StratProf \rightarrow Prop :=$

$SGPE\_leaf : \forall f:Utility\_fun, SGPE (\ll f \gg)$
CoInductive \( \text{SGPE}: \text{StratProf} \rightarrow \text{Prop} \) :=

\[
\begin{align*}
\text{SGPE}_\text{leaf}: & \quad \forall f:\text{Utility}\_\text{fun}, \text{SGPE} (\ll f \gg) \\
\text{SGPE}_\text{left}: & \quad \forall (a:\text{Agent})(u v: \text{Utility}) (sl sr: \text{StratProf}), \\
& \quad \text{AlwLeadsToLeaf} (\ll a, l, sl, sr \gg) \rightarrow \\
& \quad \text{SGPE} \ sl \rightarrow \text{SGPE} \ sr \rightarrow \\
& \quad s2u \ sl \ a \ u \rightarrow s2u \ sr \ a \ v \rightarrow (v \preceq u) \rightarrow \\
& \quad \text{SGPE} (\ll a, l, sl, sr \gg)
\end{align*}
\]
CoInductive $SGPE$: $StratProf \rightarrow Prop :=$

| $SGPE_{\text{leaf}}$: $\forall f:Utility_{\text{fun}}, SGPE (\ll f\gg)$
| $SGPE_{\text{left}}$: $\forall (a:Agent)(u v: Utility) (sl sr: StratProf)$,
  
  $\text{AlwLeadsToLeaf} (\ll a,l,sl,sr\gg) \rightarrow$

  $SGPE sl \rightarrow SGPE sr \rightarrow$

  $s2u sl a u \rightarrow s2u sr a v \rightarrow (v \preceq u) \rightarrow$

  $SGPE (\ll a,l,sl,sr\gg)$
SubGame Perfect Equilibria

\textbf{CoInductive} \textit{SGPE}: \textit{StratProf} \to \textit{Prop} :=

\begin{enumerate}
\item \textit{SGPE\_leaf}: \forall f: \textit{Utility\_fun}, \textit{SGPE} (\ll f \gg)
\item \textit{SGPE\_left}: \forall (a: \textit{Agent})(u v: \textit{Utility}) (sl sr: \textit{StratProf}),
\quad \text{AlwLeadsToLeaf} (\ll a,l,sl,sr \gg) \to
\quad \textit{SGPE} \ sl \to \textit{SGPE} \ sr \to
\quad s2u sl a u \to s2u sr a v \to (v \preceq u) \to
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\text{SGPE}_{\text{left}}: & \quad \forall (a: \text{Agent})(u \ v: \text{Utility}) (sl \ sr: \text{StratProf}), \\
& \hspace{1cm} \text{AlwLeadsToLeaf} (\langle a, l, sl, sr \rangle) \rightarrow \\
& \hspace{1cm} \text{SGPE } sl \rightarrow \text{SGPE sr} \rightarrow \\
& \hspace{1cm} s2u sl a u \rightarrow s2u sr a v \rightarrow (v \preceq u) \rightarrow \\
& \hspace{1cm} \text{SGPE} (\langle a, l, sl, sr \rangle)
\end{align*} \]
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AlwLeadsToLeaf (\langle a, l, sl, sr \rangle) \rightarrow
SGPE sl \rightarrow \text{SGPE} sr \rightarrow
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\mid \text{SGPE}_\text{right}: \forall (a: \text{Agent}) (u v: \text{Utility}) (sl sr: \text{StratProf}),
AlwLeadsToLeaf (\langle a, r, sl, sr \rangle) \rightarrow
SGPE sl \rightarrow \text{SGPE} sr \rightarrow
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CoInductive \textit{SGPE}: StratProf \rightarrow Prop \equiv

| \textit{SGPE}_\text{leaf}: \forall \ f:\text{Utility}_\text{fun}, \text{SGPE}\ (\llbracket f \rrbracket) |

| \textit{SGPE}_\text{left}: \forall \ (a:\text{Agent})(u \ v:\text{Utility})(sl \ sr:\text{StratProf}), \text{AlwLeadsToLeaf}\ (\llbracket a,l,sl,sr \rrbracket) \rightarrow \text{SGPE}\ sl \rightarrow \text{SGPE}\ sr \rightarrow \ s2u\ sl\ a\ u \rightarrow s2u\ sr\ a\ v \rightarrow (v \preceq u) \rightarrow \text{SGPE}\ (\llbracket a,l,sl,sr \rrbracket) |

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Examples of games with escalation

1. Automatic bidding

2. Escalation
   - Madness vs extreme rationality
   - Escalation in the Dollar Auction Game

3. Coinduction a tool for reasoning on infinite structures
   - Examples of infinite objects
   - Coinductives in COQ
   - How does coinduction work?

4. Applications to sequential games
   - Sequential games
   - Infinite games
   - Finite sequential games in COQ
   - Infinite sequential games in COQ

5. Examples of games with escalation
   - The 0, 1 game
   - “Illogic” conflict of escalation revisited

6. Conclusion
Examples of games with escalation

The 0, 1 game

The game 0, 1
Examples of games with escalation

The 0, 1 game

The game 0, 1

0 and 1 are costs
Examples of games with escalation

The 0, 1 game

Alice  Bob

1,0  0,1

The strategy profile Alice continues, Bob stops

0 and 1 are costs
Examples of games with escalation

The 0, 1 game

<table>
<thead>
<tr>
<th>Alice</th>
<th>Bob</th>
</tr>
</thead>
<tbody>
<tr>
<td>1,0</td>
<td>0,1</td>
</tr>
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The game 0, 1

0 and 1 are costs

The strategy profile Alice continues, Bob stops

The strategy profile Alice stops, Bob continues
Examples of games with escalation

Two SGPE’s

Theorem

*The strategy profile* Alice continues, Bob stops *is a SGPE.*
Two SGPE’s

Theorem

The strategy profile Alice continues, Bob stops is a SGPE.

Theorem

The strategy profile Alice stops, Bob continues is a SGPE.
At each step Alice is rational if she continues.
The escalation

At each step Alice is rational if she continues.

At each step Bob is rational if he continues.
The escalation

At each step Alice is rational if she continues.

At each step Bob is rational if he continues.

As a sequence of rational decisions, escalation is rational in game 0, 1.
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The *Dollar Auction* revisited

Example of games with escalation

“Illogic” conflict of escalation revisited

Alice → Bob → Alice → Bob →...

\( v + n, n \) → \( n + 1, v + n \) → \( v + n + 1, n + 1 \) → \( n + 2, v + n + 1 \) →...
Alice abandons

In Shubik’s game, we can prove that the strategy Alice abandons and Bob continues

\[ v+n, n \quad n+1, v+n \quad v+n+1, n+1 \quad n+2, v+n+1 \]

is a SubGame Perfect equilibrium.
Examples of games with escalation

"Illogic" conflict of escalation revisited

Alice abandons

In Shubik’s game, we can prove that the strategy
Alice abandons and Bob continues

\[
\begin{align*}
\text{Alice} & \rightarrow \text{Bob} \quad \text{Alice} & \rightarrow \text{Bob} \\
v+n,n & \rightarrow n+1,v+n & v+n+1,n+1 & \rightarrow n+2,v+n+1
\end{align*}
\]

is a SubGame Perfect equilibrium.

Alice takes Bob’s threat as credible and considers it is better to give up.
Examples of games with escalation

"Illogic" conflict of escalation revisited

The strategy Alice continues and Bob gives up

\[ v + n, n \quad n+1, v+n \quad v + n+1, n+1 \quad n+2, v+n+1 \]

is a **SubGame Perfect Equilibrium.**
Bob gives up

The strategy Alice continues and Bob gives up

\[
\begin{align*}
&v+n, n \\
&n+1, v+n \\
&v+n+1, n+1 \\
&n+2, v+n+1
\end{align*}
\]

is a SubGame Perfect Equilibrium.

Bob takes Alice’s threat as credible.
Always give up

The strategy always give up

\[
\begin{align*}
&n+1, v+n & v+n+1, n+1 & n+2, v+n+1 & v+n+2, n+2 \\
&\cdots & \cdots
\end{align*}
\]

is a not a SubGame Perfect Equilibrium and therefore not a Nash equilibrium.
At each turn if the agent continues she (he) is rational.
Examples of games with escalation

“Illogic” conflict of escalation revisited

Escalation in the Dollar Auction

At each turn if the agent continues she (he) is rational.

Escalation is rational in the Dollar Auction game.
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6. Conclusion
Reasoning on infinite sequential games is subtle, however necessary.
Conclusion

- Reasoning on infinite sequential games is subtle, however necessary.
- Escalation is rational in a world of infinite resources.
Any question?