

Commutative Monads for Probabilistic Programming Languages^{*}

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Abstract

A long-standing open problem in the semantics of programming languages supporting probabilistic choice is to find a commutative monad for probability on the category DCPO. In this paper we present three such monads and a general construction for finding even more. We show how to use these monads to provide a sound and adequate denotational semantics for the Probabilistic FixPoint Calculus (PFPC) – a call-by-value simply-typed lambda calculus with mixed-variance recursive types, term recursion and probabilistic choice. We also show that in the special case of continuous dcpo's, all three monads coincide with the valuations monad of Jones, and we fully characterise the induced Eilenberg-Moore categories by showing that they are all isomorphic to the category of continuous Kegelspitzen of Keimel and Plotkin.

Keywords: Probability, Recursion, Domain Theory, Valuations Monads

1. Introduction

Probabilistic methods now are a staple of computation. The initial discovery of randomized algorithms [2] was quickly followed by the definition of Probabilistic Turing machines and related complexity classes [3]. There followed advances in a number of areas, including, e.g., process calculi, probabilistic model checking and verification [4, 5, 6], right through to the recent development of statistical probabilistic programming languages (cf. [7, 8, 9]), not to mention the crucial role probability plays in quantum programming languages [10, 11].

Domain theory, a staple of denotational semantics, has struggled to keep up with these advances. Domain theory encompasses two broad classes of objects: *directed complete partial orders (dcpo's)*, based on an order-theoretic view of computation, and the smaller class of (*continuous*) *domains*, those dcpo's that

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also come equipped with a notion of approximation. However, adding probabilistic choice to the domain-theoretic approach has been a challenge. The canonical model of (sub)probability measures in domain theory is the family of *valuations* – certain maps from the lattice of open subsets of a dcpo to the unit interval. It is well-known that the valuations form a monad \mathcal{V} on **DCPO** (the category of dcpo’s and Scott-continuous functions) and on **DOM** (the full subcategory of **DCPO** consisting of domains) [12, 13].

In fact, the monad \mathcal{V} on **DOM** is commutative [13], which is important for two reasons: (1) its commutativity is equivalent to the Fubini Theorem [13], a cornerstone of integration theory and (2) computationally, commutativity of a monad together with adequacy can be used to establish contextual equivalences for effectful programs. However, in order to do so, one typically needs a Cartesian closed category for the semantic model, and **DOM** is not closed; in fact, despite repeated attempts, it remains unknown whether there is *any* Cartesian closed category of *domains* on which \mathcal{V} is an endofunctor; this is the well-known *Jung-Tix Problem* [14]. On the other hand, it also is unknown if the monad \mathcal{V} is commutative on the larger Cartesian closed category **DCPO**. In this paper, we offer a solution to this conundrum.

1.1. Our Contributions

We use *topological methods* to construct a commutative valuations monad \mathcal{M} on **DCPO**, as follows: it is straightforward to show the family \mathcal{SD} of *simple valuations* on a dcpo D can be equipped with the structure of a commutative monad, but \mathcal{SD} is not a dcpo, in general. So, we complete \mathcal{SD} by taking the smallest subdcpo $\mathcal{MD} \subseteq \mathcal{VD}$ that contains \mathcal{SD} . This defines the object-mapping of a monad \mathcal{M} on **DCPO**. The unit, multiplication and strength of the monad \mathcal{M} at D are given by the restrictions of the same operations of \mathcal{V} to \mathcal{MD} . Topological arguments then imply that \mathcal{M} is a commutative valuations monad on **DCPO**.

In fact, there are several completions of \mathcal{SD} that give rise to commutative valuations monads on **DCPO**. These completions are determined by so-called K-categories, introduced by Keimel and Lawson [15]. This observation allows us to define two additional commutative valuations monads, \mathcal{W} and \mathcal{P} , on **DCPO** simply by specifying their corresponding K-categories. Finally, while we have identified three such K-categories, there likely are more that meet our requirements, each of which would define yet another commutative monad of valuations on **DCPO** containing \mathcal{S} .

With this background, we now summarise our main results.

Commutative monads. A K-category is a full subcategory of the category \mathbf{T}_0 of T_0 -spaces satisfying properties that imply it determines a *completion* of each T_0 -space among the objects of the K-category. For example, each K-category defines a completion of a poset endowed with its Scott topology, among the objects in the K-category. In particular, each K-category determines a completion of the family \mathcal{SD} when considered as a subset of \mathcal{VD} , for each dcpo D .

By specifying an additional constraint on \mathbf{K} -categories, we can show the corresponding completions of \mathcal{S} define commutative monads on **DCPO**. We identify three commutative monads concretely: \mathcal{M} , \mathcal{W} and \mathcal{P} , corresponding to the \mathbf{K} -categories of d-spaces, that of well-filtered spaces and that of sober spaces, respectively (see Theorem 9 and Theorem 23). As part of our construction, we also prove the most general Fubini Theorem for dcpo’s yet available (see Theorem 22).

Eilenberg-Moore Algebras. All three of \mathcal{M} , \mathcal{W} and \mathcal{P} restrict to monads on **DOM**, where they coincide with \mathcal{V} . We characterize their Eilenberg-Moore categories over **DOM** by showing they are isomorphic to the category of continuous Kegelspitzen and Scott-continuous linear maps [16]; this corrects an error in [13] (see Remark 37 below).

On the larger category **DCPO**, we show the Eilenberg-Moore algebras of our monads \mathcal{M} , \mathcal{W} and \mathcal{P} are Kegelspitzen (see Subsection 3.5). It is unknown if every Kegelspitze is an \mathcal{M} -algebra.

Semantics. We consider the *Probabilistic FixPoint Calculus* (PFPC) – a call-by-value simply-typed lambda calculus with mixed-variance recursive types, term recursion and probabilistic choice (see Section 2). We show that each of the Kleisli categories of our three commutative monads is a sound and computationally adequate model of PFPC (see Section 5). Moreover, we show that adequacy holds in a strong sense (Theorem 61), i.e., the interpretation of each term is a (potentially infinite) convex sum of the values it reduces to.

1.2. Related Work

The first dcpo model for probabilistic choice was given in [17], but this preceded Moggi’s seminal work using Kleisli categories to model computational effects [18]. The work closest to ours is Jones’ thesis [13] (see also [12]), which considers the same language PFPC, but with a slightly different syntax. This work is based on an early version of FPC, and uses the Kleisli category of \mathcal{V} over **DCPO** as the semantic model. While soundness and adequacy theorems are included in [13], the proof of adequacy does not identify a semantic space on which \mathcal{V} is commutative, instead offering arguments based on the commutativity of \mathcal{S} , and on realizing the valuations needed to interpret the language as directed suprema of simple valuations. Our semantic results improve those of Jones by clearly identifying the categorical setting; the commutativity of our monads together with adequacy allows us to easily establish a large class of contextual equivalences.

Other related work includes [19], where the authors use *probabilistic coherence spaces* to provide a fully abstract model of a probabilistic call-by-push-value language with recursive types. This work builds on previous work [20], which describes a fully abstract model of probabilistic PCF also based on probabilistic coherence spaces. Recently, *quasi-Borel spaces* were introduced in [21] and they were later used to provide a model of SFPC (a statistical probabilistic programming language with recursive types, sampling and conditioning) in

[9]. Compared to probabilistic coherence spaces and quasi-Borel spaces, our methods are based on the traditional domain-theoretic approach and its well-established connections to probability theory [22, 23, 24]; we hope to exploit these connections in future work.

The paper [25] uses Kegelsptizen to provide a sound and adequate model for probabilistic PCF. The author then discusses a possible interpretation of a version of linear PFPC without contraction or a !-modality (which implies the system is strongly normalising), but [25] does not state any soundness, nor adequacy results for it.

2. Syntax and Operational Semantics

In this section we describe the syntax and operational semantics of our language, the *Probabilistic FixPoint Calculus* (PFPC). The presentation we choose for PFPC is exactly the same as FPC [26, 27, 28] together with the addition of one extra term ($M \text{ or}_p N$) for probabilistic choice. The same language is also considered by Jones [13], but with a slightly different syntax.

2.1. The Types of PFPC

Recursive types in PFPC are formed in the same way as in FPC. We use X, Y to range over *type variables* and we use Θ to range over *type contexts*. A type context $\Theta = X_1, \dots, X_n$ is *well-formed*, written $\Theta \vdash$, if all type variables within it are distinct. We write Θ_i for the i -th variable. We use A, B to range over the *types* of our language which are defined in Figure 1. We write $\Theta \vdash A$ to indicate that type A is *well-formed* in type context Θ whenever the judgement is derivable via the rules in Figure 2. A type A is *closed* when $\cdot \vdash A$. Note that there are no restrictions on the admissible logical polarities when forming recursive types.

Example 1. The *empty type* is defined as $0 \stackrel{\text{def}}{=} \mu X.X$ and the *unit type* as $1 \stackrel{\text{def}}{=} 0 \rightarrow 0$. Among others, we may also define:

- *Booleans* as $\text{Bool} \stackrel{\text{def}}{=} 1 + 1$;
- *Natural numbers* as $\text{Nat} \stackrel{\text{def}}{=} \mu X.1 + X$;
- *Lists of type A* as $\text{List}(A) \stackrel{\text{def}}{=} \mu X.1 + A \times X$;
- *Streams of type A* as $\text{Stream}(A) \stackrel{\text{def}}{=} \mu X.1 \rightarrow A \times X$.

2.2. The Terms of PFPC

We now explain the syntax we use for terms. When forming terms and term contexts, we implicitly assume that all types within are closed and well-formed. We use x, y to range over *term variables* and we use Γ to range over *term contexts*. A (well-formed) term context $\Gamma = x_1 : A_1, \dots, x_n : A_n$ is a list of (distinct) variables with their types. The terms (ranged over by M, N)

Type Variables	X, Y	Term Variables	x, y
Type Contexts	$\Theta ::= \cdot \mid \Theta, X$		
Types	$A, B ::= X \mid A + B \mid A \times B \mid A \rightarrow B \mid \mu X.A$		
Term Contexts	$\Gamma ::= \cdot \mid \Gamma, x : A$		
Terms	$M, N ::= x \mid (M, N) \mid \pi_1 M \mid \pi_2 M \mid \text{in}_1 M \mid \text{in}_2 M \mid$ $(\text{case } M \text{ of } \text{in}_1 x \Rightarrow N_1 \mid \text{in}_2 y \Rightarrow N_2) \mid$ $\lambda x.M \mid MN \mid \text{fold } M \mid \text{unfold } M \mid M \text{ or}_p N$		
Values	$V, W ::= x \mid (V, W) \mid \text{in}_1 V \mid \text{in}_2 V \mid \text{fold } V \mid \lambda x.M$		

Figure 1: Grammars for types, contexts and terms.

$$\frac{\Theta \vdash}{\Theta \vdash \Theta_i} \quad \frac{\Theta \vdash A \quad \Theta \vdash B}{\Theta \vdash A \star B} \star \in \{+, \times, \rightarrow\} \quad \frac{\Theta, X \vdash A}{\Theta \vdash \mu X.A}$$

Figure 2: Formation rules for types.

$$\frac{}{\Gamma, x : A \vdash x : A} \quad \frac{\Gamma \vdash M : A \quad \Gamma \vdash N : A}{\Gamma \vdash M \text{ or}_p N : A} \quad \frac{\Gamma \vdash M : A \quad \Gamma \vdash N : B}{\Gamma \vdash (M, N) : A \times B}$$

$$\frac{\Gamma \vdash M : A_1 \times A_2}{\Gamma \vdash \pi_i M : A_i} i \in \{1, 2\} \quad \frac{\Gamma \vdash M : A_i}{\Gamma \vdash \text{in}_i M : A_1 + A_2} i \in \{1, 2\}$$

$$\frac{\Gamma \vdash M : A_1 + A_2 \quad \Gamma, x : A_1 \vdash N_1 : B \quad \Gamma, y : A_2 \vdash N_2 : B}{\Gamma \vdash (\text{case } M \text{ of } \text{in}_1 x \Rightarrow N_1 \mid \text{in}_2 y \Rightarrow N_2) : B}$$

$$\frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x^A.M : A \rightarrow B} \quad \frac{\Gamma \vdash M : A \rightarrow B \quad \Gamma \vdash N : A}{\Gamma \vdash MN : B}$$

$$\frac{\Gamma \vdash M : A[\mu X.A/X]}{\Gamma \vdash \text{fold } M : \mu X.A} \quad \frac{\Gamma \vdash M : \mu X.A}{\Gamma \vdash \text{unfold } M : A[\mu X.A/X]}$$

Figure 3: Formation rules for terms. All types are assumed to be closed and well-formed.

$$\begin{array}{c}
\pi_1(V, W) \xrightarrow{1} V \qquad \pi_2(V, W) \xrightarrow{1} W \\
(\text{case } \text{in}_1 V \text{ of } \text{in}_1 x \Rightarrow N_1 \mid \text{in}_2 y \Rightarrow N_2) \xrightarrow{1} N_1[V/x] \\
(\text{case } \text{in}_2 V \text{ of } \text{in}_1 x \Rightarrow N_1 \mid \text{in}_2 y \Rightarrow N_2) \xrightarrow{1} N_2[V/y] \\
\text{unfold fold } V \xrightarrow{1} V \\
(\lambda x.M)V \xrightarrow{1} M[V/x] \\
M \text{ or}_p N \xrightarrow{p} M \\
M \text{ or}_p N \xrightarrow{1-p} N \\
\\
E ::= [\cdot] \mid (E, M) \mid (V, E) \mid \pi_i E \mid EM \mid VE \mid \\
\text{in}_i E \mid (\text{case } E \text{ of } \text{in}_1 x \Rightarrow N_1 \mid \text{in}_2 y \Rightarrow N_2) \mid \\
\text{fold } E \mid \text{unfold } E \\
\\
\frac{M \xrightarrow{p} M'}{E[M] \xrightarrow{p} E[M']}
\end{array}$$

Figure 4: Reduction rules for PFPC. The grammar for E defines our call-by-value evaluation contexts.

and the values (ranged over by V, W) of PFPC are specified in Figure 1 and their formation rules in Figure 3. They are completely standard. In Figure 3, the notation $A[\mu X.A/X]$ indicates type substitution which is defined in the standard way. The term $M \text{ or}_p N$ represents probabilistic choice. A term M of type A is *closed* when $\cdot \vdash M : A$ and in this case we also simply write $M : A$.

Example 2. Important closed values in PFPC include: the *unit value* $() \stackrel{\text{def}}{=} \lambda x^0.x : 1$; the *false* and *true* values given by $\text{ff} \stackrel{\text{def}}{=} \text{in}_1() : \text{Bool}$ and $\text{tt} \stackrel{\text{def}}{=} \text{in}_2() : \text{Bool}$; the *zero natural number* $\text{zero} \stackrel{\text{def}}{=} \text{fold } \text{in}_1() : \text{Nat}$ and the *successor function* $\text{succ} \stackrel{\text{def}}{=} \lambda n^{\text{Nat}}.\text{fold } \text{in}_2 n : \text{Nat} \rightarrow \text{Nat}$; among many others.

2.3. The Reduction Rules of PFPC

To describe execution of programs in PFPC, we use a small-step call-by-value operational semantics which is described in Figure 4. The reduction relation $M \xrightarrow{p} N$ should be understood as specifying that term M reduces to term N with probability $p \in [0, 1]$ in exactly one step. Our reduction rules are simply the standard rules for small-step reduction in FPC [29, §20] and small-step reduction for probabilistic choice [30]. Of course, it is well-known this system is type-safe.

Theorem 3. *If $\Gamma \vdash M : A$ and $M \xrightarrow{p} N$, then $\Gamma \vdash N : A$. In this situation, if $p < 1$, then there exists a term N' , such that $M \xrightarrow{1-p} N'$. Furthermore, if $\cdot \vdash M : A$, then either M is a value or there exists N , such that $M \xrightarrow{p} N$ for some $p \in (0, 1]$.*

Assumption 4. Throughout the rest of the paper, we implicitly assume that all types, terms and contexts are well-formed.

2.4. Recursion and Asymptotic Behaviour of Reduction

It is well-known that type recursion in FPC induces term recursion [27, 26, 29] and the same is true for PFPC. This allows us to *derive* the call-by-value fixpoint operator

$$\cdot \vdash \text{fix}_{A \rightarrow B} : ((A \rightarrow B) \rightarrow A \rightarrow B) \rightarrow A \rightarrow B$$

at any function type $A \rightarrow B$ (see [26] and [27, §8] for more details). Using $\text{fix}_{A \rightarrow B}$, we may write recursive functions.

Example 5. Consider the following program:

$$\text{coins} \stackrel{\text{def}}{=} \text{fix}_{1 \rightarrow 1} \lambda f^{1 \rightarrow 1}. \lambda x^1. \text{case}(\text{ff or}_{0.5} \text{tt}) \text{ of } \text{in}_1 z \Rightarrow () \mid \text{in}_2 z \Rightarrow fx.$$

It follows $\cdot \vdash \text{coins} : 1 \rightarrow 1$. Evaluating at $()$ shows that $\text{coins}()$ performs a fair coin toss and depending on the outcome, either terminates to $()$ or repeats the process again. We see that there is no upper bound on the number of coin tosses this program would perform. On the other hand, it is easy to see that the probability $\text{coins}()$ terminates to $()$ is precisely $\sum_{i=1}^{\infty} 2^{-i} = 1$.

The above simple example shows that a rigorous operational analysis of PFPC has to consider the *asymptotic behaviour* of terms under reduction. We do this by showing how to determine the probability that a term reduces to a value in any number of steps. We will later see that this is crucial for proving our adequacy result (Theorem 61).

We may determine the overall probability that a term M reduces to a value V in the same way as in [10]. The *probability weight* of a reduction path $\pi = (M_1 \xrightarrow{p_1} \dots \xrightarrow{p_n} M_n)$ is $P(\pi) \stackrel{\text{def}}{=} \prod_{i=1}^n p_i$. The probability that term M reduces to the value V in *at most* n steps is

$$P(M \rightarrow_{\leq n} V) \stackrel{\text{def}}{=} \sum_{\pi \in \text{Paths}_{\leq n}(M, V)} P(\pi),$$

where $\text{Paths}_{\leq n}(M, V)$ is the set of all reduction paths from M to V of length at most n . The probability that term M reduces to value V (in *any finite number* of steps) is $P(M \rightarrow_* V) \stackrel{\text{def}}{=} \sup_i P(M \rightarrow_{\leq i} V)$.

Finally, the probability that term M *terminates* is denoted $\text{Halt}(M)$ and it is determined in the following way:

$$\text{Val}(M) \stackrel{\text{def}}{=} \{V \mid V \text{ is a value and } P(M \rightarrow_* V) > 0\} \quad (1)$$

$$\text{Halt}(M) \stackrel{\text{def}}{=} \sum_{V \in \text{Val}(M)} P(M \rightarrow_* V). \quad (2)$$

Note that the sum in (2) is countably infinite, in general.

3. Commutative Monads for Probability

In this section we present a novel and general construction for commutative probabilistic monads on **DCPO** and we use it to identify three such monads.

3.1. Domain-theoretic and Topological Preliminaries

We begin by recalling some background on domain theory and topology.

3.1.1. Directed-Complete Partial Orders

A nonempty subset A of a partially ordered set (*poset*) D is *directed* if each pair of elements in A has an upper bound in A . A *directed-complete partial order*, (*dcpo*, for short) is a poset in which every directed subset A has a supremum $\sup A$. For example, the unit interval $[0, 1]$ is a dcpo in the usual ordering. A function $f: D \rightarrow E$ between two (posets) dcpo's is *Scott-continuous* if it is monotone and preserves (existing) suprema of directed subsets.

The category **DCPO** of dcpo's and Scott-continuous functions is complete, cocomplete and cartesian closed [31]. We denote with $A_1 \times A_2$ ($A_1 + A_2$) the categorical (co)product of the dcpo's A_1 and A_2 and with π_1, π_2 (in_1, in_2) the associated (co)projections. We denote with \emptyset and 1 the initial and terminal objects of **DCPO**; these are the empty dcpo and the singleton dcpo, respectively. **DCPO** is Cartesian closed, where the internal hom of A and B is $[A \rightarrow B]$, the Scott-continuous functions $f: A \rightarrow B$ ordered pointwise.

The category **DCPO**_{⊥!} of *pointed* dcpo's and *strict* Scott-continuous functions also is important. **DCPO**_{⊥!} is symmetric monoidal closed when equipped with the smash product and strict Scott-continuous function space, and it is also complete and cocomplete [31].

3.1.2. The Scott Topology

The *Scott topology* σD on a dcpo D consists of the upper subsets $U = \uparrow U = \{x \in D \mid (\exists u \in U) u \leq x\}$ that are *inaccessible by directed suprema*: i.e., if $A \subseteq D$ is directed and $\sup A \in U$, then $A \cap U \neq \emptyset$. The space $(D, \sigma D)$ is also written as ΣD . Scott-continuous functions between dcpo's D and E are exactly the continuous functions between ΣD and ΣE [32, Proposition II-2.1]. We equip $[0, 1]$ with the Scott topology induced by the usual order unless stated otherwise.

3.1.3. The d-topology

A subset B of a dcpo D is a *sub-dcpo* if every directed subset $A \subseteq B$ satisfies $\sup_D A \in B$. In this case, B is a dcpo in the induced order from D . The *d-topology* on D is the topology whose closed subsets consist of sub-dcpo's of D . Open (closed) sets in the d-topology will be called *d-open* (*d-closed*). The *d-closure* of $C \subseteq D$ is the topological closure of C with respect to the d-topology

on D , which is the intersection of all sub-dcpo's of D containing C . The d-topology on D is finer than the Scott topology. In fact D is even Hausdorff in the d-topology: for $x \not\leq y$ in D , $D \setminus \downarrow y$ and $\downarrow y$ are disjoint open sets in the d-topology, containing x and y respectively. Functions that are continuous between dcpo's equipped with the d-topology are called *d-continuous* functions. Scott-continuous functions between dcpo's are d-continuous [33, Lemma 5].

3.1.4. Monotone-convergence Spaces

The family of open sets of a topological space X , denoted $\mathcal{O}X$, is a complete lattice in the inclusion order. The *specialization order* \leq_X on X is defined as $x \leq_X y$ if and only if x is in the closure of $\{y\}$, for $x, y \in X$. We write ΩX to denote X equipped with the specialization order. It is well-known that X is T_0 if and only if ΩX is a poset. A subset of X is called *saturated* if it is an upper set in ΩX . A space X is called a *d-space* or a *monotone-convergence space* if ΩX is a dcpo and each open set of X is Scott open in ΩX . As an example, ΣD is always a d-space for each dcpo D . The full subcategory of \mathbf{T}_0 consisting of d-spaces is denoted by \mathbf{D} . There is a functor $\Sigma: \mathbf{DCPO} \rightarrow \mathbf{D}$ that assigns the space ΣD to each dcpo D , and the map $f: \Sigma D \rightarrow \Sigma E$ to the Scott-continuous map $f: D \rightarrow E$. Dually, the functor $\Omega: \mathbf{D} \rightarrow \mathbf{DCPO}$ assigns ΩX to each d-space X and the map $f: \Omega X \rightarrow \Omega Y$ to each continuous map $f: X \rightarrow Y$. In fact, $\Sigma \dashv \Omega$, i.e., Σ is left adjoint to Ω [34].

3.1.5. Sober Spaces

A T_0 space X is called *sober* if every nonempty closed irreducible subset of X is the closure of some (unique) singleton set, where $A \subseteq X$ is *irreducible* if $A \subseteq B \cup C$ with B and C nonempty closed subsets implies $A \subseteq B$ or $A \subseteq C$. The category of sober spaces and continuous functions is denoted by \mathbf{SOB} . Sober spaces are d-spaces, hence $\mathbf{SOB} \subseteq \mathbf{D}$ [15].

3.2. A Commutative Monad for Probability

In this subsection we describe a novel commutative monad of valuations \mathcal{M} which we show is a submonad of the valuations monad \mathcal{V} of Jones and Plotkin [12].

3.2.1. Valuations on Topological Spaces

To begin, a *subprobability valuation* on a topological space X is a Scott-continuous function $\nu: \mathcal{O}X \rightarrow [0, 1]$ that is strict ($\nu(\emptyset) = 0$), and modular ($\nu(U) + \nu(V) = \nu(U \cup V) + \nu(U \cap V)$). The set of subprobability valuations on X is denoted by $\mathcal{V}X$. The *stochastic order* on $\mathcal{V}X$ is defined pointwise: $\nu_1 \leq \nu_2$ if and only if $\nu_1(U) \leq \nu_2(U)$ for all $U \in \mathcal{O}X$. $\mathcal{V}X$ is a pointed dcpo in the stochastic order, with least element given by the constantly zero valuation $\mathbf{0}_X$ and where the supremum of a directed family $\{\nu_i\}_{i \in I}$ is $\sup_{i \in I} \nu_i \stackrel{\text{def}}{=} \lambda U. \sup_{i \in I} \nu_i(U)$.

The canonical examples of subprobability valuations are the *Dirac valuations* δ_x for $x \in X$, defined by $\delta_x(U) = 1$ if $x \in U$ and $\delta_x(U) = 0$ otherwise. $\mathcal{V}X$ enjoys a convex structure: if $\nu_i \in \mathcal{V}X$ and $r_i \geq 0$, with $\sum_{i=1}^n r_i \leq 1$, then the

convex sum $\sum_{i=1}^n r_i \nu_i \stackrel{\text{def}}{=} \lambda U. \sum_{i=1}^n r_i \nu_i(U)$ also is in $\mathcal{V}X$. The *simple valuations* on D are those of the form $\sum_{i=1}^n r_i \delta_{x_i}$, where $x_i \in X$, $r_i > 0$, $i = 1, \dots, n$ and $\sum_{i=1}^n r_i \leq 1$. The set of simple valuations on X is denoted by $\mathcal{S}X$. Clearly, $\mathcal{S}X \subseteq \mathcal{V}X$. Unlike $\mathcal{V}X$, $\mathcal{S}X$ is not directed-complete in the stochastic order in general.

Given $\nu \in \mathcal{V}X$ and $f: X \rightarrow [0, 1]$ continuous, we can define the integral of f against ν by the Choquet formula

$$\int_{x \in X} f(x) d\nu \stackrel{\text{def}}{=} \int_0^1 \nu(f^{-1}((t, 1])) dt,$$

where the right side is a Riemann integral of the bounded antitone function $\lambda t. \nu(f^{-1}((t, 1]))$. If no confusion occurs, we simply write $\int_{x \in X} f(x) d\nu$ as $\int f d\nu$. We note that the map $\nu \mapsto \int f d\nu: \mathcal{V}X \rightarrow [0, 1]$, for a fixed f , is Scott-continuous, and

$$\int f d \sum_i^n r_i \delta_{x_i} = \sum_{i=1}^n r_i f(x_i) \quad (3)$$

for $\sum_{i=1}^n r_i \delta_{x_i} \in \mathcal{V}X$. Some other basic properties of the integral, which can be found in [13], are described by the next proposition.

Proposition 6. *Let X be a topological space and $f: X \rightarrow [0, 1]$ be a continuous function (where $[0, 1]$ is equipped with the Scott topology). Then:*

1. *Let $r_i \in [0, 1]$ for $i = 1, \dots, n$ with $\sum_{i=1}^n r_i \leq 1$, then for any $1 \leq k \leq n$ and any collection of valuations $\nu_j \in \mathcal{V}X$ with $j \neq k$, it follows that the map $(\nu_k \mapsto \sum_{i=1}^n r_i \nu_i): \mathcal{V}X \rightarrow \mathcal{V}X$ is Scott-continuous hence d -continuous.*
2. *For $\sum_{i=1}^n r_i \nu_i \in \mathcal{V}X$, it is true that $\int f d \sum_{i=1}^n r_i \nu_i = \sum_{i=1}^n r_i \int f d\nu_i$.*
3. *For $\nu \in \mathcal{V}X$ and $f, g \in [X \rightarrow [0, 1]]$, $\int r f + s g d\nu = r \int f d\nu + s \int g d\nu$ where $r, s \in [0, 1]$ and $r + s \leq 1$.*

3.2.2. (Commutative) Valuations Monads on Dcpo's

For a dcpo D , $\mathcal{V}D$ is defined as $\mathcal{V}(D, \sigma D)$. Using Manes' description of monads (Kleisli triples) [35], Jones proved in her PhD thesis [13] that \mathcal{V} is a monad on **DCPO**:

- The *unit* of \mathcal{V} at D is $\eta_D^\mathcal{V}: D \rightarrow \mathcal{V}D :: x \mapsto \delta_x$;
- The *Kleisli extension* f^\dagger of a Scott-continuous map $f: D \rightarrow \mathcal{V}E$ maps $\nu \in \mathcal{V}D$ to $f^\dagger(\nu) \in \mathcal{V}E$ by

$$f^\dagger(\nu) \stackrel{\text{def}}{=} \lambda U \in \sigma E. \int_{x \in D} f(x)(U) d\nu.$$

Then the *multiplication* $\mu_D^\mathcal{V}: \mathcal{V}\mathcal{V}D \rightarrow \mathcal{V}D$ is given by $\text{id}_{\mathcal{V}D}^\dagger$; it maps $\varpi \in \mathcal{V}\mathcal{V}D$ to $\lambda U \in \sigma D. \int_{\nu \in \mathcal{V}D} \nu(U) d\varpi \in \mathcal{V}D$. Thus, \mathcal{V} defines an endofunctor on **DCPO** that sends a dcpo D to $\mathcal{V}D$, and a Scott-continuous map $h: D \rightarrow E$ to $\mathcal{V}(h) \stackrel{\text{def}}{=} (\eta_E \circ h)^\dagger$; concretely, $\mathcal{V}(h)$ maps $\nu \in \mathcal{V}D$ to $\lambda U \in \sigma E. \nu(h^{-1}(U))$.

Jones [13] also showed that \mathcal{V} is a strong monad over **DCPO**: its strength at (D, E) is given by

$$\tau_{DE}^{\mathcal{V}}: D \times \mathcal{V}E \rightarrow \mathcal{V}(D \times E) :: (x, \nu) \mapsto \lambda U. \int_{y \in E} \chi_U(x, y) d\nu,$$

where χ_U is the characteristic function of $U \in \sigma(D \times E)$. Whether \mathcal{V} is a commutative monad on **DCPO** has remained an open problem for decades. Proving this to be true requires showing the following Fubini-type equation holds:

$$\int_{x \in D} \int_{y \in E} \chi_U(x, y) d\xi d\nu = \int_{y \in E} \int_{x \in D} \chi_U(x, y) d\nu d\xi, \quad (4)$$

for dcpo's D and E , for $U \in \sigma(D \times E)$ and for $\nu \in \mathcal{V}D, \xi \in \mathcal{V}E$ [12, Section 6]. The difficulty lies in the well-known fact that a Scott open set $U \in \sigma(D \times E)$ might not be open in the product topology $\sigma D \times \sigma E$ in general [32, Exercise II-4.26].

However, if either ν or ξ is a simple valuation, then Equation (4) holds. For example, if $\nu = \sum_{i=1}^n r_i \delta_{x_i} \in \mathcal{S}D$, then by (3) both sides of (4) are equal to $\sum_{i=1}^n r_i \int_{y \in E} \chi_U(x_i, y) d\xi$. The Scott continuity of the integral in ν then implies Equation (4) holds for valuations that are directed suprema of simple valuations. This is why, for example, \mathcal{V} is a commutative monad on the category of domains and Scott-continuous maps [13], as we now explain.

If D is a dcpo and $x, y \in D$, we say x is *way-below* y (in symbols, $x \ll y$) if and only if for every directed set A with $y \leq \sup A$, there is some $a \in A$ such that $x \leq a$. We write $\downarrow y = \{x \in D \mid x \ll y\}$. A *basis* for a dcpo D is a subset B satisfying $\downarrow x \cap B$ is directed and $x = \sup \downarrow x \cap B$, for each $x \in D$. D is *continuous* if it has a basis. Continuous dcpo's are also called *domains*, and the category of domains and Scott-continuous maps is denoted by **DOM**. When D is a domain, then $\mathcal{V}D$ is also a domain with basis given by $\mathcal{S}D$ [13], and therefore every valuation on D is a directed supremum of simple valuations. Thus Equation (4) holds for domains and therefore \mathcal{V} is a commutative monad on **DOM**.

In general, not every valuation on a dcpo D is a directed supremum of simple valuations [36]. However, we can apply the reasoning above about simple valuations to obtain a commutative submonad of \mathcal{V} by restricting to a suitable completion of $\mathcal{S}D$ inside $\mathcal{V}D$. There are several possibilities (cf. [37]), and we choose the smallest and simplest – the d -closure of $\mathcal{S}D$ in $\mathcal{V}D$.

Definition 7. For each dcpo D , we define $\mathcal{M}D$ to be the intersection of all sub-dcpo's of $\mathcal{V}D$ that contain $\mathcal{S}D$.¹

Since $\mathcal{V}D$ itself is a dcpo containing $\mathcal{S}D$, it is immediate from the definition of sub-dcpo's that $\mathcal{M}D$ is a well-defined dcpo in the stochastic order with $\mathcal{S}D \subseteq \mathcal{M}D \subseteq \mathcal{V}D$. Note that $\mathcal{M}D$ is the smallest sub-dcpo of $\mathcal{V}D$ that contains $\mathcal{S}D$,

¹The same definition applies in the case of topological spaces.

hence $\mathcal{M}D$ is actually the topological closure of $\mathcal{S}D$ in $\mathcal{V}D$ equipped with the d-topology. Hence we also say that $\mathcal{M}D$ is the *d-closure* of $\mathcal{S}D$ inside $\mathcal{V}D$. Analogous to $\mathcal{V}D$, $\mathcal{M}D$ also enjoys a convex structure.

Lemma 8. *For $\nu_i \in \mathcal{M}D$ and $r_i \geq 0, i = 1, \dots, n$ with $\sum_{i=1}^n r_i \leq 1$, the convex sum $\sum_{i=1}^n r_i \nu_i$ is still in $\mathcal{M}D$.*

Proof. We prove the case $n = 2$ and the general case can be proved similarly. We realize that for a fixed simple valuation $s \in \mathcal{S}D$, the map $(\nu \mapsto r_1\nu + r_2s): \mathcal{V}D \rightarrow \mathcal{V}D$ maps $\mathcal{S}D$ into $\mathcal{S}D$. From the Proposition 6, Item 1, this map is d-continuous, it then maps the dcpo-closure of $\mathcal{S}D$, which is $\mathcal{M}D$, into $\mathcal{M}D$, the dcpo-closure of $\mathcal{S}D$. That is, for each simple valuation s and each $\nu \in \mathcal{M}D$, $r_1\nu + r_2s \in \mathcal{M}D$. Now we fix $\nu \in \mathcal{M}D$. Then the map $\xi \mapsto r_1\nu + r_2\xi: \mathcal{V}D \rightarrow \mathcal{V}D$ maps $\mathcal{S}D$ into $\mathcal{M}D$, hence it also maps $\mathcal{M}D$ into $\mathcal{M}D$ since it is d-continuous. This means for $\xi, \nu \in \mathcal{M}D, r_1, r_2 \in [0, 1]$ with $r_1 + r_2 \leq 1, r_1\nu + r_2\xi \in \mathcal{M}D$. \square

For the proofs of the following results, we repeatedly use the fact that Scott-continuous maps between dcpos D and E are *d-continuous*, i.e., continuous when D and E are equipped with the d-topology [33, Lemma 5].

Theorem 9. *\mathcal{M} is a commutative monad on **DCPO**.*

Proof. We sketch the key steps in showing \mathcal{M} is commutative:

Unit. The unit of \mathcal{M} at D is $\eta_D^{\mathcal{M}}: D \rightarrow \mathcal{M}D :: x \mapsto \delta_x$, the co-restriction of $\eta_D^{\mathcal{V}}$ to $\mathcal{M}D$. Obviously, it is a well-defined Scott-continuous map.

Extension. Since a Scott-continuous map $f: D \rightarrow \mathcal{M}E$ is also Scott-continuous from D to $\mathcal{V}E$, the Kleisli extension $f^\dagger: \mathcal{M}D \rightarrow \mathcal{M}E$ of f can be defined as the restriction and co-restriction of $f^\dagger: \mathcal{V}D \rightarrow \mathcal{V}E$ to $\mathcal{M}D$ and $\mathcal{M}E$, respectively. The validity of this definition requires $f^\dagger(\mathcal{M}D) \subseteq \mathcal{M}E$, which boils down to $f^\dagger(\mathcal{S}D) \subseteq \mathcal{M}E$ by d-continuity of f^\dagger , since f^\dagger is Scott-continuous. Hence we only need to check that $f^\dagger(\sum_{i=1}^n r_i \delta_{x_i}) \in \mathcal{M}E$ for each $\sum_{i=1}^n r_i \delta_{x_i} \in \mathcal{S}D$. However, $f^\dagger(\sum_{i=1}^n r_i \delta_{x_i}) = \sum_{i=1}^n r_i f(x_i)$, which is indeed in $\mathcal{M}E$ by Lemma 8.

Strength. The strength $\tau_{DE}^{\mathcal{M}}$ of \mathcal{M} at (D, E) is given by $\tau_{DE}^{\mathcal{V}}$ restricted to $D \times \mathcal{M}E$ and co-restricted to $\mathcal{M}(D \times E)$. This is well-defined provided that $\tau_{DE}^{\mathcal{V}}$ maps $D \times \mathcal{M}E$ into $\mathcal{M}(D \times E)$. Again, we only need to prove that $\tau_{DE}^{\mathcal{V}}$ maps $D \times \mathcal{S}E$ into $\mathcal{M}(D \times E)$ and conclude the proof with the d-continuity of $\tau_{DE}^{\mathcal{V}}$ in its second component. Towards this end, we pick $(a, \sum_{i=1}^n r_i \delta_{y_i}) \in D \times \mathcal{S}E$, and see

$$\begin{aligned} \tau_{DE}^{\mathcal{V}}(a, \sum_{i=1}^n r_i \delta_{y_i}) &= \lambda U. \int \chi_U(a, y) d \sum_{i=1}^n r_i \delta_{y_i} \\ &\stackrel{(3)}{=} \lambda U. \sum_{i=1}^n r_i \chi_U(a, y_i) \\ &= \lambda U. \sum_{i=1}^n r_i \delta_{(a, y_i)}(U) \stackrel{\text{def}}{=} \sum_{i=1}^n r_i \delta_{(a, y_i)} \end{aligned}$$

is indeed in $\mathcal{M}(D \times E)$.

With f^\ddagger and $\tau^{\mathcal{M}}$ well-defined, the same arguments used to prove $(\mathcal{V}, \eta^{\mathcal{V}}, _^\ddagger, \tau^{\mathcal{V}})$ is a strong monad in [13] prove $(\mathcal{M}, \eta^{\mathcal{M}}, _^\ddagger, \tau^{\mathcal{M}})$ is a strong monad on **DCPO**.

Commutativity. Finally, we show \mathcal{M} is commutative by proving Equation (4) holds for any dcpo's D and E and $\nu \in \mathcal{M}D, \xi \in \mathcal{M}E$. As commented above, this holds if ν is simple, and then the Scott-continuity of the integral in the ν -component implies Equation (4) also holds for directed suprema of simple valuations, directed suprema of directed suprema of simple valuations and so forth, transfinitely. But these are exactly the valuations $\mathcal{M}D$.

Formally, we consider for each fixed $\xi \in \mathcal{M}E$ (even for $\xi \in \mathcal{V}E$) the functions

$$F: \nu \mapsto \int_{x \in D} \int_{y \in E} \chi_U(x, y) d\xi d\nu: \mathcal{M}D \rightarrow [0, 1]$$

and

$$G: \nu \mapsto \int_{y \in E} \int_{x \in D} \chi_U(x, y) d\nu d\xi: \mathcal{M}D \rightarrow [0, 1].$$

Note that both F and G are Scott-continuous functions hence d-continuous, and they are equal on $\mathcal{S}D$ by Equation (3). Since $[0, 1]$ is Hausdorff in the d-topology, F and G are then equal on the d-closure of $\mathcal{S}D$ which is, by construction, $\mathcal{M}D$. \square

Remark 10. The multiplication $\mu_D^{\mathcal{M}}$ of \mathcal{M} at D is given by $(\text{id}_{\mathcal{M}D})^\ddagger$. Concretely, $\mu_D^{\mathcal{M}}$ maps each valuation $\varpi \in \mathcal{M}(\mathcal{M}D)$ to $\lambda U \in \sigma D$. $\int_{\nu \in \mathcal{M}D} \nu(U) d\varpi$. In particular, $\mu_D^{\mathcal{M}}$ maps each simple valuation $\sum_{i=1}^n r_i \delta_{\nu_i} \in \mathcal{M}(\mathcal{M}D)$ to $\sum_{i=1}^n r_i \nu_i$, where $\nu_i \in \mathcal{M}D, i = 1, \dots, n$, and $\sum_{i=1}^n r_i \leq 1$.

Remark 11. The double strength of \mathcal{M} at (D, E) is given by the Scott-continuous map $(\nu, \xi) \mapsto \nu \otimes \xi: \mathcal{M}(D) \times \mathcal{M}(E) \rightarrow \mathcal{M}(D \times E)$, where $\nu \otimes \xi$ is defined as $\lambda U \in \sigma(D \times E)$. $\int_{y \in E} \int_{x \in D} \chi_U(x, y) d\nu d\xi$.

Remark 12. We note that $\mathcal{M}D$ is the first example of a commutative valuations monad on **DCPO** that contains the simple valuations. And, since every valuation on a domain D is a directed supremum of simple valuations [13, Theorem 5.2], it follows that $\mathcal{M} = \mathcal{V}$ on the category **DOM**.

Readers may notice a similarity between our monad \mathcal{M} and \mathcal{T} of [9], because the object-mappings are defined as the smallest dcpo (ω QBS, respectively) that satisfy some property, of a larger monad. This is a common technique in topology and in other fields of mathematics. However, the ambient (larger) monads are different (valuations vs continuations monads) and all subsequent proofs are also very different – we use topological properties (**K**-completions), whereas [9] uses categorical factorisation systems.

3.3. Dcpo-completion versus **D**-completion

Recall that a *dcpo-completion* of a poset P is a pair (D, e) , where D is a dcpo and $e: P \rightarrow D$ is an injective Scott-continuous map, such that for any dcpo E

and Scott-continuous map $f: P \rightarrow E$, there exists a unique Scott-continuous map $f': D \rightarrow E$ satisfying $f = f' \circ e$. The dcpo-completion of posets always exists [33, Theorem 1].

As we have seen, for each dcpo D , $\mathcal{M}D$ is the smallest sub-dcpo in $\mathcal{V}D$ containing $\mathcal{S}D$, one may wonder whether $\mathcal{M}D$, together with the inclusion map from $\mathcal{S}D$ into $\mathcal{M}D$, is a dcpo-completion of $\mathcal{S}D$. The answer is “no” in general. The reason is that the inclusion of $\mathcal{S}D$ into $\mathcal{M}D$ may not be Scott-continuous, even when D is a domain (see [37, Section 6]). The construction $\mathcal{M}D$ is actually more in a topological flavour, as we now explain. For simplicity, we assume all spaces considered in the sequel are in \mathbf{T}_0 , the category of T_0 spaces and continuous maps.

Definition 13. Let X be a topological space. The *weak topology* on $\mathcal{V}X$ is generated by the sets

$$[U > r] \stackrel{\text{def}}{=} \{\nu \in \mathcal{V}X \mid \nu(U) > r\},$$

which form a subbasis, where U is open in X and $r \in [0, 1]$.

Remark 14. For each continuous map $f: X \rightarrow [0, 1]$ and $r \in [0, 1]$, the set $[f > r] \stackrel{\text{def}}{=} \{\nu \in \mathcal{V}X \mid \int f d\nu > r\}$ is open in the weak topology.

We use $\mathcal{V}_w X$ to denote the space $\mathcal{V}X$ equipped with the weak topology. We will use the fact that $\mathcal{V}_w X$ is a sober space, which follows from [38, Proposition 5.1]. It is easy to see that the specialization order on $\mathcal{V}_w X$ is just the stochastic order. Hence $\mathcal{V}X = \Omega(\mathcal{V}_w X)$.

We also use $\mathcal{S}_w X$ ($\mathcal{M}_w X$) to denote the space $\mathcal{S}X$ ($\mathcal{M}X$) endowed with the relative topology from $\mathcal{V}_w X$. Accordingly, $\mathcal{M}X = \Omega(\mathcal{M}_w X)$, and $\mathcal{S}X = \Omega(\mathcal{S}_w X)$. Although $\mathcal{M}X$ is not the dcpo-completion of $\mathcal{S}X$ in general, we do have the following proposition.

Proposition 15. *For each space X , $\mathcal{M}_w X$ is a \mathbf{D} -completion of $\mathcal{S}_w X$. That is, $\mathcal{M}_w X$ itself is a d -space, an object in \mathbf{D} ; the inclusion map $i: \mathcal{S}_w X \rightarrow \mathcal{M}_w X$ is continuous; and for any d -space Y and continuous map $f: \mathcal{S}_w X \rightarrow Y$, there exists a unique continuous map $f': \mathcal{M}_w X \rightarrow Y$ such that $f = f' \circ i$.*

The above proposition is a straightforward application of Keimel and Lawson’s K-category theory [15] to the category \mathbf{D} .

Definition 16. A K-category \mathbf{K} is a full subcategory of \mathbf{T}_0 , whose objects will be called k-spaces, satisfying:

1. Homeomorphic copies of k-spaces are k-spaces;
2. All sober spaces are k-spaces, i.e., $\mathbf{SOB} \subseteq \mathbf{K}$;
3. In a sober space S , the intersection of any family of k-subspaces, equipped with the relative topology from S , is a k-space;
4. For any continuous map $f: S \rightarrow T$ between sober spaces S and T , and any k-subspace K of T , $f^{-1}(K)$ is a k-subspace of S .

Each \mathbf{K} -category \mathbf{K} is a full reflective subcategory of \mathbf{T}_0 and the unit of the induced adjunction at X is called the \mathbf{K} -completion of X . More concretely, a possible completion process goes as follows [15, Theorem 4.4]: first, pick any $j: X \rightarrow Y$ such that Y is sober and j is a topological embedding. For example, one can take j as the embedding of X into its standard sobrification. Second, let \tilde{X} be the intersection of all \mathbf{k} -subspaces of Y containing $j(X)$ and equip it with the relative topology from Y . Then \tilde{X} , together with the co-restriction $i: X \rightarrow \tilde{X}$ of j , is a \mathbf{K} -completion of X .

Now we apply this procedure to prove Proposition 15. First, note that \mathbf{D} is indeed a \mathbf{K} -category as proved in [15, Lemma 6.4]. We embed $\mathcal{S}_w X$ into the sober space $\mathcal{V}_w X$, and notice that all \mathbf{d} -subspaces of $\mathcal{V}_w X$ are precisely sub-dcpo's of $\mathcal{V}X$. Hence $\mathcal{M}_w X$, which is the intersection of sub-dcpo's $\mathcal{V}X$ containing $\mathcal{S}X$ equipped with the relative topology from $\mathcal{V}_w X$, is a \mathbf{D} -completion of $\mathcal{S}_w X$.

3.4. A uniform construction

Proposition 15 motivates the next definition.

Definition 17. Let \mathbf{K} be a \mathbf{K} -category. For each space X , we define $\mathcal{V}_{\mathbf{K}} X$ to be the intersection of all \mathbf{k} -subspaces of $\mathcal{V}_w X$ containing $\mathcal{S}_w X$, equipped with the relative topology from $\mathcal{V}_w X$.

As discussed above, $\mathcal{V}_{\mathbf{K}} X$ is a \mathbf{K} -completion of $\mathcal{S}_w X$. It was proved in [37, Theorem 3.5]² that $\mathcal{V}_{\mathbf{K}}: \mathbf{T}_0 \rightarrow \mathbf{T}_0$ is a monad for each \mathbf{K} -category \mathbf{K} : The unit of $\mathcal{V}_{\mathbf{K}}$ at X maps $x \in X$ to δ_x , and for any continuous map $f: X \rightarrow \mathcal{V}_{\mathbf{K}} Y$, the Kleisli extension $f^\dagger: \mathcal{V}_{\mathbf{K}} X \rightarrow \mathcal{V}_{\mathbf{K}} Y$ maps ν to $\lambda U \in \mathcal{O}Y. \int_{x \in X} f(x)(U) d\nu$. Therefore, if \mathbf{K} is a full subcategory of \mathbf{D} , then according to the construction $\mathcal{V}_{\mathbf{K}} X$ is always a \mathbf{d} -space for each X , hence the monad $\mathcal{V}_{\mathbf{K}}: \mathbf{T}_0 \rightarrow \mathbf{T}_0$ can be restricted to a monad on \mathbf{D} .

Theorem 18. Let \mathbf{K} be a \mathbf{K} -category with $\mathbf{K} \subseteq \mathbf{D}$. Then $\mathcal{V}_{\mathbf{K}, \leq} \stackrel{\text{def}}{=} \Omega \circ \mathcal{V}_{\mathbf{K}} \circ \Sigma$ is a monad on \mathbf{DCPO} .

$$\begin{array}{ccccc}
 & & \mathcal{V}_{\mathbf{K}} & & \\
 & & \curvearrowright & & \\
 \mathbf{DCPO} & \xleftrightarrow[\Omega]{\Sigma} & \mathbf{D} & \xleftrightarrow[\mathcal{U}]{\mathcal{F}} & \mathbf{D}^{\mathcal{V}_{\mathbf{K}}} \\
 & \perp & & \perp & \\
 & \xleftrightarrow{\quad} & & \xleftrightarrow{\quad} &
 \end{array}$$

Proof. Let $\mathbf{D}^{\mathcal{V}_{\mathbf{K}}}$ be the Eilenberg-Moore category of $\mathcal{V}_{\mathbf{K}}$ over \mathbf{D} and $\mathcal{F} \dashv \mathcal{U}$ be the adjunction that recovers $\mathcal{V}_{\mathbf{K}}$, then $\mathcal{V}_{\mathbf{K}, \leq} = \Omega \circ \mathcal{U} \circ \mathcal{F} \circ \Sigma$. The statement follows from the standard categorical fact that adjoints compose: $\mathcal{F} \circ \Sigma \dashv \Omega \circ \mathcal{U}$. \square

Remark 19. The unit of $\mathcal{V}_{\mathbf{K}, \leq}$ at a dcpo D sends $x \in D$ to δ_x , and for dcpo's D and E , the Kleisli extension $f^\dagger: \mathcal{V}_{\mathbf{K}, \leq} D \rightarrow \mathcal{V}_{\mathbf{K}, \leq} E$ of $f: D \rightarrow \mathcal{V}_{\mathbf{K}, \leq} E$ maps ν to $\lambda U \in \sigma E. \int_{x \in D} f(x)(U) d\nu$.

²The authors allow valuations to take values in $[0, \infty]$. However, the theorem is also true for valuations with values in $[0, 1]$

Remark 20. $\mathcal{M}_w = \mathcal{V}_{\mathbf{D}}$ and $\mathcal{M} = \mathcal{V}_{\mathbf{D}, \leq}$.

Note that the category **SOB** of sober spaces is the smallest \mathbf{K} -category [15, Remark 4.1]. We denote $\mathcal{V}_{\mathbf{SOB}}$ by \mathcal{P}_w and $\mathcal{V}_{\mathbf{SOB}, \leq}$ by \mathcal{P} . The following statement is then obvious.

Proposition 21. *Let \mathbf{K} be a \mathbf{K} -category with $\mathbf{K} \subseteq \mathbf{D}$. Then for each dcpo D , we have $\mathcal{S}D \subseteq \mathcal{M}D \subseteq \mathcal{V}_{\mathbf{K}, \leq}D \subseteq \mathcal{P}D \subseteq \mathcal{V}D$.*

Heckmann [38, Theorem 5.5] proved that $\mathcal{P}D$ consists of the so-called *point-continuous* valuations on D . We show that Equation 4 holds when either ν or ξ is point-continuous.

Theorem 22. *Let D and E be dcpo's, and $U \in \sigma(D \times E)$. Then the equation*

$$\int_{x \in D} \int_{y \in E} \chi_U(x, y) d\xi d\nu = \int_{y \in E} \int_{x \in D} \chi_U(x, y) d\nu d\xi,$$

holds for $(\nu, \xi) \in \mathcal{P}D \times \mathcal{V}E$ (equivalently, $(\nu, \xi) \in \mathcal{V}D \times \mathcal{P}E$).

Proof. We first recall two results due to Heckmann [38, Theorem 2.4, Theorem 5.5]. Specifying these results to dcpo D , it implies that if ν is a point-continuous valuation in $\mathcal{P}D$, and $\nu \in \mathcal{O}$ for \mathcal{O} an open set in \mathcal{P}_wD , then there exists a simple valuation $\sum_{i=1}^n r_i \delta_{x_i} \in \mathcal{S}D$ such that $\sum_{i=1}^n r_i \delta_{x_i} \leq \nu$ and $\sum_{i=1}^n r_i \delta_{x_i} \in \mathcal{O}$.

Now we fix $\xi \in \mathcal{P}E$ and $U \in \sigma(D \times E)$, and consider the functions

$$F: \mathcal{V}_wD \rightarrow [0, \infty]: \nu \mapsto \int_{x \in D} \int_{y \in E} \chi_U(x, y) d\xi d\nu$$

and

$$G: \mathcal{V}_wD \rightarrow [0, \infty]: \nu \mapsto \int_{y \in E} \int_{x \in D} \chi_U(x, y) d\nu d\xi,$$

where $[0, \infty]$ is equipped with the Scott topology. We claim that F and G are continuous.

The fact that F is continuous is straightforward from Remark 14. To see that G is continuous, we assume that $\int_{y \in E} \int_{x \in D} \chi_U d\nu d\xi > r$ and aim to find an open set \mathcal{U} of \mathcal{V}_wD such that $\nu \in \mathcal{U}$ and for any $\nu' \in \mathcal{U}$, $\int_{y \in E} \int_{x \in D} \chi_U d\nu' d\xi > r$. To this end, we note that $g: E \rightarrow [0, 1]: y \mapsto \int_{x \in D} \chi_U(x, y) d\nu$ is Scott-continuous. Hence $[g > r] \cap \mathcal{P}E$ is an open subset of \mathcal{P}_wE that contains ξ . Applying the aforementioned result we find a simple valuation $\sum_{i=1}^n r_i \delta_{y_i} \in \mathcal{S}E$ such that $\sum_{i=1}^n r_i \delta_{y_i} \leq \xi$ and $\sum_{i=1}^n r_i \delta_{y_i} \in [g > r]$. This implies that

$$\int_{y \in E} \int_{x \in D} \chi_U(x, y) d\nu d \sum_{i=1}^n r_i \delta_{y_i} > r.$$

By applying Equation 3, this in turn implies that

$$\sum_{i=1}^n \int_{x \in D} r_i \chi_U(x, y_i) d\nu > r.$$

Obviously, we could find $s_i \geq 0, i = 1, \dots, n$ such that $\int_{x \in D} r_i \chi_U(x, y_i) d\nu > s_i$ and $\sum_{i=1}^n s_i > r$. Now we let

$$\mathcal{U} = \bigcap_{i=1}^n [r_i \chi_U(x, y_i) > s_i].$$

By Remark 14 the set \mathcal{U} is open in $\mathcal{V}_w D$ and obviously $\nu \in \mathcal{U}$. Moreover, for any $\nu' \in \mathcal{U}$, we have

$$\int_{y \in E} \int_{x \in D} \chi_U(x, y) d\nu' d\xi \geq \int_{y \in E} \int_{x \in D} \chi_U(x, y) d\nu' d \sum_{i=1}^n r_i \delta_{y_i} = \sum_{i=1}^n \int_{x \in D} r_i \chi_U(x, y_i) d\nu' \geq \sum_{i=1}^n s_i > r.$$

Hence G is continuous indeed.

The functions F and G are also linear from Proposition 6, Item 2. Hence both F and G are continuous linear map from $\mathcal{V}_w D$ to $[0, \infty]$, we now apply a varied version of the Schröder-Simpson Theorem, which can be found in [39, Corollary 2.5], to see that F and G are uniquely determined by their actions on Dirac measures $\delta_a, a \in D$. However, we note that $F(\delta_a) = \int_{y \in E} \chi_U(a, y) d\xi = G(\delta_a)$, again by Equation 3. Hence $F = G$, and we finish the proof by letting ξ range in $\mathcal{P}_w E$. \square

As far as we know, this is the most general Fubini theorem on dcpo's yet available. The proof relies on the Schröder-Simpson Theorem [39]. Hence by combining Remark 19, Proposition 21 and Theorem 22 we obtain our next theorem.

Theorem 23. *For any \mathbf{K} -category \mathbf{K} with $\mathbf{K} \subseteq \mathbf{D}$, $\mathcal{V}_{\mathbf{K}, \leq}$ is a commutative monad on \mathbf{DCPO} .*

Proof. We only need to prove that the strength of $\mathcal{V}_{\mathbf{K}, \leq}$ exists, and is of the same form as $\tau^{\mathcal{V}}$, the strength of \mathcal{V} , and then conclude with Theorem 22.

We know that for each \mathbf{K} -category $\mathbf{K} \subseteq \mathbf{D}$, $\mathcal{V}_{\mathbf{K}, \leq}$ is a monad on \mathbf{DCPO} . Hence, for any dcpo's D and E , and any Scott-continuous map $f: D \rightarrow \mathcal{V}_{\mathbf{K}, \leq} E$, the function

$$f^\dagger: \mathcal{V}_{\mathbf{K}, \leq} D \rightarrow \mathcal{V}_{\mathbf{K}, \leq} E: \nu \mapsto \lambda U \in \sigma E. \int_{x \in D} f(x)(U) d\nu$$

is a well-defined Scott-continuous map.

Now we apply this fact to the map $g: E \rightarrow \mathcal{V}_{\mathbf{K}, \leq}(D \times E): y \mapsto \delta_{(a, y)}$, where a is any fixed element in D . The map g is obviously Scott-continuous. Hence for any $\nu \in \mathcal{V}_{\mathbf{K}, \leq} E$,

$$g^\dagger(\nu) = \lambda U \in \sigma(D \times E). \int_{y \in E} \delta_{(a, y)}(U) d\nu = \lambda U \in \sigma(D \times E). \int_{y \in E} \chi_U(a, y) d\nu$$

is in $\mathcal{V}_{\mathbf{K}, \leq}(D \times E)$. This implies the map

$$\tau_{D, E}: D \times \mathcal{V}_{\mathbf{K}, \leq} E \rightarrow \mathcal{V}_{\mathbf{K}, \leq}(D \times E): (a, \nu) \mapsto \lambda U \in \sigma(D \times E). \int_{y \in E} \chi_U(a, y) d\nu$$

is well-defined, and it is obviously Scott-continuous. Note that apart from the domain and codomain, the map $\tau_{D,E}$ is same to the strength $\tau_{D,E}^{\mathcal{V}}$ of \mathcal{V} at (D, E) . Then the same arguments as in Jones' thesis would show that $\tau_{D,E}$ is the strength of $\mathcal{V}_{\mathbf{K}, \leq}$ at (D, E) . Hence $\mathcal{V}_{\mathbf{K}, \leq}$ is a strong monad. \square

As promised, we conclude this subsection with a third commutative monad \mathcal{W} on **DCPO** by describing a \mathbf{K} -category lying between **SOB** and **D**, the category **WF** consisting of well-filtered spaces and continuous maps. A T_0 space X is *well-filtered* if, given any filtered family $\{K_a\}_{a \in A}$ of compact saturated subsets of X with $\bigcap_{a \in A} K_a \subseteq U$, with U open, there is some $a \in A$ with $K_a \subseteq U$. A proof that **WF** is a \mathbf{K} -category between **SOB** and **D** can be found in [40]. Hence $\mathcal{W} \stackrel{\text{def}}{=} \mathcal{V}_{\mathbf{WF}, \leq}$ is a commutative monad on **DCPO** and $SD \subseteq MD \subseteq WD \subseteq PD \subseteq \mathcal{V}D$ for every dcpo D . Moreover, we can show that $MD \subsetneq PD$ when D is taken to be Johnstone's non-sober dcpo [41], but we leave further separation results for future work.

A natural question is which monad to use for semantic models. This paper shows \mathcal{M} suffices for *discrete* probabilistic choice and it is the smallest and simplest commutative submonad of \mathcal{V} over dcpo's. For *continuous* probabilistic choice, we do not yet know if \mathcal{M} is large enough, so \mathcal{W} or \mathcal{P} may be needed. We plan to address this in future work.

Remark 24. All subsequent results we present in this paper hold for the three monads \mathcal{M}, \mathcal{W} and \mathcal{P} . To avoid cumbersome repetition, we explicitly state them for \mathcal{M} .

3.5. Continuous Kegelspitzen and \mathcal{M} -algebras

Kegelspitzen [16] are dcpo's that enjoy a convex structure. In this section, we show every *continuous* Kegelspitze K has a *linear barycenter map* $\beta: MK \rightarrow K$ making (K, β) an \mathcal{M} -algebra and conversely, every \mathcal{M} -algebra (K, β) on **DCPO** admits a Kegelspitze structure on K making $\beta: MK \rightarrow K$ a linear map. We begin with the notion of a barycentric algebra.

Definition 25. A *barycentric algebra* is a set A equipped with a binary operation $a +_r b$ for every real number $r \in [0, 1]$ such that for all $a, b, c \in A$ and $r, p \in [0, 1]$, the following equations hold:

$$\begin{aligned} a +_1 b &= a; & a +_r b &= b +_{1-r} a; & a +_r a &= a; \\ (a +_p b) +_r c &= a +_{pr} (b +_{\frac{r-pr}{1-pr}} c) & & \text{provided } r, p < 1. \end{aligned}$$

Definition 26. A *pointed barycentric algebra* is a barycentric algebra A with a distinguished element \perp . For $a \in A$ and $r \in [0, 1]$, we define $r \cdot a \stackrel{\text{def}}{=} a +_r \perp$. A map $f: A \rightarrow B$ between pointed barycentric algebras is called *linear* if $f(\perp_A) = \perp_B$ and $f(a +_r b) = f(a) +_r f(b)$ for all $a, b \in A, r \in [0, 1]$.

Definition 27. A *Kegelspitze* is a pointed barycentric algebra K equipped with a directed-complete partial order such that, for every r in the unit interval, the functions determined by convex combination $(a, b) \mapsto a +_r b: K \times K \rightarrow K$ and

scalar multiplication $(r, a) \mapsto r \cdot a: [0, 1] \times K \rightarrow K$ are Scott-continuous in both arguments. A *continuous Kegelspitze* is a Kegelspitze that is a domain in the equipped order.

Remark 28. In a Kegelspitze K , the map $(r, a) \mapsto r \cdot a = a +_r \perp$ is Scott-continuous, hence monotone, in the r -component, which implies $\perp = \perp +_1 a = a +_0 \perp = 0 \cdot a \leq 1 \cdot a = a$ for each $a \in K$, i.e., \perp is the least element of K .

Example 29. For each dcpo D , $\mathcal{M}D$ is a Kegelspitze: for $\nu_1, \nu_2 \in \mathcal{M}D$ and $r \in [0, 1]$, $\nu_1 +_r \nu_2$ is defined as $r\nu_1 + (1-r)\nu_2$. Lemma 8 implies this is well-defined.³ The constantly zero valuation $\mathbf{0}_D$ is the distinguished element. Verifying that $\mathcal{M}D$ is a Kegelspitze is then straightforward.

As a consequence, for each Scott-continuous map $f: D \rightarrow E$, the map $\mathcal{M}(f): \mathcal{M}D \rightarrow \mathcal{M}E :: \nu \mapsto \lambda U \in \sigma E. \nu(f^{-1}(U))$ is obviously linear.

Definition 30. In each pointed barycentric algebra K , for $a_i \in K, r_i \in [0, 1], i = 1, \dots, n$ with $\sum_{i=1}^n r_i \leq 1$, we define the convex sum inductively

$$\sum_{i=1}^n r_i a_i \stackrel{\text{def}}{=} \begin{cases} a_1 & , \text{ if } r_1 = 1, \\ a_1 +_{r_1} (\sum_{i=2}^n \frac{r_i}{1-r_1} a_i) & , \text{ if } r_1 < 1. \end{cases}$$

This is invariant under index-permutation: for π a permutation of $\{1, \dots, n\}$, $\sum_{i=1}^n r_i a_i = \sum_{i=1}^n r_{\pi(i)} a_{\pi(i)}$ [13, Lemma 5.6]. If K is a Kegelspitze, then the expression $\sum_{i=1}^n r_i a_i$ is Scott-continuous in each r_i and a_i . A countable convex sum may also be defined: given $a_i \in K$ and $r_i \in [0, 1]$, for $i \in I$, with $\sum_{i \in I} r_i \leq 1$, let $\sum_{i \in I} r_i a_i \stackrel{\text{def}}{=} \sup\{\sum_{j \in J} r_j a_j \mid J \subseteq I \text{ and } J \text{ is finite}\}$.

Lemma 31. A function $f: K_1 \rightarrow K_2$ between pointed barycentric algebras K_1 and K_2 is linear if and only if $f(\sum_{i=1}^n r_i a_i) = \sum_{i=1}^n r_i f(a_i)$ for $a_i \in K_1, i = 1, \dots, n$ and $\sum_{i=1}^n r_i \leq 1$.

Definition 32. Let K be a Kegelspitze and $s = \sum_{i=1}^n r_i \delta_{x_i}$ be a simple valuation on K . The *barycenter* of s is defined as $\beta_*(s) \stackrel{\text{def}}{=} \sum_{i=1}^n r_i x_i$.

As a straightforward consequence of Jones' Splitting Lemma ([32, Proposition IV-9.18]), the map $\beta_*(s)$ is monotone from $\mathcal{S}K$ to K . If K is continuous, then $\mathcal{M}K = \mathcal{V}K$ and $\mathcal{S}K$ is a basis for $\mathcal{M}K$ (see Remark 12). We extend β_* to the *barycenter map*

$$\beta: \mathcal{M}K \rightarrow K \text{ by } \beta(\nu) \stackrel{\text{def}}{=} \sup\{\beta_*(s) \mid s \in \mathcal{S}K \text{ and } s \ll \nu\}.$$

Note that for each simple valuation $s = \sum_{i=1}^n r_i \delta_{x_i} \in \mathcal{S}K$, there exists a directed set A of $\mathcal{S}K$ with supremum s consisting of simple valuations way-below s . For example, one can choose $A = \{\sum_{i=1}^n \frac{mr_i}{m+1} \delta_{y_i} \mid m \in \mathbb{N} \text{ and } y_i \ll x_i\}$. By [32, Lemma IV-9.23.], the map β , as defined above, is a Scott-continuous map extending β_* , i.e., $\beta(\nu) = \beta_*(\nu)$ for $\nu \in \mathcal{S}K$. Moreover, β is a linear map since β_* is.

³Note that Lemma 8 is stated only for \mathcal{M} , but it also holds for \mathcal{W} and \mathcal{P} .

Proposition 33. *Each continuous Kegelspitze K admits a linear barycenter map $\beta: \mathcal{MK} \rightarrow K$ (as above) for which the pair (K, β) is an Eilenberg-Moore algebra of \mathcal{M} .*

Proof. Clearly, $\beta \circ \eta_K^{\mathcal{M}} = \text{id}_K$. To prove that $\beta \circ \mu_K^{\mathcal{M}} = \beta \circ \mathcal{M}(\beta)$, we only need to prove both sides are equal on simple valuations in $\mathcal{M}(\mathcal{MK})$, since $\mathcal{S}(\mathcal{MK})$ is dense in $\mathcal{M}(\mathcal{MK})$ in the d-topology, and both sides of the equation are d-continuous functions. However, when applied to the simple valuation $\sum_{i=1}^n r_i \delta_{\nu_i} \in \mathcal{S}(\mathcal{MK})$, both sides equal $\sum_{i=1}^n r_i \beta(\nu_i)$. This follows from direct computation by employing Remark 10 and linearity of β . \square

We next show that every Eilenberg-Moore algebra (K, β) of \mathcal{M} on **DCPO** admits a Kegelspitze structure on K making $\beta: \mathcal{MK} \rightarrow K$ a linear map.

Proposition 34. *Let (K, β) be an \mathcal{M} -algebra on **DCPO**. For $a, b \in K$ and $r \in [0, 1]$, define $a +_r b \stackrel{\text{def}}{=} \beta(\delta_a +_r \delta_b)$. Then with the operation $+_r$, K is a Kegelspitze and $\beta: \mathcal{MK} \rightarrow K$ is linear.*

Proof. We first prove that K is a pointed barycentric algebra. It is easy to see that $\beta(\mathbf{0}_K)$ is the least element in K , since for any $x \in K$, $\beta(\mathbf{0}_K) \leq \beta(\delta_x) = x$. It is also easy to see that $a +_1 b = a$, $a +_r b = b +_{1-r} a$ and $a +_r a = a$. We now proceed to prove that $(a +_p b) +_r c = a +_{pr} (b +_{\frac{r-pr}{1-pr}} c)$ for any $r, p < 1$ and $a, b, c \in K$. We reason as follows:

$$\begin{aligned}
& (a +_p b) +_r c && \\
& = \beta(\delta_{a+_p b} +_r \delta_c) && \text{(Definition of } +_r) \\
& = \beta(\delta_{\beta(\delta_a +_p \delta_b)} +_r \delta_{\beta(\delta_c)}) && \text{(Definition of } +_p \text{ and } \beta(\delta_c) = c) \\
& = \beta(\mathcal{M}(\beta)(\delta_{\delta_a +_p \delta_b} +_r \delta_{\delta_c})) && (\mathcal{M}(\beta) \text{ is linear and } \mathcal{M}(\beta)(\delta_\nu) = \delta_{\beta(\nu)}) \\
& = \beta(\mu_K^{\mathcal{M}}(\delta_{\delta_a +_p \delta_b} +_r \delta_{\delta_c})) && ((K, \beta) \text{ is an } \mathcal{M}\text{-algebra}) \\
& = \beta((\delta_a +_p \delta_b) +_r \delta_c) && (\mu_K^{\mathcal{M}} \text{ is the multiplication of } \mathcal{M} \text{ at } K) \\
& = \beta(\delta_a +_{pr} (\delta_b +_{\frac{r-pr}{1-pr}} \delta_c)) && (\mathcal{MK} \text{ is a Kegelspitze}) \\
& = \beta(\mu_K^{\mathcal{M}}(\delta_{\delta_a} +_{pr} \delta_{(\delta_b +_{\frac{r-pr}{1-pr}} \delta_c)})) && (\mu_K^{\mathcal{M}} \text{ is the multiplication of } \mathcal{M} \text{ at } K) \\
& = \beta(\mathcal{M}(\beta)(\delta_{\delta_a} +_{pr} \delta_{(\delta_b +_{\frac{r-pr}{1-pr}} \delta_c)})) && ((K, \beta) \text{ is an } \mathcal{M}\text{-algebra}) \\
& = \beta(\delta_{\beta(\delta_a)} +_{pr} \delta_{\beta(\delta_b +_{\frac{r-pr}{1-pr}} \delta_c)}) && (\mathcal{M}(\beta) \text{ is linear and } \mathcal{M}(\beta)(\delta_\nu) = \delta_{\beta(\nu)}) \\
& = \beta(\delta_a +_{pr} \delta_{(b +_{\frac{r-pr}{1-pr}} c)}) && \text{(Definition of } b +_{\frac{r-pr}{1-pr}} c \text{ and } \beta(\delta_a) = a) \\
& = a +_{pr} (b +_{\frac{r-pr}{1-pr}} c). && \text{(Definition of } +_{pr})
\end{aligned}$$

The map $(a, b) \mapsto a +_r b = \beta(\delta_a +_r \delta_b): K \times K \rightarrow K$ is Scott-continuous since β and δ are Scott-continuous and \mathcal{MK} is a Kegelspitze. The map $(r, a) \mapsto ra = a +_r \beta(\mathbf{0}_K) = \beta(\delta_a +_r \delta_{\beta(\mathbf{0}_K)}): [0, 1] \times K \rightarrow K$ is Scott-continuous in a for exactly the same reasons; to see that it also is Scott-continuous in r , we only

need to show that $r \mapsto \delta_a +_r \delta_{\beta(\mathbf{0}_K)}: [0, 1] \rightarrow \mathcal{MK}$ is Scott-continuous for any fixed $a \in K$. This is true if $\beta(\mathbf{0}_K) \leq a$. However, we already see that $\beta(\mathbf{0}_K)$ is the least element in K . Hence we have proved that K is a Kegelspitze. The map β is clearly linear. \square

Proposition 35. *Let (K_1, β_1) and (K_2, β_2) be \mathcal{M} -algebras on \mathbf{DOM} . A Scott-continuous function $f: K_1 \rightarrow K_2$ is an algebra morphism from (K_1, β_1) to (K_2, β_2) if and only if f is linear with respect to the Kegelspitze structure on K_1 and K_2 introduced by β_1 and β_2 , respectively, as in Proposition 34.*

Proof. The “if” direction. Assume that $f: K_1 \rightarrow K_2$ is linear. We need to prove that $f \circ \beta_1 = \beta_2 \circ \mathcal{M}(f)$. Since both sides are Scott-continuous hence d-continuous and K_2 is Hausdorff in the d-topology (if K_2 has more than one element). We only need to prove they are equal on simple valuations on K_1 . Towards this end, we pick $\sum_{i=1}^n r_i \delta_{x_i} \in \mathcal{MK}_1$, and see that

$$\begin{aligned}
f(\beta_1(\sum_{i=1}^n r_i \delta_{x_i})) &= f(\sum_{i=1}^n r_i x_i) && (\beta_1 \text{ is linear and } \beta_1(\delta_{x_i}) = x_i) \\
&= \sum_{i=1}^n r_i f(x_i) && (f \text{ is linear}) \\
&= \beta_2(\sum_{i=1}^n r_i \delta_{f(x_i)}) && (\beta_2 \text{ is linear and } \beta_2(\delta_{f(x_i)}) = f(x_i)) \\
&= \beta_2(\mathcal{M}(f)(\sum_{i=1}^n r_i \delta_{x_i})) && (\mathcal{M}(f) \text{ is linear and } \mathcal{M}(f)(\delta_{x_i}) = \delta_{f(x_i)}).
\end{aligned}$$

The “only if” direction. Assume that $f: K_1 \rightarrow K_2$ is an algebra morphism from (K_1, β_1) to (K_2, β_2) . Then we know that $f \circ \beta_1 = \beta_2 \circ \mathcal{M}(f)$. We prove that f is linear. First, for $a, b \in K_1$ and $r \in [0, 1]$, we have

$$\begin{aligned}
f(a +_r b) &= f(\beta_1(\delta_a +_r \delta_b)) && (\text{definition of } a +_r b) \\
&= \beta_2(\mathcal{M}(f)(\delta_a +_r \delta_b)) && (f \text{ is an algebra morphism}) \\
&= \beta_2(\delta_{f(a)} +_r \delta_{f(b)}) && (\mathcal{M}(f) \text{ is linear and } \mathcal{M}(f)(\delta_x) = \delta_{f(x)}) \\
&= f(a) +_r f(b). && (\text{definition of } f(a) +_r f(b))
\end{aligned}$$

Second, to prove that f maps $\beta(\mathbf{0}_{K_1})$ to $\beta_2(\mathbf{0}_{K_2})$, we see that $f(\beta_1(\mathbf{0}_{K_1})) = \beta_2(\mathcal{M}(f)(\mathbf{0}_{K_1})) = \beta_2(\mathbf{0}_{K_2})$ because $\mathcal{M}(f)$ is linear. \square

Theorem 36. *The Eilenberg-Moore category $\mathbf{DOM}^{\mathcal{M}}$ of \mathcal{M} over \mathbf{DOM} is isomorphic to the category of continuous Kegelspitzen and Scott-continuous linear maps.*

Proof. Combine Propositions 33, 34 and 35. \square

Remark 37. Theorem 36 characterises $\mathbf{DOM}^{\mathcal{M}}$, which equals $\mathbf{DOM}^{\mathcal{V}_{\mathbf{K}, \leq}}$ for any \mathbf{K} -category \mathbf{K} with $\mathbf{K} \subseteq \mathbf{D}$ since $\mathcal{V} = \mathcal{M}$ on domains (see Remark 12 and

Proposition 21). This corrects an error in [13], where it is proved that *continuous abstract probabilistic domains* and linear maps form a full subcategory of $\mathbf{DOM}^{\mathcal{V}}$. But the claim in [13] is that all objects in $\mathbf{DOM}^{\mathcal{V}}$ are abstract probabilistic domains. A separating example is the extended non-negative reals $[0, \infty]$, which is a continuous Kegelspitze but not an abstract probabilistic domain.

4. Categorical Model

In this section we describe the categorical properties of the Kleisli adjunction of our monad \mathcal{M} (which also has the structure of a Freyd category [42]). Everything we say in this section is also true for our other two monads as well.

We write $\mathbf{DCPO}_{\mathcal{M}}$ for the Kleisli category of our monad $\mathcal{M} : \mathbf{DCPO} \rightarrow \mathbf{DCPO}$. In order to distinguish between the categorical primitives of \mathbf{DCPO} and $\mathbf{DCPO}_{\mathcal{M}}$, we indicate with $f : A \rightarrow B$ the morphisms of $\mathbf{DCPO}_{\mathcal{M}}$ and we write $f \circ g \stackrel{\text{def}}{=} \mu \circ \mathcal{M}(f) \circ g$ for the Kleisli composition of morphisms in $\mathbf{DCPO}_{\mathcal{M}}$. We write $\mathbf{id}_A : A \rightarrow A$ with $\mathbf{id}_A = \eta_A : A \rightarrow \mathcal{M}A$ for the identity morphisms in $\mathbf{DCPO}_{\mathcal{M}}$. The monad \mathcal{M} induces an adjunction $\mathcal{J} \dashv \mathcal{U} : \mathbf{DCPO}_{\mathcal{M}} \rightarrow \mathbf{DCPO}$, where:

$$\mathcal{J}A \stackrel{\text{def}}{=} A, \quad \mathcal{J}f \stackrel{\text{def}}{=} \eta \circ f, \quad \mathcal{U}A \stackrel{\text{def}}{=} \mathcal{M}A, \quad \mathcal{U}f \stackrel{\text{def}}{=} \mu \circ \mathcal{M}f.$$

4.1. Coproducts

The category $\mathbf{DCPO}_{\mathcal{M}}$ inherits (small) coproducts from \mathbf{DCPO} in the standard way [43, pp. 264] and we write $A_1 \dot{+} A_2 \stackrel{\text{def}}{=} A_1 + A_2$ for the induced (binary) coproduct. The induced coprojections are given by $\mathcal{J}(in_1) : A_1 \rightarrow A_1 \dot{+} A_2$ and $\mathcal{J}(in_2) : A_2 \rightarrow A_1 \dot{+} A_2$. Then for $f : A \rightarrow C$ and $g : B \rightarrow D$, $f \dot{+} g = [\mathcal{M}(in_C) \circ f, \mathcal{M}(in_D) \circ g]$.

4.2. Symmetric monoidal structure

Because our monad \mathcal{M} is *commutative*, it induces a symmetric monoidal structure on $\mathbf{DCPO}_{\mathcal{M}}$ in a canonical way [44, pp. 462]. The induced tensor product is $A \dot{\times} B \stackrel{\text{def}}{=} A \times B$ and the Kleisli projections are $\mathcal{J}(\pi_A) : A \dot{\times} B \rightarrow A$ and $\mathcal{J}(\pi_B) : A \dot{\times} B \rightarrow B$ ⁴. For $f : A \rightarrow C$ and $g : B \rightarrow D$, their tensor product is given by $f \dot{\times} g = \lambda(a, b).f(a) \otimes g(b)$. Note that the last expression uses the double strength of \mathcal{M} , see Remark 11.

Standard categorical arguments now show that the Kleisli products distribute over the Kleisli coproducts. We write $d_{A,B,C} : A \dot{\times} (B \dot{+} C) \cong (A \dot{\times} B) \dot{+} (A \dot{\times} C)$ for this natural isomorphism.

⁴These projections do *not* satisfy the universal property of a product.

4.3. The left adjoint \mathcal{J}

The functor \mathcal{J} , whose action is the identity on objects, preserves the monoidal structure and the coproduct structure up to equality (and not merely up to isomorphism). That is, $\mathcal{J}(A \star B) = JA \star JB$ and $\mathcal{J}(f \star g) = Jf \star Jg$, where $\star \in \{\times, +\}$.

4.4. Kleisli Exponential

Our Kleisli adjunction also contains the structure of a *Kleisli-exponential* (which is also known as a \mathcal{M} -exponential). Following Moggi [18], we will use this to interpret higher-order function types. Next, we describe this structure in greater detail.

The functor $J(-) \times B : \mathbf{DCPO} \rightarrow \mathbf{DCPO}_{\mathcal{M}}$ has a right adjoint, which we write as $[B \rightarrow -] : \mathbf{DCPO}_{\mathcal{M}} \rightarrow \mathbf{DCPO}$, for each dcpo B . In particular $[B \rightarrow -] \stackrel{\text{def}}{=} [B \rightarrow \mathcal{U}(-)]$, which means that, on objects, $[B \rightarrow C] = [B \rightarrow \mathcal{M}C]$. This data provides us with a family of Scott-continuous bijections

$$\lambda : \mathbf{DCPO}_{\mathcal{M}}(JA \times B, C) \cong \mathbf{DCPO}(A, [B \rightarrow C]) \quad (5)$$

natural in A and C , called *currying*. We also denote with $\epsilon : \mathcal{J}[B \rightarrow -] \times B \Rightarrow \text{Id}$, the counit of the adjunctions (5), often called *evaluation*. Because this family of adjunctions is parameterised by objects B of $\mathbf{DCPO}_{\mathcal{M}}$, it follows using standard categorical results [45, §IV.7] that the assignment $[B \rightarrow -] : \mathbf{DCPO}_{\mathcal{M}} \rightarrow \mathbf{DCPO}$ may be extended uniquely to a bifunctor $[- \rightarrow -] : \mathbf{DCPO}_{\mathcal{M}}^{\text{op}} \times \mathbf{DCPO}_{\mathcal{M}} \rightarrow \mathbf{DCPO}$, such that the bijections λ in (5) are natural in all components⁵.

Remark 38. Some authors describe currying and evaluation for Kleisli exponentials without referring to the functor \mathcal{J} . This cannot lead to confusion on the object level, but to be fully precise, one has to specify that the naturality properties on the A -component hold only for *total* maps. We make this explicit by including \mathcal{J} in our presentation.

4.5. Enrichment Structure

The category $\mathbf{DCPO}_{\mathcal{M}}$ is enriched over $\mathbf{DCPO}_{\perp 1}$: for all dcpo's A, B and C , the Kleisli exponential $[A \rightarrow B] = [A \rightarrow \mathcal{M}B] = \mathbf{DCPO}_{\mathcal{M}}(A, B)$ is a pointed dcpo in the pointwise order, and the Kleisli composition

$$\circ : [A \rightarrow B] \times [B \rightarrow C] \rightarrow [A \rightarrow C] : (f, g) \mapsto g \circ f = g^{\dagger} \circ f$$

is obviously a strict Scott-continuous map. Moreover, the adjunction $\mathcal{J} \dashv \mathcal{U} : \mathbf{DCPO}_{\mathcal{M}} \rightarrow \mathbf{DCPO}$ is also \mathbf{DCPO} -enriched (see [46, Definition 6.7.1] for definition) and so are the bifunctors $(- \times -)$, $(- \dot{+} -)$ and $[- \rightarrow -]$.

We interpret probabilistic effects using the convex structure of our model which we now describe. For each dcpo B , $\mathcal{M}B$ is a Kegelspitze in the stochastic

⁵This extension is canonically given by $[f \rightarrow g] \stackrel{\text{def}}{=} \lambda(g \circ \epsilon_{C_1} \circ (\text{id} \times f))$.

order (Example 29) : for $r \in [0, 1]$ and $\nu_1, \nu_2 \in \mathcal{MB}$, $\nu_1 +_r \nu_2$ is defined as $r\nu_1 + (1-r)\nu_2$; the zero-valuation $\mathbf{0}_B$ is the distinguished element (which is also least). It follows that $[A \rightarrow B] = \mathbf{DCPO}_{\mathcal{M}}(A, B)$ is a Kegelspitze in the pointwise order: for $f, g \in [A \rightarrow B]$, $f +_r g$ is defined as $\lambda x. f(x) +_r g(x)$. Next, we note that this convex structure is preserved by Kleisli composition \circ , Kleisli coproduct $\dot{+}$ and Kleisli product $\dot{\times}$.

Proposition 39. *For $f, h: A \rightarrow \mathcal{MC}$, $g: B \rightarrow \mathcal{MD}$ and $r \in [0, 1]$, we have*

1. $(f +_r h) \dot{\times} g = f \dot{\times} g +_r h \dot{\times} g$;
2. $g \dot{\times} (f +_r h) = g \dot{\times} f +_r g \dot{\times} h$.

Proof. We only prove Item 1, the second item can be proved similarly.

For each $(a, b) \in A \times B$, we reason as follows:

$$\begin{aligned}
& ((f +_r h) \dot{\times} g)(a, b) \\
&= (f +_r h)(a) \otimes g(b) && \text{(Definition of } \dot{\times} \text{)} \\
&= (f(a) +_r h(a)) \otimes g(b) && \text{(Definition of } +_r \text{)} \\
&= \lambda U \in \sigma(C \dot{\times} D). \int_{y \in D} \int_{x \in C} \chi_U(x, y) d(f(a) +_r h(a)) dg(b) && \text{(Definition of } \otimes \text{)} \\
&= \lambda U. \int_{y \in D} \left(\int_{x \in C} \chi_U(x, y) df(a) +_r \right. \\
&\quad \left. \int_{x \in C} \chi_U(x, y) dh(a) \right) dg(b) && \text{(Proposition 6, Item 2)} \\
&= \lambda U. \int_{y \in D} \int_{x \in C} \chi_U(x, y) df(a) dg(b) +_r \\
&\quad \int_{y \in D} \int_{x \in C} \chi_U(x, y) dh(a) dg(b) && \text{(Proposition 6, Item 3)} \\
&= \lambda U. \int_{y \in D} \int_{x \in C} \chi_U(x, y) df(a) dg(b) +_r \\
&\quad \lambda U. \int_{y \in D} \int_{x \in C} \chi_U(x, y) dh(a) dg(b) && \text{(Definition of } +_r \text{ for valuations)} \\
&= (f \dot{\times} g)(a, b) +_r (h \dot{\times} g)(a, b) && \text{(Definition of } \dot{\times} \text{)} \\
&= (f \dot{\times} g +_r h \dot{\times} g)(a, b) && \text{(Definition of } +_r \text{ for functions).}
\end{aligned}$$

Hence the proof is completed. \square

Proposition 40. *For $f, h: A \rightarrow \mathcal{MC}$, $g: B \rightarrow \mathcal{MD}$ and $r \in [0, 1]$, we have*

1. $(f +_r h) \dot{+} g = (f \dot{+} g) +_r (h \dot{+} g)$;
2. $g \dot{+} (f +_r h) = (g \dot{+} f) +_r (g \dot{+} h)$.

Proof. Again, we only prove the first claim as the second one can be proved

using similar arguments. If $a \in A$, then:

$$\begin{aligned}
& ((f +_r h) \dot{+} g)(in_A(a)) \\
&= [\mathcal{M}(in_C) \circ (f +_r h), \mathcal{M}(i_D) \circ g](in_A(a)) && \text{(Definition of } \dot{+} \text{)} \\
&= \mathcal{M}(in_C)((f +_r h)(a)) && \text{(Obvious)} \\
&= \mathcal{M}(in_C)(f(a) +_r h(a)) && \text{(Definition of } f +_r h \text{)} \\
&= \lambda U. (f(a) +_r h(a))(i_C^{-1}(U)) && \text{(Definition of } \mathcal{M}(in_C) \text{)} \\
&= \lambda U. f(a)(i_C^{-1}(U)) +_r h(a)(i_C^{-1}(U)) && \text{(Definition of } f(a) +_r h(a) \text{)} \\
&= \lambda U. f(a)(i_C^{-1}(U)) +_r \lambda U. h(a)(i_C^{-1}(U)) && \text{(Definition of } +_r \text{ for valuations)} \\
&= \mathcal{M}(in_C)(f(a)) +_r \mathcal{M}(in_C)(h(a)), && \text{(Definition of } \mathcal{M}(in_C) \text{)} \\
&= (f \dot{+} g)(in_A(a)) +_r (h \dot{+} g)(in_A(a)) && \text{(Definition of } \dot{+} \text{)} \\
&= ((f \dot{+} g) +_r (h \dot{+} g))(in_A(a)) && \text{(Definition of } +_r \text{)}.
\end{aligned}$$

Moreover, it is easy to see that for $b \in B$, $((f +_r h) \dot{+} g)(in_B(b)) = \mathcal{M}(in_D)(g(b)) = \mathcal{M}(in_D)(g(b)) +_r \mathcal{M}(in_D)(g(b)) = ((f \dot{+} g) +_r (h \dot{+} g))(in_B(b))$, which completes the proof. \square

Proposition 41. For $f, h: A \rightarrow \mathcal{M}B, g, k: B \rightarrow \mathcal{M}C$ and $r \in [0, 1]$, we have

1. $g \circ (f +_r h) = g \circ f +_r g \circ h$;
2. $(g +_r k) \circ f = g \circ f +_r k \circ f$.

Proof. For the first claim, let $a \in A$. Then:

$$\begin{aligned}
& g \circ (f +_r h)(a) \\
&= (g^\ddagger \circ (f +_r h))(a) && \text{(Definition of } \circ \text{)} \\
&= g^\ddagger(f(a) +_r h(a)) && \text{(Definition of } f +_r h \text{)} \\
&= \lambda U. \int_{x \in B} g(x)(U) d(f(a) +_r h(a)) && \text{(Definition of } g^\ddagger \text{)} \\
&= \lambda U. \int_{x \in B} g(x)(U) df(a) +_r \lambda U. \int_{x \in B} g(x)(U) dh(a) && \text{(Proposition 6, Item 2)} \\
&= g^\ddagger(f(a)) +_r g^\ddagger(h(a)) && \text{(Definition of } g^\ddagger \text{)} \\
&= (g \circ f +_r g \circ h)(a).
\end{aligned}$$

For the second claim, let $a \in A$. Then:

$$\begin{aligned}
& ((g +_r k) \circ f)(a) \\
&= (g +_r k)^\ddagger(f(a)) && \text{(Definition of } \circ \text{)} \\
&= \lambda U. \int_{x \in B} (g +_r k)(x)(U) df(a) && \text{(Definition of } _^\ddagger \text{)} \\
&= \lambda U. \int_{x \in B} g(x)(U) df(a) +_r \lambda U. \int_{x \in B} k(x)(U) df(a) && \text{(Proposition 6, Item 3)} \\
&= g^\ddagger(f(a)) +_r k^\ddagger(f(a)) && \text{(Definition of } _^\ddagger \text{)} \\
&= (g \circ f +_r k \circ f)(a).
\end{aligned}$$

□

4.6. Important Subcategories

In order to describe our denotational semantics, we have to identify two important subcategories of $\mathbf{DCPO}_{\mathcal{M}}$.

Definition 42. The subcategory of *deterministic total maps*, denoted \mathbf{TD} , is the full-on-objects subcategory of $\mathbf{DCPO}_{\mathcal{M}}$ each of whose morphisms $f: X \rightarrow Y$ admits a factorisation $f = \mathcal{J}(f') = \left(X \xrightarrow{f'} Y \xrightarrow{\eta_Y} \mathcal{M}Y \right)$.

Therefore, by definition, each map $f: X \rightarrow Y$ in \mathbf{TD} satisfies $f(x) = \delta_y$ for some $y \in Y$. These maps are *deterministic* in the sense that they carry no interesting convex structure and they are *total* in the sense that they map all inputs $x \in X$ to non-zero valuations. The importance of this subcategory is that all values of our language admit an interpretation within \mathbf{TD} . Moreover, the categorical structure of \mathbf{TD} is very easy to describe, as our next proposition shows.

Proposition 43. *The functor $\mathcal{J}: \mathbf{DCPO} \rightarrow \mathbf{DCPO}_{\mathcal{M}}$ corestricts to a \mathbf{DCPO} -enriched isomorphism $\mathbf{DCPO} \cong \mathbf{TD}$.*

In our model, the canonical copy map at an object A is given by the map $\mathcal{J}(\text{id}_A, \text{id}_A): A \rightarrow A \times A$ and the canonical discarding map at A is the map $\mathcal{J}(1_A): A \rightarrow 1$, where $1_A: A \rightarrow 1$ is the terminal map of \mathbf{DCPO} . Because maps in \mathbf{TD} are in the image of \mathcal{J} , it follows that they are compatible with the copy and discard maps and thus also with weakening and contraction [47, 48].

The next subcategory we introduce is important, because we will use it for the interpretation of open types. It has sufficient structure to solve recursive domain equations.

Definition 44. The subcategory of *deterministic partial maps*, denoted \mathbf{PD} , is the full-on-objects subcategory of $\mathbf{DCPO}_{\mathcal{M}}$ each of whose morphisms $f: X \rightarrow Y$ admits a factorisation $f = \left(X \xrightarrow{f'} Y_{\perp} \xrightarrow{\phi_Y} \mathcal{M}Y \right)$, where Y_{\perp} is the dcpo obtained from Y by freely adding a least element \perp , and ϕ_Y is the map:

$$\phi_Y: Y_{\perp} \rightarrow \mathcal{M}Y :: y \mapsto \begin{cases} \mathbf{0}_Y & , \text{ if } y = \perp \\ \delta_y & , \text{ if } y \neq \perp . \end{cases}$$

These maps are *partial* because some inputs are mapped to $\mathbf{0}$, but also deterministic, because the convex structure is trivial in both cases. Viewing these maps as partial and deterministic is further justified by the fact that the category \mathbf{PD} is isomorphic to the Kleisli category of the lift monad on \mathbf{DCPO} , as we now show.

Recall that we use $(\mathcal{M}, \eta, \mu, \tau)$ to indicate our commutative monad. We write $(\mathcal{L}, \eta^{\mathcal{L}}, \mu^{\mathcal{L}}, \tau^{\mathcal{L}})$ to indicate the lift monad on \mathbf{DCPO} , which is also commutative. The lift monad $\mathcal{L}: \mathbf{DCPO} \rightarrow \mathbf{DCPO}$ freely adds a new least element, often

denoted \perp , to a dcpo X . The resulting dcpo is $\mathcal{L}X \stackrel{\text{def}}{=} X_\perp$. The monad structure of \mathcal{L} is defined by the following assignments:

$$\eta_X^\mathcal{L} : X \rightarrow X_\perp :: x \mapsto x$$

$$\mu_X^\mathcal{L} : (X_{\perp_1})_{\perp_2} \rightarrow X_\perp :: x \mapsto \begin{cases} \perp & , \text{ if } x = \perp_1 \\ \perp & , \text{ if } x = \perp_2 \\ x & , \text{ if } \perp_1 \neq x \neq \perp_2 \end{cases}$$

$$\tau_{XY}^\mathcal{L} : X_\perp \times Y \rightarrow (X \times Y)_\perp :: (x, y) \mapsto \begin{cases} \perp & , \text{ if } x = \perp \\ (x, y) & , \text{ if } x \neq \perp \end{cases}$$

We write $\mathbf{DCPO}_\mathcal{L}$ for the Kleisli category of \mathcal{L} and we write its morphisms as $f : X \rightarrow Y$, which is by definition a morphism $f : X \rightarrow Y_\perp$ in \mathbf{DCPO} . We write $X \otimes Y$ and $X \oplus Y$ for the symmetric monoidal product and coproduct, respectively, which are (canonically) induced by the commutative monad \mathcal{L} .

Proposition 45. *The assignment $\phi : \mathcal{L} \Rightarrow \mathcal{M}$ from Definition 44 is a strong map of monads (see [43, Definition 5.2.9] for more details).*

Proof. To see that ϕ is a natural transformation, we need to show, for any Scott-continuous map $f : X \rightarrow Y$, $\phi_Y \circ \mathcal{L}f = \mathcal{M}f \circ \phi_X : X_\perp \rightarrow \mathcal{M}Y$. However, it is easy to see that both sides send \perp to $\mathbf{0}_Y$ and x that is not \perp to $\delta_{f(x)}$.

Now, we first verify that ϕ is a map of monads. That is, for each dcpo X , we need to prove that $\phi_X \circ \eta_X^\mathcal{L} = \eta_X$ and $\phi_X \circ \mu_X^\mathcal{L} = \mu_X \circ \mathcal{M}(\phi_X) \circ \phi_{X_\perp} : (X_{\perp_1})_{\perp_2} \rightarrow \mathcal{M}(X)$. The first equation is trivial, hence we proceed to prove the second. For this, we see

$$\phi_X \circ \mu_X^\mathcal{L}(x) = \begin{cases} \phi_X(\perp) = \mathbf{0}_X & , \text{ if } x = \perp_1 \text{ or } x = \perp_2 \\ \phi_X(x) = \delta_x & , \text{ if } \perp_1 \neq x \neq \perp_2 \end{cases}$$

and

$$\mu_X \circ \mathcal{M}(\phi_X) \circ \phi_{X_\perp}(x) = \begin{cases} \mu_X \circ \mathcal{M}(\phi_X)(\mathbf{0}_{X_\perp}) = \mu_X(\mathbf{0}_{\mathcal{M}X}) = \mathbf{0}_X & , \text{ if } x = \perp_2 \\ \mu_X \circ \mathcal{M}(\phi_X)(\delta_\perp) = \mu_X(\delta_{\phi_X(\perp)}) = \mu_X(\delta_{\mathbf{0}_X}) = \mathbf{0}_X & , \text{ if } x = \perp_1 \\ \mu_X \circ \mathcal{M}(\phi_X)(\delta_x) = \mu_X(\delta_{\phi_X(x)}) = \mu_X(\delta_{\delta_x}) = \delta_x & , \text{ if } \perp_1 \neq x \neq \perp_2 \end{cases}.$$

Hence $\phi : \mathcal{L} \Rightarrow \mathcal{M}$ is a map of monads. To prove that ϕ is a strong map of monads, we need to show that for any dcpos X and Y ,

$$\tau_{XY} \circ (\phi_X \times \text{id}_Y) = \phi_{XY} \circ \tau_{XY}^\mathcal{L} : X_\perp \times Y \rightarrow \mathcal{M}(X \times Y).$$

The strength τ of \mathcal{M} at (X, Y) is defined as follows:

$$\tau_{XY} : \mathcal{M}X \times Y \rightarrow \mathcal{M}(X \times Y) : (\nu, y) \mapsto \lambda U. \int_{x \in X} \chi_U(x, y) d\nu,$$

where χ_U is the characteristic function of $U \in \sigma(X \times Y)$, i.e., $\chi_U(x, y) = 1$ if $(x, y) \in U$ and $\chi_U(x, y) = 0$, otherwise. Now we check:

$$\tau_{XY} \circ (\phi_X \times \text{id}_Y)(x, y) = \begin{cases} \tau_{XY}(\mathbf{0}_X, y) = \lambda U. \int_{x \in X} \chi_U(x, y) d\mathbf{0}_X = \lambda U. 0 = \mathbf{0}_{X \times Y} & , \text{ if } x = \perp \\ \tau_{XY}(\delta_x, y) = \lambda U. \int_{x \in X} \chi_U(x, y) d\delta_x = \lambda U. \chi_U(x, y) = \delta_{(x, y)} & , \text{ if } x \neq \perp \end{cases}$$

and

$$\phi_{XY} \circ \tau_{XY}^{\mathcal{L}}(x, y) = \begin{cases} \phi_{XY}(\perp) = \mathbf{0}_{X \times Y} & , \text{ if } x = \perp \\ \phi_{XY}((x, y)) = \delta_{(x, y)} & , \text{ if } x \neq \perp \end{cases}$$

which concludes the proof. \square

Recall that any map of monads induces a functor between the corresponding Kleisli categories of the two monads (see [43, Exercise 5.2.1]). This allows us to show the next corollary.

Corollary 46. *The functor $\mathcal{F} : \mathbf{DCPO}_{\mathcal{L}} \rightarrow \mathbf{DCPO}_{\mathcal{M}}$, induced by $\phi : \mathcal{L} \Rightarrow \mathcal{M}$, and defined by:*

$$\begin{aligned} \mathcal{F}X &\stackrel{\text{def}}{=} X \\ \mathcal{F}(f : X \rightarrow Y) &\stackrel{\text{def}}{=} \phi_Y \circ f \end{aligned}$$

strictly preserves the monoidal and coproduct structures in the sense that the following equalities:

$$\begin{aligned} \mathcal{F}(X \otimes Y) &= \mathcal{F}X \dot{\times} \mathcal{F}Y & \mathcal{F}(X \oplus Y) &= \mathcal{F}X \dot{+} \mathcal{F}Y \\ \mathcal{F}(f \otimes g) &= \mathcal{F}f \dot{\times} \mathcal{F}g & \mathcal{F}(f \oplus g) &= \mathcal{F}f \dot{+} \mathcal{F}g \end{aligned}$$

hold.

Proof. This follows by canonical categorical arguments and is just a straightforward verification. \square

We may now describe the isomorphism between \mathbf{PD} and $\mathbf{DCPO}_{\mathcal{L}}$.

Proposition 47. *There exists a $\mathbf{DCPO}_{\perp!}$ -enriched isomorphism of categories $\mathbf{DCPO}_{\mathcal{L}} \cong \mathbf{PD}$.*

Proof. Consider the functor $\mathcal{F} : \mathbf{DCPO}_{\mathcal{L}} \rightarrow \mathbf{DCPO}_{\mathcal{M}}$ from Corollary 46. Each ϕ_Y is injective, so the corestriction of \mathcal{F} to \mathbf{PD} is the required isomorphism. \square

4.7. Solving Recursive Domain Equations

In order to interpret recursive types, we solve the required recursive domain equations by constructing *parameterised initial algebras* [27, 28] within (the subcategory of embeddings of) \mathbf{PD} using the limit-colimit coincidence theorem [49].

Definition 48 (see [27, §6.1]). Given a category \mathbf{C} and a functor $\mathcal{T} : \mathbf{C}^{n+1} \rightarrow \mathbf{C}$, a *parameterised initial algebra* for \mathcal{T} is a pair $(\mathcal{T}^{\sharp}, \iota^{\mathcal{T}})$, such that:

- $\mathcal{T}^\sharp: \mathbf{C}^n \rightarrow \mathbf{C}$ is a functor;
- $\iota^\mathcal{T}: \mathcal{T} \circ \langle \text{Id}, \mathcal{T}^\sharp \rangle \Rightarrow \mathcal{T}^\sharp: \mathbf{C}^n \rightarrow \mathbf{C}$ is a natural transformation;
- For every $\vec{C} \in \text{Ob}(\mathbf{C}^n)$, the pair $(\mathcal{T}^\sharp \vec{C}, \iota_{\vec{C}}^\mathcal{T})$ is an initial $\mathcal{T}(\vec{C}, -)$ -algebra.

In the special case when $n = 1$, we recover the usual notion of initial algebra. We consider *parameterised* initial algebras because we need to interpret mutual type recursion. Similarly, one can also define the dual notion of *parameterised final coalgebra*.

Proposition 49 (see [50, §4.3]). *Let \mathbf{C} be a category with an initial object and all ω -colimits and let $\mathcal{T}: \mathbf{C}^{n+1} \rightarrow \mathbf{C}$ be an ω -cocontinuous functor. Then \mathcal{T} has a parameterised initial algebra $(\mathcal{T}^\sharp, \iota^\mathcal{T})$ and the functor $\mathcal{T}^\sharp: \mathbf{C}^n \rightarrow \mathbf{C}$ is also ω -cocontinuous.*

The next proposition shows that the subcategory \mathbf{PD} has sufficient structure to solve recursive domain equations.

Proposition 50. *The subcategory \mathbf{PD} is (parameterised) \mathbf{DCPO} -algebraically compact. More specifically, every \mathbf{DCPO} -enriched functor $\mathcal{T}: \mathbf{PD}^{n+1} \rightarrow \mathbf{PD}$ has a parameterised compact algebra, i.e., a parameterised initial algebra whose inverse is a parameterised final coalgebra for \mathcal{T} .*

Proof. By Proposition 47, we have $\mathbf{PD} \cong \mathbf{DCPO}_{\mathcal{L}} \cong \mathbf{DCPO}_{\perp!}$ and the latter two categories are well-known to be \mathbf{DCPO} -algebraically compact (which may be easily established using [27, Corollary 7.2.4]). \square

Therefore, every \mathbf{DCPO} -enriched *covariant* functor on $\mathbf{DCPO}_{\mathcal{M}}$ which restricts to \mathbf{PD} can be equipped with a parameterised compact algebra. In order to solve equations involving *mixed-variance* functors (induced by function types), we use the limit-colimit coincidence theorem [49]. In particular, an important observation made by Smyth and Plotkin in [49] allows us to interpret all type expressions (including function spaces) as covariant functors on *subcategories of embeddings*. These ideas are developed in detail in [51, 50] and here we also follow this approach.

Definition 51. Given a \mathbf{DCPO} -enriched category \mathbf{C} , an *embedding* of \mathbf{C} is a morphism $e: X \rightarrow Y$, such that there exists (a necessarily unique) morphism $e^p: Y \rightarrow X$, called a *projection*, with the properties: $e^p \circ e = \text{id}_X$ and $e \circ e^p \leq \text{id}_Y$. We denote with \mathbf{C}_e the full-on-objects subcategory of \mathbf{C} whose morphisms are the embeddings of \mathbf{C} .

Next, let us recall an important result from [49].

Proposition 52. *Let \mathbf{A} , \mathbf{B} and \mathbf{C} be \mathbf{DCPO} -enriched categories. Assume further that \mathbf{A} and \mathbf{B} have all ω -colimits (or all ω^{op} -limits). If $\mathcal{T}: \mathbf{A}^{\text{op}} \times \mathbf{B} \rightarrow \mathbf{C}$*

is a **DCPO**-enriched functor, then the assignment

$$\begin{aligned}\mathcal{T}^E &: \mathbf{A}_e \times \mathbf{B}_e \rightarrow \mathbf{C}_e \\ \mathcal{T}^E(A, B) &\stackrel{\text{def}}{=} \mathcal{T}(A, B) \\ \mathcal{T}^E(e_1, e_2) &\stackrel{\text{def}}{=} \mathcal{T}(e_1^p, e_2)\end{aligned}$$

defines a covariant ω -cocontinuous functor.

Proof. This follows by combining several results from [49], namely Theorem 2, the corollary after it and Theorem 3. \square

Therefore, by trivialising the category \mathbf{A} , we may obtain results for purely covariant functors. When neither category is trivialised, this allows us to interpret mixed-variance functors (such as function space) as covariant functors on subcategories of embeddings.

Proposition 53. *The category \mathbf{PD}_e has an initial object and all ω -colimits, and the following assignments:*

- $\dot{\times}_e : \mathbf{PD}_e \times \mathbf{PD}_e \rightarrow \mathbf{PD}_e$ by
 $X \dot{\times}_e Y \stackrel{\text{def}}{=} X \times Y$ and $e_1 \dot{\times}_e e_2 \stackrel{\text{def}}{=} e_1 \times e_2$
- $\dot{+}_e : \mathbf{PD}_e \times \mathbf{PD}_e \rightarrow \mathbf{PD}_e$ by
 $X \dot{+}_e Y \stackrel{\text{def}}{=} X \dot{+} Y$ and $e_1 \dot{+}_e e_2 \stackrel{\text{def}}{=} e_1 \dot{+} e_2$
- $[-\rightarrow]_e^{\mathcal{J}} : \mathbf{PD}_e \times \mathbf{PD}_e \rightarrow \mathbf{PD}_e$
 $[X \rightarrow Y]_e^{\mathcal{J}} \stackrel{\text{def}}{=} \mathcal{J}[X \rightarrow Y]$ and $[e_1 \rightarrow e_2]_e^{\mathcal{J}} \stackrel{\text{def}}{=} \mathcal{J}[e_1^p \rightarrow e_2]$

define covariant ω -cocontinuous bifunctors on \mathbf{PD}_e .

Proof. The empty dcpo \emptyset is a zero object in \mathbf{PD} such that each map $e : \emptyset \rightarrow X$ is an embedding and each map $p : X \rightarrow \emptyset$ is a projection. Therefore, \emptyset is initial in \mathbf{PD}_e . The existence of all ω -colimits in \mathbf{PD}_e follows from the existence of all ω -colimits of \mathbf{PD} together with results from [49].

Next, we show that $\dot{\times} : \mathbf{DCPO}_{\mathcal{M}} \times \mathbf{DCPO}_{\mathcal{M}} \rightarrow \mathbf{DCPO}_{\mathcal{M}}$ restricts to a functor $\dot{\times}^{\mathbf{PD}} : \mathbf{PD} \times \mathbf{PD} \rightarrow \mathbf{PD}$. On objects, this is obvious. For morphisms, observe that the morphisms of \mathbf{PD} are exactly those which are in the image of \mathcal{F} . Therefore $\dot{\times}^{\mathbf{PD}}$ restricts as indicated because $\mathcal{F}f \dot{\times} \mathcal{F}g = \mathcal{F}(f \otimes g)$ by Corollary 46. Then, by Proposition 52, it follows that $(\dot{\times}^{\mathbf{PD}})^E : \mathbf{PD}_e \times \mathbf{PD}_e \rightarrow \mathbf{PD}_e$ is a covariant ω -cocontinuous functor. However, by definition, $\dot{\times}_e = (\dot{\times}^{\mathbf{PD}})^E$ which shows the result for $\dot{\times}_e$.

Exactly the same argument (swapping $\dot{\times}$ for $\dot{+}$ and \otimes for \oplus) shows the result for $\dot{+}_e$.

For function spaces, consider the functor $\mathcal{J} \circ [-\rightarrow] : \mathbf{DCPO}_{\mathcal{M}}^{\text{op}} \times \mathbf{DCPO}_{\mathcal{M}} \rightarrow \mathbf{DCPO}_{\mathcal{M}}$. This composition (co)restricts to a functor $(\mathcal{J} \circ [-\rightarrow])^{\mathbf{PD}} : \mathbf{PD}^{\text{op}} \times$

$\mathbf{PD} \rightarrow \mathbf{PD}$, because $\mathcal{J}(f \rightarrow g) = \eta \circ (f \rightarrow g) = \phi \circ \eta^{\mathcal{L}} \circ (f \rightarrow g) = \mathcal{F}(\eta^{\mathcal{L}} \circ (f \rightarrow g))$. By Proposition 52, it follows $((\mathcal{J} \circ [-\rightarrow])^{\mathbf{PD}})^E: \mathbf{PD}_e \times \mathbf{PD}_e \rightarrow \mathbf{PD}_e$ is a covariant ω -cocontinuous functor. Finally, by definition, $[-\rightarrow]_e^{\mathcal{J}} = ((\mathcal{J} \circ [-\rightarrow])^{\mathbf{PD}})^E$ which concludes the proof. \square

Therefore, by Proposition 49 and Proposition 53 we can solve recursive domain equations induced by all well-formed type expressions (with no restrictions on the admissible logical polarities of the types) within \mathbf{PD}_e . However, since our judgements support weakening and contraction, we have an extra proof obligation: showing each isomorphism that is a solution to a recursive domain equation can be copied and discarded. This is indeed true (for any isomorphism in \mathbf{PD}) because of the next proposition.

Proposition 54. *Every isomorphism of \mathbf{PD} (and \mathbf{PD}_e) is also an isomorphism of \mathbf{TD} .*

Proof. Observe that, by definition, the morphisms of \mathbf{TD} are those in the image of $\mathcal{J}: \mathbf{DCPO} \rightarrow \mathbf{DCPO}_{\mathcal{M}}$ and the morphisms of \mathbf{PD} are those in the image of $\mathcal{F}: \mathbf{DCPO}_{\mathcal{L}} \rightarrow \mathbf{DCPO}_{\mathcal{M}}$. Then, it is easy to see that the following diagram:

$$\begin{array}{ccc} \mathbf{TD} & \hookrightarrow & \mathbf{PD} \\ \wr \parallel & & \wr \parallel \\ \mathbf{DCPO} & \xrightarrow{\mathcal{J}^{\mathcal{L}}} & \mathbf{DCPO}_{\mathcal{L}} \end{array}$$

commutes, where:

- the top arrow is the subcategory inclusion $\mathbf{TD} \hookrightarrow \mathbf{PD}$;
- the left vertical isomorphism is the corestriction of \mathcal{J} to \mathbf{TD} ;
- the right vertical isomorphism is the corestriction of \mathcal{F} to \mathbf{PD} ;
- the functor $\mathcal{J}^{\mathcal{L}}$ is the Kleisli inclusion of \mathbf{DCPO} into $\mathbf{DCPO}_{\mathcal{L}}$, defined by $\mathcal{J}^{\mathcal{L}}(X) \stackrel{\text{def}}{=} X$ and $\mathcal{J}^{\mathcal{L}}(f) \stackrel{\text{def}}{=} \eta^{\mathcal{L}} \circ f$.

It is well-known (and easy to prove) that if $f: X \rightarrow Y$ in $\mathbf{DCPO}_{\mathcal{L}}$ is an isomorphism, then there exists $f': X \rightarrow Y$ in \mathbf{DCPO} which is also an isomorphism and $f = \mathcal{J}^{\mathcal{L}}(f')$. The proof is finished by a simple diagram chase using this fact. \square

We have already explained that morphisms of \mathbf{TD} are compatible with weakening and contraction, so the above proposition suffices for our purposes.

5. Denotational Semantics

We now give the denotational semantics of our language by using ideas from [51, 50].

$$\begin{aligned}
\llbracket \Theta \vdash A \rrbracket &: \mathbf{PD}_e^{|\Theta|} \rightarrow \mathbf{PD}_e \\
\llbracket \Theta \vdash \Theta_i \rrbracket &\stackrel{\text{def}}{=} \Pi_i \\
\llbracket \Theta \vdash A + B \rrbracket &\stackrel{\text{def}}{=} \dot{+}_e \circ \langle \llbracket \Theta \vdash A \rrbracket, \llbracket \Theta \vdash B \rrbracket \rangle \\
\llbracket \Theta \vdash A \times B \rrbracket &\stackrel{\text{def}}{=} \dot{\times}_e \circ \langle \llbracket \Theta \vdash A \rrbracket, \llbracket \Theta \vdash B \rrbracket \rangle \\
\llbracket \Theta \vdash A \rightarrow B \rrbracket &\stackrel{\text{def}}{=} [-\rightarrow]_e^{\mathcal{J}} \circ \langle \llbracket \Theta \vdash A \rrbracket, \llbracket \Theta \vdash B \rrbracket \rangle \\
\llbracket \Theta \vdash \mu X.A \rrbracket &\stackrel{\text{def}}{=} \llbracket \Theta, X \vdash A \rrbracket^\#
\end{aligned}$$

Figure 5: Interpretation of types.

$$\begin{aligned}
\llbracket A \times B \rrbracket &= \llbracket A \rrbracket \times \llbracket B \rrbracket \\
\llbracket A + B \rrbracket &= \llbracket A \rrbracket + \llbracket B \rrbracket \\
\llbracket A \rightarrow B \rrbracket &= \llbracket A \rrbracket \rightarrow \mathcal{M}[\llbracket B \rrbracket] \\
\llbracket \mu X.A \rrbracket &\cong \llbracket A[\mu X.A/X] \rrbracket
\end{aligned}$$

Figure 6: Derived equations for closed types.

5.1. Interpretation of Types

We begin with the interpretation of (open) types. Every type $\Theta \vdash A$ is interpreted as a functor $\llbracket \Theta \vdash A \rrbracket: \mathbf{PD}_e^{|\Theta|} \rightarrow \mathbf{PD}_e$ and its interpretation is defined by induction on the derivation of $\Theta \vdash A$ in Figure 5. The validity of this definition is justified by the next proposition.

Proposition 55. *The assignments $\llbracket \Theta \vdash A \rrbracket: \mathbf{PD}_e^{|\Theta|} \rightarrow \mathbf{PD}_e$ are ω -cocontinuous functors.*

Proof. By induction using Propositions 49 and 53. \square

We are primarily interested in closed types and for them we simply write $\llbracket A \rrbracket \stackrel{\text{def}}{=} \llbracket \cdot \vdash A \rrbracket(*)$, where $*$ is the unique object of the terminal category $\mathbf{1} = \mathbf{PD}_e^0$. For closed types, it follows that $\llbracket A \rrbracket \in \text{Ob}(\mathbf{PD}_e) = \text{Ob}(\mathbf{DCPO})$.

We proceed by defining the folding/unfolding isomorphisms for recursive types and proving a necessary lemma.

Lemma 56 (Substitution). *If $\Theta, X \vdash A$ and $\Theta \vdash B$, then:*

$$\llbracket \Theta \vdash A[B/X] \rrbracket = \llbracket \Theta, X \vdash A \rrbracket \circ \langle \text{Id}, \llbracket \Theta \vdash B \rrbracket \rangle.$$

Definition 57. For closed types $\mu X.A$, we define:

$$\text{fold}_{\mu X.A} : \llbracket A[\mu X.A/X] \rrbracket = \llbracket X \vdash A \rrbracket \llbracket \mu X.A \rrbracket \cong \llbracket \mu X.A \rrbracket,$$

$$\begin{aligned}
\llbracket \Gamma \vdash M : A \rrbracket &: \llbracket \Gamma \rrbracket \rightarrow \llbracket A \rrbracket \text{ in } \mathbf{DCPO}_{\mathcal{M}} \\
\llbracket \Gamma, x : A \vdash x : A \rrbracket &\stackrel{\text{def}}{=} \mathcal{J}\pi_2 \\
\llbracket \Gamma \vdash (M, N) : A \times B \rrbracket &\stackrel{\text{def}}{=} (\llbracket M \rrbracket \times \llbracket N \rrbracket) \circ \mathcal{J}\langle \text{id}, \text{id} \rangle \\
\llbracket \Gamma \vdash \pi_i M : A_i \rrbracket &\stackrel{\text{def}}{=} \mathcal{J}\pi_i \circ \llbracket M \rrbracket, \text{ for } i \in \{1, 2\} \\
\llbracket \Gamma \vdash \text{in}_i M : A_1 + A_2 \rrbracket &\stackrel{\text{def}}{=} \mathcal{J}\text{in}_i \circ \llbracket M \rrbracket, \text{ for } i \in \{1, 2\} \\
\llbracket \Gamma \vdash (\text{case } M \text{ of } \text{in}_1 x \Rightarrow N_1 \mid \text{in}_2 y \Rightarrow N_2) : B \rrbracket &\stackrel{\text{def}}{=} \\
&\llbracket \llbracket N_1 \rrbracket, \llbracket N_2 \rrbracket \rrbracket \circ d \circ (\text{id} \times \llbracket M \rrbracket) \circ \mathcal{J}\langle \text{id}, \text{id} \rangle \\
\llbracket \Gamma \vdash \lambda x^A. M : A \rightarrow B \rrbracket &\stackrel{\text{def}}{=} \mathcal{J}\lambda(\llbracket M \rrbracket) \\
\llbracket \Gamma \vdash MN : B \rrbracket &\stackrel{\text{def}}{=} \epsilon \circ (\llbracket M \rrbracket \times \llbracket N \rrbracket) \circ \mathcal{J}\langle \text{id}, \text{id} \rangle \\
\llbracket \Gamma \vdash \text{fold } M : \mu X. A \rrbracket &\stackrel{\text{def}}{=} \text{fold} \circ \llbracket M \rrbracket \\
\llbracket \Gamma \vdash \text{unfold } M : A[\mu X. A/X] \rrbracket &\stackrel{\text{def}}{=} \text{unfold} \circ \llbracket M \rrbracket \\
\llbracket \Gamma \vdash M \text{ or}_p N : A \rrbracket &\stackrel{\text{def}}{=} \llbracket M \rrbracket +_p \llbracket N \rrbracket
\end{aligned}$$

Figure 7: Interpretation of term judgements.

where the equality is Lemma 56 and the isomorphism is the initial algebra. We write $\text{unfold}_{\mu X. A}$ for the inverse isomorphism. Note that both of them are isomorphisms in **TD**.

Now the equations for closed types in Figure 6 follow immediately.

5.2. Interpretation of Terms

A context $\Gamma = x_1 : A_1, \dots, x_n : A_n$ is interpreted as the dcpo $\llbracket \Gamma \rrbracket \stackrel{\text{def}}{=} \llbracket A_1 \rrbracket \times \dots \times \llbracket A_n \rrbracket$. A term $\Gamma \vdash M : A$ is, as usual, interpreted as a morphism $\llbracket \Gamma \vdash M : A \rrbracket : \llbracket \Gamma \rrbracket \rightarrow \llbracket A \rrbracket$ in $\mathbf{DCPO}_{\mathcal{M}}$ and we will abbreviate this by writing $\llbracket M \rrbracket$ when its type and context are clear. The interpretation of term judgements is defined by induction in Figure 7. This interpretation is defined in the standard categorical way using the structure of $\mathbf{DCPO}_{\mathcal{M}}$ and using the structure of the Kleisli exponential following Moggi [18]. To interpret probabilistic choice, we use the convex structure of $\mathbf{DCPO}_{\mathcal{M}}$. All the notation used in Figure 7 is introduced in Section 4 and Section 5.

5.3. Soundness and Computational Adequacy

In this subsection we prove the main semantic results for our model – soundness and (strong) adequacy. In order to do so, we first have to prove some useful lemmas.

As usual, the interpretation of values enjoys additional structural properties.

Lemma 58. *For any value $\Gamma \vdash V : A$, its interpretation $\llbracket V \rrbracket$ is a morphism of **TD**. Equivalently, it is in the image of \mathcal{J} .*

This means the interpretation of each closed value may be seen as a Dirac valuation. Next, we prove a substitution lemma.

Lemma 59 (Substitution). *Let $\Gamma \vdash V : A$ be a value and $\Gamma, x : A \vdash M : B$ a term. Then:*

$$\llbracket M[V/x] \rrbracket = \llbracket M \rrbracket \circ (\text{id}_{[\Gamma]} \dot{\times} \llbracket V \rrbracket) \circ \mathcal{J}(\text{id}_{[\Gamma]}, \text{id}_{[\Gamma]}).$$

Soundness and (strong) adequacy are formulated in terms of convex sums of the interpretations of terms. For a collection of terms M_i with $\Gamma \vdash M_i : A$, for each $i \in I$, each interpretation $\llbracket M_i \rrbracket$ is a map in the Kegelspitze $\mathbf{DCPO}_{\mathcal{M}}([\Gamma], [A])$, so, we may form convex sums of these maps.

Soundness is the statement that our interpretation is invariant under single-step reduction (in a probabilistic sense).

Theorem 60 (Soundness). *For any term $\Gamma \vdash M : A$,*

$$\llbracket M \rrbracket = \sum_{M \xrightarrow{p} M'} p \llbracket M' \rrbracket,$$

assuming $M \xrightarrow{p} M'$ for some rule from Figure 4 and where the convex sum ranges over all such rules.

Proof. Straightforward induction using Lemma 59. □

In the above theorem, the convex sum has at most two summands which are reached after a single reduction step. The next, considerably stronger statement, generalises this result to reductions involving an arbitrary number of steps. *Strong adequacy* is the statement that the denotational interpretation is invariant with respect to reduction in a *big-step* sense (see [52], [9, 13] where such results are proven).

Theorem 61 (Strong Adequacy). *For any term $\cdot \vdash M : A$,*

$$\llbracket M \rrbracket = \sum_{V \in \text{Val}(M)} P(M \rightarrow_* V) \llbracket V \rrbracket.$$

Proof. The proof of this theorem requires considerable effort and it is presented in Section 6. □

This theorem implies that every denotable element in our model is a *discrete valuation*, because the interpretation of every value $\cdot \vdash V : A$ is a Dirac valuation. Theorem 61 is also true to its name, because it immediately implies the usual notion of adequacy, by using the fact that if $\cdot \vdash V : 1$ is a value, then $\llbracket V \rrbracket(\ast)(\{\ast\}) = 1 \in \mathbb{R}$.

Corollary 62 (Adequacy). *Let $\cdot \vdash M : 1$ be a term. Then*

$$\llbracket M \rrbracket(\ast)(\{\ast\}) = \text{Halt}(M), \quad (\text{see (2)})$$

where \ast is the unique element of the singleton dcpo 1.

The commutativity of our monad \mathcal{M} implies that given any well-formed terms $\Gamma \vdash M_1 : A_1$ and $\Gamma \vdash M_2 : A_2$ and $\Gamma, x_1 : A_1, x_2 : A_2 \vdash N : B$, then

$$\llbracket \text{let } x_1 = M_1 \text{ in let } x_2 = M_2 \text{ in } N \rrbracket = \llbracket \text{let } x_2 = M_2 \text{ in let } x_1 = M_1 \text{ in } N \rrbracket, \quad (6)$$

where $\text{let } x = M \text{ in } N$ may be defined using the usual syntactic sugar. This, together with adequacy (Corollary 62) and some standard arguments (see [9]) implies that the programs in (6) are contextually equivalent.

We finally note that all results in this section also hold for the monads \mathcal{W} and \mathcal{P} .

6. Proof of Strong Adequacy

The purpose of this section is to provide a proof Theorem 61. We begin by stating a corollary for the soundness theorem.

Corollary 63. *For any closed term $\cdot \vdash M : A$, we have:*

$$\llbracket M \rrbracket \geq \sum_{V \in \text{Val}(M)} P(M \rightarrow_* V) \llbracket V \rrbracket.$$

Proof. First, let us decompose the convex sum on the right-hand side.

$$\begin{aligned} & \sum_{V \in \text{Val}(M)} P(M \rightarrow_* V) \llbracket V \rrbracket \\ = & \{ \text{Definition} \} \\ & \sup_{\substack{F \subseteq \text{Val}(M) \\ F \text{ finite}}} \sum_{V \in F} P(M \rightarrow_* V) \llbracket V \rrbracket \\ = & \{ \text{Definition} \} \\ & \sup_{\substack{F \subseteq \text{Val}(M) \\ F \text{ finite}}} \sum_{V \in F} \left(\sup_{i \in \mathbb{N}} P(M \rightarrow_{\leq i} V) \right) \llbracket V \rrbracket \\ = & \left\{ \text{Scott-continuity of } \sum_i r_i a_i \text{ in each } r_i \right\} \\ & \sup_{\substack{F \subseteq \text{Val}(M) \\ F \text{ finite}}} \sup_{i \in \mathbb{N}} \sum_{V \in F} P(M \rightarrow_{\leq i} V) \llbracket V \rrbracket. \end{aligned}$$

Therefore, it suffices to show that

$$\llbracket M \rrbracket \geq \sum_{V \in F} P(M \rightarrow_{\leq i} V) \llbracket V \rrbracket \quad (7)$$

for any choice of finite $F \subseteq \text{Val}(M)$ and $i \in \mathbb{N}$. This can now be shown by induction on i . If $M \in F$ (which means M is a value), then (7) is a strict equality. Assume $M \notin F$. If $i = 0$, then the right-hand side of (7) is 0 and so

the inequality holds. For the step case, if M is a value, then RHS is 0 and the inequality holds. Otherwise:

$$\begin{aligned}
\sum_{V \in F} P(M \rightarrow_{\leq i+1} V) \llbracket V \rrbracket &= \sum_{V \in F} \sum_{M \xrightarrow{p} M'} p \cdot P(M' \rightarrow_{\leq i} V) \llbracket V \rrbracket \\
&= \sum_{M \xrightarrow{p} M'} p \cdot \sum_{V \in F} P(M' \rightarrow_{\leq i} V) \llbracket V \rrbracket \\
&\leq \sum_{M \xrightarrow{p} M'} p \cdot \llbracket M' \rrbracket && \text{(IH for } M') \\
&= \llbracket M \rrbracket && \text{(Soundness)}
\end{aligned}$$

where we also implicitly used the fact that $\text{Val}(M') \subseteq \text{Val}(M)$. \square

The remainder of the section is dedicated to showing the converse inequality, which is considerably more difficult to prove.

6.1. Overview of the Proof Strategy

The proof of strong adequacy requires considerable effort. Our proof strategy consists in formulating logical relations that we use to prove our adequacy result. These logical relations are described in Theorem 103 and the design of our logical relations follows that of Claire Jones in her thesis [13]. Once this theorem is proved, the proof of adequacy is fairly straightforward. We use the logical relations to establish some useful closure properties in Subsection 6.6 and this allows us to easily prove Lemma 113, which is often called the Fundamental Lemma. This lemma easily implies Strong Adequacy as we show.

Most of the effort in proving our Strong Adequacy result lies in the proof of Theorem 103. It is not possible to use the properties (A1) – (A4) as a definition of the relations, because then condition (A4) would be defined via non-well-founded induction. The proof of the existence of this family of relations is not obvious. We use techniques from [51, 50] (which are in turn based on ideas from [27]) to show the existence of these relations. The main idea of the proof of existence is to define, for every type A , a category $\mathbf{R}(A)$ of logical relations with a suitable notion of morphism. We then show that every such category has sufficient structure to construct parameterised initial algebras (Proposition 83). We may then define functors on these categories (Definition 90) which construct logical relations in the same manner as they are needed in Theorem 103. These functors are ω -cocontinuous (Proposition 92) which means that we may form (parameterised) initial algebras using them. This allows us to define an *augmented interpretation of types* on the categories $\mathbf{R}(A)$ which satisfies some important coherence conditions with respect to the standard interpretation of types (Corollary 101). These coherence conditions show that each augmented interpretation $\|A\|$ of a type A contains the standard interpretation $\llbracket A \rrbracket$, together with the logical relation that we need, as shown in Theorem 103.

6.2. Logical Relations

Assumption 64. Throughout this section, we assume that all types are closed, unless otherwise noted.

Definition 65. For each type A , we write:

- $\text{Val}(A) \stackrel{\text{def}}{=} \{V \mid V \text{ is a value and } \cdot \vdash V : A\}$.
- $\text{Prog}(A) \stackrel{\text{def}}{=} \{M \mid M \text{ is a term and } \cdot \vdash M : A\}$.

Next, we define sets of relations that are parameterised by dcpo's X from our semantic category, types A from our language and partial deterministic embeddings $e_X : X \rightarrow \llbracket A \rrbracket$ which show how X approximates $\llbracket A \rrbracket$. We shall write relation membership in infix notation, that is, for a binary relation \triangleleft , we write $v \triangleleft V$ to indicate $(v, V) \in \triangleleft$.

Definition 66. For any dcpo X , type A and morphism $e : X \rightarrow \llbracket A \rrbracket$ in \mathbf{PD}_e , let:

$$\begin{aligned} \text{ValRel}(X, A, e) = & \{ \triangleleft_{X,A}^e \subseteq \mathbf{TD}(1, X) \times \text{Val}(A) \mid \\ & \forall V \in \text{Val}(A). (-) \triangleleft_{X,A}^e V \text{ is a Scott closed subset of } \mathbf{TD}(1, X) \text{ and} \\ & \forall V \in \text{Val}(A). v \triangleleft_{X,A}^e V \Rightarrow e \circ v \leq \llbracket V \rrbracket \}. \end{aligned}$$

Remark 67. In the above definition, relations $\triangleleft_{X,A}^e \in \text{ValRel}(X, A, e)$ can be seen as ternary relations $\triangleleft_{X,A}^e \subseteq \mathbf{TD}(1, X) \times \text{Val}(A) \times \{e\}$. However, since there is no choice for the third component, we prefer to see them as binary relations that are parameterised by the embeddings e . Indeed, this leads to a much nicer notation. We shall also sometimes indicate the parameters X, A and e of the relation in order to avoid confusion as to which set $\text{ValRel}(X, A, e)$ it belongs to.

The relations we need for the adequacy proof inhabit the sets $\text{ValRel}(\llbracket A \rrbracket, A, \mathbf{id}_{\llbracket A \rrbracket})$. In the remainder of the section, we will show how to choose exactly one relation (the one we need) from each of those sets.

Before we may define the relation constructors we need, we have to introduce some auxiliary definitions.

Definition 68. Let $M : A$ and $N : A$ be closed terms of the same type. We define

$$\text{Paths}(M, N) \stackrel{\text{def}}{=} \left\{ \pi \mid \pi = \left(M = M_0 \xrightarrow{p_0} M_1 \xrightarrow{p_1} M_2 \xrightarrow{p_2} \dots \xrightarrow{p_n} M_n = N \right) \text{ is a reduction path} \right\}.$$

In other words, $\text{Paths}(M, N)$ is the set of all reduction paths from M to N . The *probability weight* of a path $\pi \in \text{Paths}(M, N)$ is $P(\pi) \stackrel{\text{def}}{=} \prod_{i=0}^n p_i$, i.e., it is simply the product of all the probabilities of single-step reductions within the path. The *set of terminal reduction paths of M* is

$$\text{TPaths}(M) \stackrel{\text{def}}{=} \bigcup_{V \in \text{Val}(A)} \text{Paths}(M, V).$$

Thus the endpoint of any path $\pi \in \text{TPaths}(M)$ is a value. If $\pi \in \text{Paths}(M, W)$, where W is a value, then we shall write $V_\pi \stackrel{\text{def}}{=} W$. That is, for a path $\pi \in \text{TPaths}(M)$, the notation V_π indicates the endpoint of the path π which is indeed a value.

Remark 69. We also note that for each closed term M , the set $\text{TPaths}(M)$ is countable.

The next definition we introduce is crucial for the proof of strong adequacy.

Definition 70. Given a relation $\triangleleft_{X,A}^e \in \text{ValRel}(X, A, e)$ and a term $\cdot \vdash M : A$, let $\mathcal{S}(\triangleleft_{X,A}^e; M)$ be the Scott-closure in $\mathbf{DCPO}_{\mathcal{M}}(1, X)$ of the set

$$\mathcal{S}_0(\triangleleft_{X,A}^e; M) \stackrel{\text{def}}{=} \left\{ \sum_{\pi \in F} P(\pi)v_\pi \mid F \subseteq \text{TPaths}(M), F \text{ is finite and } v_\pi \triangleleft_{X,A}^e V_\pi \text{ for each } \pi \in F \right\}. \quad (8)$$

In other words, $\mathcal{S}(\triangleleft_{X,A}^e; M)$ is the smallest Scott-closed subset of $\mathbf{DCPO}_{\mathcal{M}}(1, X)$ which contains all morphisms of the form in (8). For a subset $U \subseteq \mathbf{DCPO}_{\mathcal{M}}(1, X)$, we write \overline{U} to indicate its Scott-closure in $\mathbf{DCPO}_{\mathcal{M}}(1, X)$.

Lemma 71. *For any value V , we have $\mathcal{S}(\triangleleft_{X,A}^e; V) = \overline{\{v \mid v \triangleleft_{X,A}^e V\}} \cup \{0\} = \overline{\{v \mid v \triangleleft_{X,A}^e V\}} \cup \{0\}$.*

Proof. This is because all of the sums in (8) are singleton sums or the empty sum. \square

Lemma 72 ([13, Lemma 8.4]). *Let Y be a dcpo and let $\{X_i\}_{i \in F}$ be a finite collection of dcpo's. Let $f: \prod_i X_i \rightarrow Y$ be a Scott-continuous function. Let C_Y be a Scott-closed subset of Y . Let $U_i \subseteq X_i$ be arbitrary subsets, such that $f(\prod_i U_i) \subseteq C_Y$. Then $f(\prod_i \overline{U_i}) \subseteq C_Y$, where $\overline{U_i}$ is the Scott-closure of U_i in X_i .*

Lemma 73. *Let $\triangleleft_{X_1,A}^{e_1}$ and $\triangleleft_{X_2,A}^{e_2}$ be two logical relations and $\cdot \vdash M : A$ a term. Assume that $g : X_1 \rightarrow X_2$ is a morphism, such that $v \triangleleft_{X_1,A}^{e_1} V$ implies $g \circ v \in \mathcal{S}(\triangleleft_{X_2,A}^{e_2}; V)$, for any $V \in \text{Val}(M)$. If $m \in \mathcal{S}(\triangleleft_{X_1,A}^{e_1}; M)$, then $g \circ m \in \mathcal{S}(\triangleleft_{X_2,A}^{e_2}; M)$.*

Proof. By Lemma 72, it suffices to show that

$$\left(g \circ \sum_{\pi \in F} P(\pi)v_\pi \right) \in \mathcal{S}(\triangleleft_{X_2,A}^{e_2}; M)$$

for any choice of finite $F \subseteq \text{TPaths}(M)$ and morphisms v_π with $v_\pi \triangleleft_{X_1,A}^{e_1} V_\pi$. We have

$$g \circ \sum_{\pi \in F} P(\pi)v_\pi = \sum_{\pi \in F} P(\pi)(g \circ v_\pi),$$

where the equality follows by linearity of $(g \circ -)$. Next, for each v_π , by assumption $g \circ v_\pi \in \mathcal{S}(\triangleleft_{X_2, A}^{e_2}; V_\pi)$. Therefore by applying Lemma 71, it follows $g \circ v_\pi \in \overline{\{v' \mid v' \triangleleft_{X_2, A}^{e_2} V_\pi\}} \cup \{0\}$. Now, consider the function

$$\sum_{\pi \in F} P(\pi)(-): \prod_{|F|} \mathbf{DCPO}_{\mathcal{M}}(1, X_2) \rightarrow \mathbf{DCPO}_{\mathcal{M}}(1, X_2).$$

This function is continuous, so by Lemma 72 again, it suffices to show that

$$\sum_{\pi \in F} P(\pi)m'_\pi = \sum_{\substack{\pi \in F \\ m'_\pi \neq 0}} P(\pi)m'_\pi \in \mathcal{S}(\triangleleft_{X_2, A}^{e_2}; M),$$

where either $m'_\pi = 0$ or $m'_\pi \triangleleft_{X_2, A}^{e_2} V_\pi$ for each $\pi \in F$. Since the summands where $m'_\pi = 0$ do not affect the sum, it suffices to show that this is true under the assumption that $m'_\pi \triangleleft_{X_2, A}^{e_2} V_\pi$. But this is true by definition of $\mathcal{S}(\triangleleft_{X_2, A}^{e_2}; M)$. \square

Next, we define important *closure relations* which we use for terms.

Definition 74. If $\triangleleft_{X, A}^e \in \text{ValRel}(X, A, e)$, let $\overline{\triangleleft_{X, A}^e} \subseteq \mathbf{DCPO}_{\mathcal{M}}(1, X) \times \text{Prog}(A)$ be the relation defined by

$$m \overline{\triangleleft_{X, A}^e} M \text{ iff } m \in \mathcal{S}(\triangleleft_{X, A}^e; M).$$

Lemma 75. For any term $\cdot \vdash M : A$ and $\triangleleft_{X, A}^e \in \text{ValRel}(X, A, e)$, the set $(-) \overline{\triangleleft_{X, A}^e} M$ is a Scott-closed subset of $\mathbf{DCPO}_{\mathcal{M}}(1, X)$.

Proof. This follows immediately by definition, because $\mathcal{S}(\triangleleft_{X, A}^e; M)$ is Scott-closed. \square

Lemma 76. Let C be a Scott-closed subset of a dcpo X . Let $W \stackrel{\text{def}}{=} \{\delta_x \mid x \in C\} \subseteq \mathcal{M}X$ and let \overline{W} be the Scott-closure of W in $\mathcal{M}X$. Then, $\delta_y \in \overline{W}$ iff $y \in C$.

Proof. The “if” direction is straightforward. The “only if” direction is trivial when $C = X$. We now prove the case that C is a proper subset of X , and let U be the complement of C . Hence U is a nonempty Scott open subset of X . Let us assume that $\delta_y \in \overline{W}$ but $y \in U$, then we know that $[U > 0] \stackrel{\text{def}}{=} \{\nu \in \mathcal{M}X \mid \nu(U) > 0\}$ is a Scott open subset of $\mathcal{M}X$ containing δ_y , hence we would have that $[U > 0] \cap W \neq \emptyset$ since by assumption $\delta_y \in \overline{W}$. However, this is impossible since for any $x \in C$, $\delta_x(U) = 0$. \square

Lemma 77. Let X be a dcpo, let $v \in \mathbf{TD}(1, X)$ and let V be a value. Then $v \triangleleft_{X, A}^e V$ iff $v \overline{\triangleleft_{X, A}^e} V$.

Proof. The left-to-right direction follows immediately by Lemma 71. For the other direction, we first observe that since $v \in \mathbf{TD}(1, X)$, then $v \neq 0$. Therefore by Lemma 71, it follows $v \in \overline{\{w \mid w \triangleleft_{X, A}^e V\}}$ and then by Lemma 76 we complete the proof. \square

Lemma 78. For any value $\cdot \vdash V : A$ and $\triangleleft_{X,A}^e \in \text{ValRel}(X, A, e)$, if $m \overline{\triangleleft_{X,A}^e} V$ then $e \circ m \leq \llbracket V \rrbracket$.

Proof. We know $m \in \mathcal{S}(\triangleleft_{X,A}^e; V) = \overline{\{v \mid v \triangleleft_{X,A}^e V\}} \cup \{0\}$ and clearly $e \circ m \leq \llbracket V \rrbracket$ is equivalent to $(e \circ m) \in \downarrow \llbracket V \rrbracket$, which is a Scott-closed subset. If $m = 0$, then the statement is obviously true. So, assume that $m \in \overline{\{v \mid v \triangleleft_{X,A}^e V\}}$. Composition with e is a Scott-continuous function and therefore using Lemma 72, to finish the proof it suffices to show $e \circ v \leq \llbracket V \rrbracket$ for each choice of $v \triangleleft_{X,A}^e V$. But this is true by assumption on $\triangleleft_{X,A}^e$. \square

6.3. Categories of Logical Relations

Definition 79. For any type A , we define a category $\mathbf{R}(A)$ where:

- Each object is a triple $(X, e_X, \triangleleft_X)$, where X is a depo, $e_X : X \rightarrow \llbracket A \rrbracket$ is a morphism in \mathbf{PD}_e and $\triangleleft_X \in \text{ValRel}(X, A, e_X)$.
- A morphism $f : (X, e_X, \triangleleft_X) \rightarrow (Y, e_Y, \triangleleft_Y)$ is a morphism $f : X \rightarrow Y$ in \mathbf{PD}_e , which satisfies the three additional conditions:
 - If $v \triangleleft_X V$, then $f \circ v \overline{\triangleleft_Y} V$.
 - If $v \triangleleft_Y V$, then $f^p \circ v \overline{\triangleleft_X} V$.
 - $e_X = e_Y \circ f$.
- Composition and identities coincide with those in \mathbf{PD}_e .

Lemma 80. For every type A , the category $\mathbf{R}(A)$ is indeed well-defined.

Proof. We have to show that $\text{id} : (X, e_X, \triangleleft_X) \rightarrow (X, e_X, \triangleleft_X)$ is indeed a morphism in $\mathbf{R}(A)$. This follows from Lemma 77. Next, we have to show that if $f : (X, e_X, \triangleleft_X) \rightarrow (Y, e_Y, \triangleleft_Y)$ and $g : (Y, e_Y, \triangleleft_Y) \rightarrow (Z, e_Z, \triangleleft_Z)$, then we also have $g \circ f : (X, e_X, \triangleleft_X) \rightarrow (Z, e_Z, \triangleleft_Z)$. But this follows by Lemma 73. \square

Lemma 81. Let $\cdot \vdash M : A$ be a term and let $g : (X, e_X, \triangleleft_X) \rightarrow (Y, e_Y, \triangleleft_Y)$ be a morphism in $\mathbf{R}(A)$. If $m \overline{\triangleleft_X} M$ then $g \circ m \overline{\triangleleft_Y} M$. Moreover, if $n \overline{\triangleleft_Y} N$, then $g^p \circ n \overline{\triangleleft_X} N$.

Proof. This follows immediately by Lemma 73. \square

Definition 82. For every type A , we define the obvious forgetful functor $U^A : \mathbf{R}(A) \rightarrow \mathbf{PD}_e$ by

$$\begin{aligned} U^A(X, e, \triangleleft) &= X \\ U^A(f) &= f. \end{aligned}$$

Proposition 83. For each type A , the category $\mathbf{R}(A)$ has an initial object and all ω -colimits. Furthermore, the forgetful functor $U^A : \mathbf{R}(A) \rightarrow \mathbf{PD}_e$ preserves and reflects ω -colimits (and also the initial objects).

Proof. We begin with the initial object.

Initial object. For any dcpo's X and Y , we write $0_{X,Y} : X \rightarrow Y$ for the zero morphism in \mathbf{PD} . Notice that $0_{\emptyset,X}$ is an embedding with projection counterpart given by $0_{X,\emptyset}$.

The object $(\emptyset, 0_{\emptyset, \llbracket A \rrbracket}, \emptyset)$ is initial in $\mathbf{R}(A)$. Indeed, let $(X, e_X, \triangleleft_X)$ be any other object of $\mathbf{R}(A)$. It suffices to show that $0_{\emptyset,X} : (\emptyset, 0_{\emptyset, \llbracket A \rrbracket}, \emptyset) \rightarrow (X, e_X, \triangleleft_X)$ is a morphism in $\mathbf{R}(A)$, because if it exists, then it is clearly unique. The first and third conditions of Definition 79 are trivially satisfied. The second condition is also satisfied, because $0_{\emptyset,X}^p \circ v = 0_{1,\emptyset}$, which is the least (and only) element in $\mathbf{DCPO}_{\mathcal{M}}(1, \emptyset)$ and this element is contained in every relation $\overline{\triangleleft_Y}$, including \emptyset .

The diagram. For the rest of the proof, let $D : \omega \rightarrow \mathbf{R}(A)$ be an ω -diagram in $\mathbf{R}(A)$. Let $D(i) = (X_i, e_i, \triangleleft_i)$ and let $D(i \leq j) = f_{i,j}$.

Construction of the colimiting object. Consider the ω -diagram UD in \mathbf{PD}_e . This category has all ω -colimits, so let $\tau : UD \Rightarrow X_\omega$ be its colimiting cocone. Next, consider the cocone $\epsilon : UD \Rightarrow \llbracket A \rrbracket$ defined by $\epsilon_i \stackrel{\text{def}}{=} e_i : X_i \rightarrow \llbracket A \rrbracket$. Let $e_\omega : X_\omega \rightarrow \llbracket A \rrbracket$ be the unique cocone morphism $e_\omega : \tau \rightarrow \epsilon$ induced by the colimit τ in \mathbf{PD}_e . We now define a relation

$$\begin{aligned} \triangleleft_\omega \in \text{ValRel}(X_\omega, A, e_\omega) \text{ by:} \\ v \triangleleft_\omega V \text{ iff } \forall k \in \mathbb{N}. \tau_k^p \circ v \overline{\triangleleft_k} V. \end{aligned}$$

We have to show that $\triangleleft_\omega \in \text{ValRel}(X_\omega, A, e_\omega)$, as claimed above. We begin with downwards-closure. Assume $v \triangleleft_\omega V$ and that $v' \leq v$ in $\mathbf{TD}(1, X_\omega)$. Then, $\forall k \in \mathbb{N}. \tau_k^p \circ v \overline{\triangleleft_k} V$ and therefore $\tau_k^p \circ v' \overline{\triangleleft_k} V$, because $(-\overline{\triangleleft_k} V)$ is downwards-closed and so by definition $v' \triangleleft_\omega V$, as required.

Next, we show that $(-\triangleleft_\omega V)$ preserves directed suprema and is therefore Scott-closed in $\mathbf{TD}(1, X_\omega)$. Assume that $\{v_d\}_{d \in D}$ is a directed set, such that $v_d \triangleleft_\omega V$ for each $d \in D$. Therefore, $\forall k \in \mathbb{N}. \forall d \in D. \tau_k^p \circ v_d \overline{\triangleleft_k} V$. Scott-closure of $(-\overline{\triangleleft_k} V)$ implies that $\tau_k^p \circ (\sup_{d \in D} v_d) = \sup_{d \in D} \tau_k^p \circ v_d \overline{\triangleleft_k} V$ holds for all $k \in \mathbb{N}$. Therefore, by definition $\sup_{d \in D} v_d \triangleleft_\omega V$.

We also have to show that if $v \triangleleft_\omega V$, then $e_\omega \circ v \leq \llbracket V \rrbracket$. If $v \triangleleft_\omega V$, then $\forall k \in \mathbb{N}. \tau_k^p \circ v \overline{\triangleleft_k} V$ and so by Lemma 78 we get $e_k \circ \tau_k^p \circ v \leq \llbracket V \rrbracket$. But $e_k \circ \tau_k^p \circ v = e_\omega \circ \tau_k \circ \tau_k^p \circ v$. The limit-colimit coincidence theorem in the category \mathbf{PD} , shows that this forms an increasing sequence and that

$$\llbracket V \rrbracket \geq \sup_{k \in \mathbb{N}} e_\omega \circ \tau_k \circ \tau_k^p \circ v = e_\omega \circ \left(\sup_{k \in \mathbb{N}} \tau_k \circ \tau_k^p \right) \circ v = e_\omega \circ \mathbf{id} \circ v = e_\omega \circ v,$$

as required. We will show that the object $(X_\omega, e_\omega, \triangleleft_\omega)$ is the colimiting object of D in $\mathbf{R}(A)$. Before we can do this, we first have to construct the colimiting cocone in $\mathbf{R}(A)$.

Construction of the colimiting cocone. We show that $\tau : D \Rightarrow X_\omega$ is a cocone in $\mathbf{R}(A)$. The commutativity requirements are clearly satisfied, so it suffices to show that each $\tau_i : X_i \rightarrow X_\omega$ is a morphism $\tau_i : (X_i, e_i, \triangleleft_i) \rightarrow$

$(X_\omega, e_\omega, \triangleleft_\omega)$ in $\mathbf{R}(A)$. Towards that end, assume that $v \triangleleft_i V$. We have to show that $\tau_i \circ v \overline{\triangleleft}_\omega V$, but by Lemma 77, it suffices to show that $\tau_i \circ v \triangleleft_\omega V$ ⁶. Showing this is equivalent to showing that $\forall k \in \mathbb{N}$. $\tau_k^p \circ \tau_i \circ v \overline{\triangleleft}_k V$. For any $k \geq i$, we get:

$$\tau_k^p \circ \tau_i \circ v = \tau_k^p \circ \tau_k \circ f_{i,k} \circ v = f_{i,k} \circ v \overline{\triangleleft}_k V$$

because $f_{i,k}$ is a morphism $f_{i,k}: (X_i, e_i, \triangleleft_i) \rightarrow (X_k, e_k, \triangleleft_k)$ and $v \triangleleft_i V$ by assumption. For any $k < i$, we get:

$$\tau_k^p \circ \tau_i \circ v = f_{k,i}^p \circ \tau_i^p \circ \tau_i \circ v = f_{k,i}^p \circ v \overline{\triangleleft}_k V$$

because $f_{k,i}$ is a morphism $f_{k,i}: (X_k, e_k, \triangleleft_k) \rightarrow (X_i, e_i, \triangleleft_i)$ and $v \triangleleft_i V$ by assumption (and Lemma 81).

To show that $\tau_i: (X_i, e_i, \triangleleft_i) \rightarrow (X_\omega, e_\omega, \triangleleft_\omega)$ is a morphism, we have to show that if $v \triangleleft_\omega V$, then also $\tau_i^p \circ v \overline{\triangleleft}_i V$. But this is true by definition of \triangleleft_ω .

Finally we have to show that $e_i = e_\omega \circ \tau_i$. But this is true by construction of e_ω .

Therefore, $\tau: D \Rightarrow (X_\omega, e_\omega, \triangleleft_\omega)$ is indeed a cocone of D in $\mathbf{R}(A)$.

Couniversality of the cocone. For the rest of the proof, assume that $\alpha: D \Rightarrow (Y, e_Y, \triangleleft_Y)$ is some other cocone of D in $\mathbf{R}(A)$. Next, consider the cocone $U\alpha$ in \mathbf{PD}_e and let $a: X_\omega \rightarrow Y$ be the unique cocone morphism $a: U\tau \rightarrow U\alpha$ induced by the colimit in \mathbf{PD}_e . By the limit-colimit coincidence theorem in \mathbf{PD} , we get

$$a = a \circ \mathbf{id} = a \circ \sup_{i \in \mathbb{N}} \tau_i \circ \tau_i^p = \sup_{i \in \mathbb{N}} a \circ \tau_i \circ \tau_i^p = \sup_{i \in \mathbb{N}} \alpha_i \circ \tau_i^p$$

We will show that $a: (X_\omega, e_\omega, \triangleleft_\omega) \rightarrow (Y, e_Y, \triangleleft_Y)$ is a morphism in $\mathbf{R}(A)$. Towards this end, assume that $v \triangleleft_\omega V$. Then $\forall k \in \mathbb{N}$. $\tau_k^p \circ v \overline{\triangleleft}_k V$ and therefore $\alpha_k \circ \tau_k^p \circ v \overline{\triangleleft}_Y V$, because by assumption $\alpha_k: (X_k, e_k, \triangleleft_k) \rightarrow (Y, e_Y, \triangleleft_Y)$. Since $(-\overline{\triangleleft}_Y V)$ is closed under suprema, it follows

$$\sup_{k \in \mathbb{N}} \alpha_k \circ \tau_k^p \circ v = \left(\sup_{k \in \mathbb{N}} \alpha_k \circ \tau_k^p \right) \circ v = a \circ v \overline{\triangleleft}_Y V,$$

which shows that a satisfies one of the requirements for being a morphism in $\mathbf{R}(A)$.

For the second requirement, assume that $v \triangleleft_Y V$. Then, $\forall k \in \mathbb{N}$. $\alpha_k^p \circ v \overline{\triangleleft}_k V$, by assumption on α_k . The same argument shows that $\forall k \in \mathbb{N}$. $\tau_k \circ \alpha_k^p \circ v \overline{\triangleleft}_\omega V$, because τ_k is also a morphism in the category. Since $(-\overline{\triangleleft}_\omega V)$ is closed under suprema, we get:

$$\sup_{k \in \mathbb{N}} \tau_k \circ \alpha_k^p \circ v = \sup_{k \in \mathbb{N}} \tau_k \circ \tau_k^p \circ a^p \circ v = \left(\sup_{k \in \mathbb{N}} \tau_k \circ \tau_k^p \right) \circ a^p \circ v = a^p \circ v \overline{\triangleleft}_\omega V$$

⁶Note that $\tau_i \circ v$ is a morphism of \mathbf{TD} , because v is one and because $\tau_i \in \mathbf{PD}_e$ which is a subcategory of \mathbf{TD} .

as required.

For the third requirement, we have to show that $e_\omega = e_Y \circ a$. By assumption on the cone $\alpha : D \Rightarrow (Y, e_Y, \triangleleft_Y)$, we have that $\forall i \in \mathbb{N}. e_i = e_Y \circ \alpha_i$ and by construction of a , we know $\alpha_i = a \circ \tau_i$. Therefore $\forall i \in \mathbb{N}. e_i = e_Y \circ a \circ \tau_i$. However, e_ω is by construction the unique morphism in \mathbf{PD}_e , such that $\forall i. e_i = e_\omega \circ \tau_i$, which shows that $e_\omega = e_Y \circ a$, as required. Therefore, we have shown that $a : (X_\omega, e_\omega, \triangleleft_\omega) \rightarrow (Y, e_Y, \triangleleft_Y)$ is indeed a morphism in $\mathbf{R}(A)$.

That $a : \tau \rightarrow \alpha$ is the unique cocone morphism is now obvious, because if $a' : \tau \rightarrow \alpha$ is another one, then Ua and Ua' are both cocone morphisms between $U\tau$ and $U\alpha$ in \mathbf{PD}_e and therefore $a = Ua = Ua' = a'$. Therefore, $\tau : D \Rightarrow (X_\omega, e_\omega, \triangleleft_\omega)$ is indeed the colimiting cocone of D in $\mathbf{R}(A)$, which shows that $\mathbf{R}(A)$ has all ω -colimits.

U^A **preserves ω -colimits.** Assume that the cocone $\alpha : D \Rightarrow (Y, e_Y, \triangleleft_Y)$ from above is colimiting in $\mathbf{R}(A)$. But, we know that $\tau : D \Rightarrow (X_\omega, e_\omega, \triangleleft_\omega)$ is also a colimiting cocone of D . Therefore, there exists a unique cocone isomorphism $i : \tau \rightarrow \alpha$. Then, $Ui : U\tau \rightarrow U\alpha$ is a cocone isomorphism in \mathbf{PD}_e . However, by construction, $U\tau$ is a colimiting cocone of UD in \mathbf{PD}_e and therefore so is $U\alpha$.

U^A **reflects ω -colimits.** Assume that the cocone $\alpha : D \Rightarrow (Y, e_Y, \triangleleft_Y)$ from above is such that $U\alpha : UD \Rightarrow Y$ is colimiting in \mathbf{PD}_e . Then the morphism $a : X_\omega \rightarrow Y$ from above is an isomorphism in \mathbf{PD}_e . We have already shown that $a : (X_\omega, e_\omega, \triangleleft_\omega) \rightarrow (Y, e_Y, \triangleleft_Y)$ is a morphism in $\mathbf{R}(A)$. Thus, to finish the proof, it suffices to show that a^{-1} is a morphism in $\mathbf{R}(A)$ in the opposite direction. But this is obviously true, because $a^{-1} = a^p$ and $(a^{-1})^p = a$ and we have shown above that these morphisms satisfy the logical requirements and clearly $e_Y = e_\omega \circ a^{-1}$. □

Next, we introduce important relation constructors and some new notation.

Notation 84. Given morphisms $m_i : 1 \rightarrow X_i$, for $i \in \{1, \dots, n\}$, we define

$$\langle\langle m_1, \dots, m_n \rangle\rangle \stackrel{\text{def}}{=} (m_1 \dot{\times} \dots \dot{\times} m_n) \circ \mathcal{J}\langle \text{id}_1, \dots, \text{id}_1 \rangle : 1 \rightarrow X_1 \times \dots \times X_n.$$

Notation 85. Given morphisms $x : 1 \rightarrow X$ and $f : 1 \rightarrow [X \rightarrow Y]$ in $\mathbf{DCPO}_{\mathcal{M}}$, let $f[x] : 1 \rightarrow Y$ be the morphism defined by

$$f[x] \stackrel{\text{def}}{=} \epsilon \circ (f \dot{\times} x) \circ \mathcal{J}\langle \text{id}_1, \text{id}_1 \rangle.$$

Definition 86 (Relation Constructions). We define relation constructors:

- If $\triangleleft_{X_1, A_1}^{e_1} \in \text{ValRel}(X_1, A_1, e_1)$ and $\triangleleft_{X_2, A_2}^{e_2} \in \text{ValRel}(X_2, A_2, e_2)$, define

$$\begin{aligned} & (\triangleleft_{X_1, A_1}^{e_1} + \triangleleft_{X_2, A_2}^{e_2}) \in \text{ValRel}(X_1 + X_2, A_1 + A_2, e_1 + e_2) \text{ by:} \\ & \mathcal{J}in_i \circ v (\triangleleft_{X_1, A_1}^{e_1} + \triangleleft_{X_2, A_2}^{e_2}) \mathbf{in}_i V \text{ iff } v \triangleleft_{X_i, A_i}^{e_i} V \quad (\text{for } i \in \{1, 2\}). \end{aligned}$$

- If $\triangleleft_{X_1, A_1}^{e_1} \in \text{ValRel}(X_1, A_1, e_1)$ and $\triangleleft_{X_2, A_2}^{e_2} \in \text{ValRel}(X_2, A_2, e_2)$, define

$$\begin{aligned} & (\triangleleft_{X_1, A_1}^{e_1} \times \triangleleft_{X_2, A_2}^{e_2}) \in \text{ValRel}(X_1 \times X_2, A_1 \times A_2, e_1 \dot{\times} e_2) \text{ by:} \\ & \langle\langle v_1, v_2 \rangle\rangle (\triangleleft_{X_1, A_1}^{e_1} \times \triangleleft_{X_2, A_2}^{e_2}) (V_1, V_2) \text{ iff } v_1 \triangleleft_{X_1, A_1}^{e_1} V_1 \text{ and } v_2 \triangleleft_{X_2, A_2}^{e_2} V_2. \end{aligned}$$

- If $\triangleleft_{X_1, A_1}^{e_1} \in \text{ValRel}(X_1, A_1, e_1)$ and $\triangleleft_{X_2, A_2}^{e_2} \in \text{ValRel}(X_2, A_2, e_2)$, define

$$\begin{aligned} & (\triangleleft_{X_1, A_1}^{e_1} \rightarrow \triangleleft_{X_2, A_2}^{e_2}) \in \text{ValRel}([X_1 \rightarrow X_2], A_1 \rightarrow A_2, \mathcal{J}[e_1^p \rightarrow e_2]) \text{ by:} \\ & f (\triangleleft_{X_1, A_1}^{e_1} \rightarrow \triangleleft_{X_2, A_2}^{e_2}) \lambda x. M \text{ iff } \mathcal{J}[e_1^p \rightarrow e_2] \circ f \leq \llbracket \lambda x. M \rrbracket \text{ and } \forall (v \triangleleft_{X_1, A_1}^{e_1} V). f[v] \overline{\triangleleft_{X_2, A_2}^{e_2}} (\lambda x. M)V. \end{aligned}$$

Lemma 87. *The assignments in Definition 86 are indeed well-defined.*

Proof. Straightforward verification. \square

Next, a simple lemma that we use later.

Lemma 88. *Assume we are given morphisms $f : 1 \rightarrow [C \rightarrow D]$, $h : A \rightarrow C$, $g : D \rightarrow B$ and $v : 1 \rightarrow A$. Then*

$$(\mathcal{J}[h \rightarrow g] \circ f)[v] = g \circ f[h \circ v].$$

Proof.

$$\begin{aligned} & (\mathcal{J}[h \rightarrow g] \circ f)[v] \\ &= \{ \text{Definition} \} \\ & \quad \epsilon \circ ((\mathcal{J}[h \rightarrow g] \circ f) \dot{\times} v) \circ \mathcal{J}\langle \text{id}, \text{id} \rangle \\ &= \epsilon \circ (\mathcal{J}[h \rightarrow g] \dot{\times} \mathbf{id}) \circ (f \dot{\times} v) \circ \mathcal{J}\langle \text{id}, \text{id} \rangle \\ &= \epsilon \circ (\mathcal{J}[\mathbf{id} \rightarrow g] \dot{\times} \mathbf{id}) \circ (\mathcal{J}[h \rightarrow \mathbf{id}] \dot{\times} \mathbf{id}) \circ (f \dot{\times} v) \circ \mathcal{J}\langle \text{id}, \text{id} \rangle \\ &= \{ \text{Naturality of } \epsilon \} \\ & \quad g \circ \epsilon \circ (\mathcal{J}[h \rightarrow \mathbf{id}] \dot{\times} \mathbf{id}) \circ (f \dot{\times} v) \circ \mathcal{J}\langle \text{id}, \text{id} \rangle \\ &= \{ \text{Parameterised adjunction [45, pp.102]} \} \\ & \quad g \circ \epsilon \circ (\mathbf{id} \dot{\times} h) \circ (f \dot{\times} v) \circ \mathcal{J}\langle \text{id}, \text{id} \rangle \\ &= \{ \text{Definition} \} \\ & \quad g \circ f[h \circ v] \end{aligned}$$

\square

Notation 89. Throughout the rest of the paper we shall write $(- \rightarrow_e -) \stackrel{\text{def}}{=} [- \rightarrow -]_e^{\mathcal{J}} : \mathbf{PD}_e \times \mathbf{PD}_e \rightarrow \mathbf{PD}_e$. That is, we just introduce a more concise notation for the functor $[- \rightarrow -]_e^{\mathcal{J}}$ from Proposition 53.

The next definition is crucial. Given two logical relations, it is used to define the product, coproduct and function space logical relations. Moreover, this is done in a functorial sense on the categories $\mathbf{R}(A)$.

Definition 90. Let A and B be types. We define *covariant* functors in the following way (recall Definition 86):

1. $\times^{A,B}: \mathbf{R}(A) \times \mathbf{R}(B) \rightarrow \mathbf{R}(A \times B)$ by

$$(X, e_X, \triangleleft_X) \times^{A,B} (Y, e_Y, \triangleleft_Y) \stackrel{\text{def}}{=} (X \times Y, e_X \dot{\times}_e e_Y, \triangleleft_X \times \triangleleft_Y)$$

$$f \times^{A,B} g \stackrel{\text{def}}{=} f \dot{\times}_e g$$

2. $+^{A,B}: \mathbf{R}(A) \times \mathbf{R}(B) \rightarrow \mathbf{R}(A + B)$ by

$$(X, e_X, \triangleleft_X) +^{A,B} (Y, e_Y, \triangleleft_Y) \stackrel{\text{def}}{=} (X + Y, e_X \dot{+}_e e_Y, \triangleleft_X + \triangleleft_Y)$$

$$f +^{A,B} g \stackrel{\text{def}}{=} f \dot{+}_e g$$

3. $\rightarrow^{A,B}: \mathbf{R}(A) \times \mathbf{R}(B) \rightarrow \mathbf{R}(A \rightarrow B)$ by

$$(X, e_X, \triangleleft_X) \rightarrow^{A,B} (Y, e_Y, \triangleleft_Y) \stackrel{\text{def}}{=} ([X \rightarrow Y], e_X \rightarrow_e e_Y, \triangleleft_X \rightarrow \triangleleft_Y)$$

$$f \rightarrow^{A,B} g \stackrel{\text{def}}{=} f \rightarrow_e g$$

Proposition 91. *Each of the functors from Definition 90 is well-defined.*

Proof. We will show the case for function types which is the most complicated. The other cases follow by a straightforward verification using similar arguments.

Function types. Let

$$f_1 : (X_1, e_1^X, \triangleleft_1^X) \rightarrow (Y_1, e_1^Y, \triangleleft_1^Y)$$

$$f_2 : (X_2, e_2^X, \triangleleft_2^X) \rightarrow (Y_2, e_2^Y, \triangleleft_2^Y)$$

We have to show

$$f_1 \rightarrow_e f_2 : (X_1 \rightarrow_e X_2, e_1^X \rightarrow_e e_2^X, \triangleleft_1^X \rightarrow \triangleleft_2^X) \rightarrow (Y_1 \rightarrow_e Y_2, e_1^Y \rightarrow_e e_2^Y, \triangleleft_1^Y \rightarrow \triangleleft_2^Y)$$

is a morphism in $\mathbf{R}(A \rightarrow B)$.

First, we show that $f_1 \rightarrow_e f_2$ respects the embedding component. Indeed:

$$e_1^X \rightarrow_e e_2^X = (e_1^Y \circ f_1) \rightarrow_e (e_2^Y \circ f_2) = (e_1^Y \rightarrow_e e_2^Y) \circ (f_1 \rightarrow_e f_2).$$

Next, assume that $v \langle \triangleleft_1^X \rightarrow \triangleleft_2^X \rangle V$. Assume further that $v' \triangleleft_1^Y V'$. Then, clearly $f_1^p \circ v' \overline{\triangleleft_1^X} V'$. If $f_1^p \circ v' = 0$, then it trivially follows that $v[f_1^p \circ v'] = 0 \overline{\triangleleft_2^X} VV'$. Otherwise, $f_1^p \circ v' \in \mathbf{TD}$ and so $f_1^p \circ v' \triangleleft_1^X V'$ and therefore $v[f_1^p \circ v'] \overline{\triangleleft_2^X} VV'$. In all cases, $v[f_1^p \circ v'] \overline{\triangleleft_2^X} VV'$ and therefore $f_2 \circ v[f_1^p \circ v'] \overline{\triangleleft_2^Y} VV'$. But then, by Lemma 88 we have:

$$f_2 \circ v[f_1^p \circ v'] = (\mathcal{J}[f_1^p \rightarrow f_2] \circ v)[v'] = ((f_1 \rightarrow_e f_2) \circ v)[v'] \overline{\triangleleft_2^Y} VV'.$$

Furthemore

$$(e_1^Y \rightarrow_e e_2^Y) \circ (f_1 \rightarrow_e f_2) \circ v = (e_1^X \rightarrow_e e_2^X) \circ v \leq \llbracket V \rrbracket$$

and therefore by definition $(f_1 \rightarrow_e f_2) \circ v \langle \triangleleft_1^Y \rightarrow \triangleleft_2^Y \rangle V$ and therefore also $(f_1 \rightarrow_e f_2) \circ v \overline{\langle \triangleleft_1^Y \rightarrow \triangleleft_2^Y \rangle} V$, as required.

For the other direction, assume that $v \langle \triangleleft_1^Y \rightarrow \triangleleft_2^Y \rangle V$. Assume further that $v' \triangleleft_1^X V'$. Then, clearly $f_1 \circ v' \triangleleft_1^Y V'$. If $f_1 \circ v' = 0$, then it trivially follows that $v[f_1 \circ v'] = 0 \overline{\triangleleft_2^Y} VV'$. Otherwise, $f_1 \circ v' \in \mathbf{TD}$ and so $f_1 \circ v' \triangleleft_1^Y V'$ and therefore $v[f_1 \circ v'] \overline{\triangleleft_2^Y} VV'$. In all cases, $v[f_1 \circ v'] \overline{\triangleleft_2^Y} VV'$ and therefore $f_2^p \circ v[f_1 \circ v'] \overline{\triangleleft_2^X} VV'$. But then, by Lemma 88 we have:

$$f_2^p \circ v[f_1 \circ v'] = (\mathcal{J}[f_1 \rightarrow f_2^p] \circ v)[v'] = ((f_1 \rightarrow_e f_2)^p \circ v)[v'] \overline{\triangleleft_2^X} VV'.$$

Furthemore

$$\begin{aligned} (e_1^X \rightarrow_e e_2^X) \circ (f_1 \rightarrow_e f_2)^p \circ v &= \mathcal{J}[(e_1^X)^p \rightarrow e_2^X] \circ \mathcal{J}[f_1 \rightarrow f_2^p] \circ v \\ &= \mathcal{J}[(f_1 \circ (e_1^X)^p) \rightarrow (e_2^X \circ f_2^p)] \circ v \\ &\leq \mathcal{J}[(f_1 \circ (e_1^X)^p) \rightarrow e_2^Y] \circ v \\ &\leq \mathcal{J}[(e_1^Y)^p \rightarrow e_2^Y] \circ v \\ &\leq \llbracket V \rrbracket. \end{aligned}$$

If $(f_1 \rightarrow_e f_2)^p \circ v \in \mathbf{TD}$, then $(f_1 \rightarrow_e f_2)^p \circ v \langle \triangleleft_1^X \rightarrow \triangleleft_2^X \rangle V$ by definition. Otherwise, $(f_1 \rightarrow_e f_2)^p \circ v = 0$ and then trivially $(f_1 \rightarrow_e f_2)^p \circ v = 0 \overline{\langle \triangleleft_1^X \rightarrow \triangleleft_2^X \rangle} V$. Therefore, in all cases $(f_1 \rightarrow_e f_2)^p \circ v \overline{\langle \triangleleft_1^X \rightarrow \triangleleft_2^X \rangle} V$, as required.

Therefore, the functor $\rightarrow^{A,B}$ is indeed well-defined. \square

Observe that Definition 90 lifts the functors that we use to interpret our types in the category $\mathbf{DCPO}_{\mathcal{M}}$ to the categories $\mathbf{R}(A)$. Next, we show that the functors we just defined are also suitable for forming (parameterised) initial algebras.

Proposition 92. *For $\star \in \{\times, +, \rightarrow\}$, for all types A and B , the functor $\star^{A,B} : \mathbf{R}(A) \times \mathbf{R}(B) \rightarrow \mathbf{R}(A \star B)$ is ω -cocontinuous and the following diagram:*

$$\begin{array}{ccc} \mathbf{R}(A) \times \mathbf{R}(B) & \xrightarrow{\star^{A,B}} & \mathbf{R}(A \star B) \\ \downarrow U^A \times U^B & & \downarrow U^{A \star B} \\ \mathbf{PD}_e \times \mathbf{PD}_e & \xrightarrow{\star_e} & \mathbf{PD}_e \end{array}$$

commutes.

Proof. Commutativity of the diagram is immediate from the definitions. To see ω -cocontinuity, let D be an ω -diagram in $\mathbf{R}(A) \times \mathbf{R}(B)$ and let τ be its colimiting cocone. Because the functors U^A, U^B and \star_e are ω -cocontinuous, it

follows that :

$$\begin{aligned}
& (\star_e \circ U^A \times U^B)_\tau \text{ is colimiting in } \mathbf{PD}_e \\
\implies (U^{A \star B} \circ \star^{A,B})_\tau \text{ is colimiting in } \mathbf{PD}_e & \quad (\text{Commutativity of the above diagram}) \\
\implies \star^{A,B}_\tau \text{ is colimiting in } \mathbf{R}(A \star B) & \quad (U \text{ reflects } \omega\text{-colimits})
\end{aligned}$$

which shows that $\star^{A,B}$ is ω -cocontinuous. \square

Next, we establish an isomorphism between the categories $\mathbf{R}(\mu X.A)$ and $\mathbf{R}(A[\mu X.A/X])$.

Definition 93. We define constructors for folding and unfolding logical relations as follows:

- If $\triangleleft_{X,A[\mu Y.A/Y]}^e \in \text{ValRel}(X, A[\mu Y.A/Y], e)$, define

$$\begin{aligned}
& (\mathbb{I}^{\mu Y.A} \triangleleft_{X,A[\mu Y.A/Y]}^e) \in \text{ValRel}(X, \mu Y.A, \text{fold} \circ e) \text{ by:} \\
v (\mathbb{I}^{\mu Y.A} \triangleleft_{X,A[\mu Y.A/Y]}^e) \text{ fold } V & \text{ iff } v \triangleleft_{X,A[\mu Y.A/Y]}^e V.
\end{aligned}$$

- If $\triangleleft_{X,\mu Y.A}^e \in \text{ValRel}(X, \mu Y.A, e)$, define

$$\begin{aligned}
& (\mathbb{E}^{\mu Y.A} \triangleleft_{X,\mu Y.A}^e) \in \text{ValRel}(X, A[\mu Y.A/Y], \text{unfold} \circ e) \text{ by:} \\
v (\mathbb{E}^{\mu Y.A} \triangleleft_{X,\mu Y.A}^e) V & \text{ iff } v \triangleleft_{X,\mu Y.A}^e \text{ fold } V.
\end{aligned}$$

Proposition 94. *The above assignments are indeed well-defined.*

Proof. Straightforward verification. \square

Proposition 95. *For every type $\cdot \vdash \mu X.A$, we have an isomorphism of categories*

$$\mathbb{I}^{\mu X.A} : \mathbf{R}(A[\mu X.A/X]) \cong \mathbf{R}(\mu X.A) : \mathbb{E}^{\mu X.A},$$

where the functors are defined by

$$\begin{aligned}
\mathbb{I}^{\mu X.A} : \mathbf{R}(A[\mu X.A/X]) & \rightarrow \mathbf{R}(\mu X.A) & \mathbb{E}^{\mu X.A} : \mathbf{R}(\mu X.A) & \rightarrow \mathbf{R}(A[\mu X.A/X]) \\
\mathbb{I}^{\mu X.A}(Y, e, \triangleleft) & = (Y, \text{fold} \circ e, \mathbb{I}^{\mu X.A} \triangleleft) & \mathbb{E}^{\mu X.A}(Y, e, \triangleleft) & = (Y, \text{unfold} \circ e, \mathbb{E}^{\mu X.A} \triangleleft) \\
\mathbb{I}^{\mu X.A}(f) & = f & \mathbb{E}^{\mu X.A}(f) & = f
\end{aligned}$$

Proof. The proof is essentially the same as [50, Lemma 7.23], with one extra proof obligation, namely we have to show that our functorial assignments respect the embedding components. But this is obviously true. \square

This finishes the categorical development of the categories $\mathbf{R}(A)$.

6.4. Augmented Interpretation of Types

We have now established sufficient categorical structure in order to construct parameterised initial algebras in the categories $\mathbf{R}(A)$. Furthermore, we have sufficient structure to also define an *augmented* interpretation of types in these categories. The main idea behind providing the augmented interpretation is to show how to pick out the logical relations we need from all those that exist in the categories $\mathbf{R}(A)$.

Notation 96. Given any type context $\Theta = X_1, \dots, X_n$ and closed types $\cdot \vdash C_i$ with $i \in \{1, \dots, n\}$, we shall write \vec{C} for C_1, \dots, C_n and we also write $[\vec{C}/\Theta]$ for $[C_1/X_1, \dots, C_n/X_n]$.

Definition 97. For any type $\Theta \vdash A$ and closed types \vec{C} , we define their *augmented interpretation* to be the functor

$$\|\Theta \vdash A\|^{\vec{C}} : \mathbf{R}(C_1) \times \dots \times \mathbf{R}(C_n) \rightarrow \mathbf{R}(A[\vec{C}/\Theta])$$

defined by induction on the derivation of $\Theta \vdash A$:

$$\begin{aligned} \|\Theta \vdash \Theta_i\|^{\vec{C}} &:= \Pi_i \\ \|\Theta \vdash A \star B\|^{\vec{C}} &:= \star^{A[\vec{C}/\Theta], B[\vec{C}/\Theta]} \circ \langle \|\Theta \vdash A\|^{\vec{C}}, \|\Theta \vdash B\|^{\vec{C}} \rangle \quad (\text{for } \star \in \{+, \times, \rightarrow\}) \\ \|\Theta \vdash \mu X.A\|^{\vec{C}} &:= \left(\mathbb{I}^{\mu X.A[\vec{C}/\Theta]} \circ \|\Theta, X \vdash A\|^{\vec{C}, \mu X.A[\vec{C}/\Theta]} \right)^\sharp, \end{aligned}$$

where the $(-)^{\sharp}$ operation is from Definition 48.

Proposition 98. *Each functor $\|\Theta \vdash A\|^{\vec{C}}$ is well-defined and ω -cocontinuous. Moreover, the following diagram:*

$$\begin{array}{ccc} \mathbf{R}(C_1) \times \dots \times \mathbf{R}(C_n) & \xrightarrow{\|\Theta \vdash A\|^{\vec{C}}} & \mathbf{R}(A[\vec{C}/\Theta]) \\ \downarrow U^{C_1} \times \dots \times U^{C_n} & & \downarrow U^{A[\vec{C}/\Theta]} \\ \mathbf{PD}_e \times \dots \times \mathbf{PD}_e & \xrightarrow{\llbracket \Theta \vdash A \rrbracket} & \mathbf{PD}_e \end{array}$$

commutes.

Proof. The proof is essentially the same as [50, Proposition 7.26]. \square

Next, a corollary which shows that parameterised initial algebras for our type expressions are constructed in the same way in both categories.

Corollary 99. *The 2-categorical diagram:*

$$\begin{array}{ccc}
& \mathbb{I}^{\mu X.A[\vec{C}/\Theta]} \circ \|\Theta, X \vdash A\|^{\vec{C}, \mu X.A[\vec{C}/\Theta]} \circ \langle \text{Id}, \|\Theta \vdash \mu X.A\|^{\vec{C}} \rangle & \\
& \curvearrowright & \\
\mathbf{R}(C_1) \times \dots \times \mathbf{R}(C_n) & \xrightarrow{\quad} & \mathbf{R}(A[\vec{C}/\Theta]) \\
& \Downarrow \iota & \\
& \|\Theta \vdash \mu X.A\|^{\vec{C}} & \\
& \curvearrowleft & \\
& \mathbb{I}^{\Theta, X \vdash A} \circ \langle \text{Id}, \mathbb{I}^{\Theta \vdash \mu X.A} \rangle & \\
U^{C_1} \times \dots \times U^{C_n} \downarrow & & \downarrow U^{A[\vec{C}/\Theta]} \\
\mathbf{PD}_e \times \dots \times \mathbf{PD}_e & \xrightarrow{\quad} & \mathbf{PD}_e \\
& \Downarrow \iota & \\
& \mathbb{I}^{\Theta \vdash \mu X.A} & \\
& \curvearrowleft &
\end{array}$$

commutes, where ι is the parameterised initial algebra isomorphism (see Definition 48).

Proof. The proof is the same as [50, Corollary 7.27]. \square

Proposition 98 shows that the first component of the augmented interpretation coincides with the standard interpretation. This is true for all types, including open ones. In the special case for closed types, let $\|A\| \stackrel{\text{def}}{=} \|\cdot \vdash A\|(\ast)$, where \ast is the unique object of the terminal category $\mathbf{1} = \mathbf{R}(A)^0$. Proposition 98 therefore shows that $U\|A\| = \llbracket A \rrbracket$, which means that $\|A\|$ has the form $\|A\| = (\llbracket A \rrbracket, e, \triangleleft)$, where $e : \llbracket A \rrbracket \rightarrow \llbracket A \rrbracket$ is some embedding. Next, we show that $e = \text{id}$. In order to do this, we prove a stronger proposition first. We show that the action of the functor $\|\Theta \vdash A\|^{\vec{C}}$ on the embedding component is also completely determined by the action of $\llbracket \Theta \vdash A \rrbracket$ on embeddings.

Proposition 100. *For every functor $\|\Theta \vdash A\|^{\vec{C}}$ and objects $(X_i, e_i, \triangleleft_i)$ with $i \in \{1, \dots, n\}$, we have:*

$$\pi_e \left(\|\Theta \vdash A\|^{\vec{C}} \left((X_1, e_1, \triangleleft_1), \dots, (X_n, e_n, \triangleleft_n) \right) \right) = \llbracket \Theta \vdash A \rrbracket (e_1, \dots, e_n),$$

where for an object $(Z, e_Z, \triangleleft_Z)$ in any category $\mathbf{R}(B)$, we define $\pi_e(Z, e_Z, \triangleleft_Z) \stackrel{\text{def}}{=} e_Z$.

Proof. By induction on the derivation of $\Theta \vdash A$.

Case Θ_i . This is obviously true.

Case $A = A_1 \star A_2$, for $\star \in \{\times, +, \rightarrow\}$. The statement follows easily by induction and the fact that for every pair of objects $(Y, e_Y, \triangleleft_Y)$ and $(Z, e_Z, \triangleleft_Z)$ we have

$$\pi_e \left((Y, e_Y, \triangleleft_Y) \star^{A_1, A_2} (Z, e_Z, \triangleleft_Z) \right) = e_Y \star_e e_Z$$

which follows by definition of the relevant functors.

Case $\mu X.A$. First we introduce some abbreviations to simplify notation. We define:

- $T \stackrel{\text{def}}{=} \|\Theta, X \vdash A\|_{\vec{C}, \mu X.A[\vec{C}/\Theta]}$.
- $H \stackrel{\text{def}}{=} \llbracket \Theta, X \vdash A \rrbracket$.
- $\mathbb{I} \stackrel{\text{def}}{=} \llbracket \mu X.A[\vec{C}/\Theta] \rrbracket$.
- $\overrightarrow{(X, e, \triangleleft)} \stackrel{\text{def}}{=} ((X_1, e_1, \triangleleft_1), \dots, (X_n, e_n, \triangleleft_n))$.
- $\vec{X} \stackrel{\text{def}}{=} (X_1, \dots, X_n)$.
- $\vec{e} \stackrel{\text{def}}{=} (e_1, \dots, e_n)$.

Now, let $(Y, e_Y, \triangleleft_Y) \stackrel{\text{def}}{=} (\mathbb{I} \circ T)^\# \overrightarrow{(X, e, \triangleleft)}$. To finish the proof, we have to show that $H^\#(\vec{e}) = e_Y$. From Proposition 98 we know that $Y = H^\#(\vec{X})$. From Corollary 99, we have a parameterised initial algebra isomorphism

$$\iota: \mathbb{I}T \left(\overrightarrow{(X, e, \triangleleft)}, (H^\#\vec{X}, e_Y, \triangleleft_Y) \right) \rightarrow (H^\#\vec{X}, e_Y, \triangleleft_Y) \quad (9)$$

which is also a parameterised initial algebra isomorphism

$$\iota: H \left(\vec{X}, H^\#\vec{X} \right) \rightarrow H^\#\vec{X} \quad (10)$$

in \mathbf{PD}_e . By the induction hypothesis for T and H and Proposition 98, we get

$$T \left(\overrightarrow{(X, e, \triangleleft)}, (H^\#\vec{X}, e_Y, \triangleleft_Y) \right) = \left(H(\vec{X}, H^\#\vec{X}), H(\vec{e}, e_Y), \blacktriangleleft \right),$$

where \blacktriangleleft is some (unimportant) logical relation. Therefore by (9) and definition of \mathbb{I} , we get that

$$\iota: \left(H(\vec{X}, H^\#\vec{X}), \text{fold} \circ H(\vec{e}, e_Y), \mathbb{I} \blacktriangleleft \right) \rightarrow (H^\#\vec{X}, e_Y, \triangleleft_Y) \quad (11)$$

is an isomorphism with the indicated type. This means that in the category \mathbf{PD}_e , we have:

$$\text{fold} \circ H(\vec{e}, e_Y) = e_Y \circ \iota \quad (12)$$

where we already know that $\iota = \iota_{X_1, \dots, X_n}$ is the parameterised initial algebra in \mathbf{PD}_e of H . But, by definition, so is fold and in fact $\text{fold} = \iota_{[C_1], \dots, [C_n]}$. However, $H^\#\vec{e}$ is the unique morphism, such that

$$\iota_{[C_1], \dots, [C_n]} \circ H(\vec{e}, H^\#\vec{e}) = H^\#\vec{e} \circ \iota_{X_1, \dots, X_n}$$

which is the universal property of a parameterised initial algebra (see [50, Remark 4.6]) and therefore by equation (12) it follows that $e_Y = H^\#\vec{e}$, as required. \square

Corollary 101. *For every closed type A , we have $\|A\| = (\llbracket A \rrbracket, \mathbf{id}_{\llbracket A \rrbracket}, \triangleleft_A)$ for some logical relation \triangleleft_A .*

Proof. We already know that the first component is $\llbracket A \rrbracket$. For the second component, the previous proposition shows that $\pi_e \|A\| = \pi_e \|\cdot \vdash A\| \cdot (*) = \llbracket \cdot \vdash A \rrbracket (\mathbf{id}_*) = \mathbf{id}_{\llbracket A \rrbracket}$, where $*$ denotes the empty tuple of objects and \mathbf{id}_* the empty tuple of embeddings. \square

Finally, we want to show that the third component of $\|A\|$ is the logical relation that we need to carry out the adequacy proof. For this, we have to prove a substitution lemma first.

Lemma 102 (Substitution). *For any types $\Theta, X \vdash A$ and $\Theta \vdash B$ and closed types C_1, \dots, C_n , we have:*

$$\|\Theta \vdash A[B/X]\|^{C_i} = \|\Theta, X \vdash A\|^{C_i, B[C_i/\Theta]} \circ \langle \text{Id}, \|\Theta \vdash B\|^{C_i} \rangle.$$

Proof. The proof is the same as [50, Lemma 7.30]. \square

For each type A , we have now provided an augmented interpretation $\|A\|$ of A in the category $\mathbf{R}(A)$. The interpretation $\|-\|$ satisfies all the fundamental properties of $\llbracket - \rrbracket$, as we have now shown. It should now be clear that this augmented interpretation is true to its name, because it carries strictly more information compared to the standard interpretation of types. The additional information that $\|A\|$ carries is precisely the logical relation that we need at type A , as we show in the next subsection.

6.5. Existence of the Logical Relations

We can now show that the logical relations we need for the adequacy proof exist.

Theorem 103. *For each closed type A , there exist formal approximation relations:*

$$\begin{aligned} \triangleleft_A &\subseteq \mathbf{TD}(1, \llbracket A \rrbracket) \times \text{Val}(A) \\ \overline{\triangleleft}_A &\subseteq \mathbf{DCPO}_{\mathcal{M}}(1, \llbracket A \rrbracket) \times \text{Prog}(A) \end{aligned}$$

which satisfy the following properties:

- (A1) $\mathcal{J} \text{in}_i \circ v \triangleleft_{A_1 + A_2} \text{in}_i V$ iff $v \triangleleft_{A_i} V$, where $i \in \{1, 2\}$.
 - (A2) $\langle\langle v_1, v_2 \rangle\rangle \triangleleft_{A_1 \times A_2} \langle\langle V_1, V_2 \rangle\rangle$ iff $v_1 \triangleleft_{A_1} V_1$ and $v_2 \triangleleft_{A_2} V_2$.
 - (A3) $f \triangleleft_{A \rightarrow B} \lambda x. M$ iff $f \leq \llbracket \lambda x. M \rrbracket$ and $\forall (v \triangleleft_A V). f[v] \overline{\triangleleft}_B (\lambda x. M)V$.
 - (A4) $v \triangleleft_{\mu X. A} \text{fold } V$ iff $\text{unfold} \circ v \triangleleft_{A[\mu X. A/X]} V$.
- (B) $m \overline{\triangleleft}_A M$ iff $m \in \mathcal{S}(\triangleleft_A; M)$, where $\mathcal{S}(\triangleleft_A; M)$ is the Scott-closure in $\mathbf{DCPO}_{\mathcal{M}}(1, \llbracket A \rrbracket)$ of the set

$$\mathcal{S}_0(\triangleleft_A; M) \stackrel{\text{def}}{=} \left\{ \sum_{\pi \in F} P(\pi) v_\pi \mid F \subseteq \text{TPaths}(M), F \text{ is finite and } v_\pi \triangleleft_A V_\pi \text{ for each } \pi \in F \right\}.$$

- (C1) If $v \triangleleft_A V$, then $v \leq \llbracket V \rrbracket$.
(C2) $(-\triangleleft_A V)$ is a Scott-closed subset of $\mathbf{TD}(1, \llbracket A \rrbracket)$.
(C3) If $m \overline{\triangleleft_A} M$, then $m \leq \llbracket M \rrbracket$.
(C4) $(-\overline{\triangleleft_A} M)$ is a Scott-closed subset of $\mathbf{DCPO}_{\mathcal{M}}(1, \llbracket A \rrbracket)$.
(C5) If $v \in \mathbf{TD}(1, \llbracket A \rrbracket)$ and V is a value, then $v \triangleleft_A V$ iff $v \overline{\triangleleft_A} V$.

Proof. Consider the object $\llbracket A \rrbracket \in \mathbf{R}(A)$. We have already shown that $\llbracket A \rrbracket = (\llbracket A \rrbracket, \mathbf{id}_{\llbracket A \rrbracket}, \triangleleft_A)$ for some logical relation $\triangleleft_A \in \mathbf{ValRel}(\llbracket A \rrbracket, A, \mathbf{id}_{\llbracket A \rrbracket})$. We now show that \triangleleft_A satisfies the required properties. Notice that the embedding components are just identities.

Property (B) is satisfied by construction (Definition 74). Properties (C1) and (C2) are also satisfied by construction (Definition 66). Property (C4) is satisfied by construction and property (B). Property (C3) is satisfied, because if $m \overline{\triangleleft_A} M$, then by Corollary 63 and property (C1) it follows that $\mathcal{S}_0(\triangleleft_A; M) \subseteq \downarrow \llbracket M \rrbracket$. The latter set is Scott-closed and therefore $m \in \mathcal{S}(\triangleleft_A; M) \subseteq \downarrow \llbracket M \rrbracket$, as required. Property (C5) is satisfied by Lemma 77.

Properties (A1), (A2) and (A3) are satisfied, because for $\star \in \{+, \times, \rightarrow\}$, we have that $\triangleleft_{A \star B} = \triangleleft_A \star \triangleleft_B$ and then by Definition 86.

To show that property (A4) is also satisfied, we reason as follows. Consider the isomorphism

$$\mathbf{unfold}_{\mu X.A} : \llbracket \mu X.A \rrbracket \cong \llbracket X \vdash A \rrbracket \llbracket \mu X.A \rrbracket = \llbracket A[\mu X.A/X] \rrbracket : \mathbf{fold}_{\mu X.A}$$

from Definition 57. By Corollary 99 and Lemma 102 (when $\Theta = \cdot$) it follows that this isomorphism lifts to an isomorphism

$$\mathbf{unfold}_{\mu X.A} : \llbracket \mu X.A \rrbracket \cong \mathbb{I}^{\mu X.A} (\llbracket X \vdash A \rrbracket \llbracket \mu X.A \rrbracket) = \mathbb{I}^{\mu X.A} (\llbracket A[\mu X.A/X] \rrbracket) : \mathbf{fold}_{\mu X.A}$$

in the category $\mathbf{R}(\mu X.A)$. Expanding definitions, this means we have an isomorphism

$$\begin{aligned} \mathbf{unfold}_{\mu X.A} : (\llbracket \mu X.A \rrbracket, \mathbf{id}, \triangleleft_{\mu X.A}) &= \llbracket \mu X.A \rrbracket \\ &\cong \mathbb{I}^{\mu X.A} (\llbracket A[\mu X.A/X] \rrbracket) \\ &= (\llbracket A[\mu X.A/X] \rrbracket, \mathbf{fold}_{\mu X.A}, \mathbb{I}^{\mu X.A} \triangleleft_{A[\mu X.A/X]}) : \mathbf{fold}_{\mu X.A} \end{aligned} \tag{13}$$

in the category $\mathbf{R}(\mu X.A)$. The notion of morphism in this category (Definition 79), construction of \mathbb{I} (Definition 93) and property (C5) allow us to conclude that property (A4) is satisfied. Indeed:

$$\begin{aligned} &v \triangleleft_{\mu X.A} \mathbf{fold} V \\ \implies &\mathbf{unfold}_{\mu X.A} \circ v (\mathbb{I}^{\mu X.A} \triangleleft_{A[\mu X.A/X]}) \mathbf{fold} V \\ \implies &\mathbf{unfold}_{\mu X.A} \circ v \triangleleft_{A[\mu X.A/X]} V \end{aligned}$$

and for the other direction of (A4):

$$\begin{aligned}
& \text{unfold}_{\mu.X.A} \circ v \triangleleft_{A[\mu.X.A/X]} V \\
\implies & \text{unfold}_{\mu.X.A} \circ v (\mathbb{I}^{\mu.X.A} \triangleleft_{A[\mu.X.A/X]}) \text{fold } V \\
\implies & v = \text{fold}_{\mu.X.A} \circ \text{unfold}_{\mu.X.A} \circ v \triangleleft_{\mu.X.A} \text{fold } V. \quad \square
\end{aligned}$$

6.6. Closure Properties of the Logical Relations

Here we establish some important closure properties of the relations $\overline{\triangleleft}_A$ from Theorem 103.

Lemma 104. *Let $\cdot \vdash M : A$ be a term and let F be some finite index set. Assume that we are given morphisms m_i and terms M_i such that $m_i \overline{\triangleleft}_A M_i$ for $i \in F$. Assume further that for each $i \in F$, we are given a reduction path $\pi_i \in \text{Paths}(M, M_i)$, such that all paths π_i are distinct. Then*

$$\sum_{i \in F} P(\pi_i) m_i \overline{\triangleleft}_A M.$$

Proof. By assumption, for every $i \in F$, we know that $m_i \in \mathcal{S}(\triangleleft_A; M_i)$. Next, consider the function

$$g \stackrel{\text{def}}{=} \sum_{i \in F} P(\pi_i)(-): \prod_{|F|} \mathbf{DCPO}_{\mathcal{M}}(1, \llbracket A \rrbracket) \rightarrow \mathbf{DCPO}_{\mathcal{M}}(1, \llbracket A \rrbracket).$$

This function is Scott continuous and therefore by Lemma 72, it suffices to show that $g(\prod_i s_i) \in \mathcal{S}(\triangleleft_A; M)$ for any choice of $s_i \in \mathcal{S}_0(\triangleleft_A; M_i)$. Next, for every $i \in F$, let

$$s_i = \left(\sum_{\pi \in F_i} P(\pi) v_\pi \right) \in \mathcal{S}_0(\triangleleft_A; M_i)$$

where $F_i \subseteq \text{TPaths}(M_i)$ is a finite subset and such that $v_\pi \triangleleft_A V_\pi$, for each $\pi \in F_i$. Then, we have

$$\begin{aligned}
g \left(\prod_i s_i \right) &= \sum_{i \in F} P(\pi_i) \left(\sum_{\pi \in F_i} P(\pi) v_\pi \right) \\
&= \sum_{i \in F} \sum_{\pi \in F_i} (P(\pi_i) \cdot P(\pi)) v_\pi \\
&= \sum_{i \in F} \sum_{\pi \in F_i} P(\pi_i \pi) v_\pi \\
&\in \mathcal{S}_0(\triangleleft_A; M),
\end{aligned}$$

where $\pi_i \pi \in \text{Paths}(M, V_\pi)$ is the path constructed by concatenating the path π_i to π . \square

Lemma 105. *If $m \overline{\triangleleft}_A M$ and $n \overline{\triangleleft}_A N$, then $p \cdot m + (1 - p) \cdot n \overline{\triangleleft}_A M \text{ or }_p N$.*

Proof. This is just a special case of Lemma 104. \square

Lemma 106. For $i \in \{1, 2\}$: if $m \overline{\triangleleft_{A_i}} M$, then $\mathcal{J}in_i \circ m \overline{\triangleleft_{A_1+A_2}} \mathbf{in}_i M$.

Proof. Assume, without loss of generality, that $i = 1$. By definition we know that $m \in \mathcal{S}(\triangleleft_{A_1}; M) = \overline{\mathcal{S}_0(\triangleleft_{A_1}; M)}$. By Lemma 72, it suffices to show

$$\mathcal{J}in_1 \circ \sum_{\pi \in F} P(\pi)v_\pi \in \mathcal{S}(\triangleleft_{A_1+A_2}; \mathbf{in}_1 M)$$

for any $\sum_{\pi \in F} P(\pi)v_\pi \in \mathcal{S}_0(\triangleleft_{A_1}; M)$. Since $(\mathcal{J}in_1 \circ -)$ is linear, we see

$$\mathcal{J}in_1 \circ \sum_{\pi \in F} P(\pi)v_\pi = \sum_{\pi \in F} P(\pi)(\mathcal{J}in_1 \circ v_\pi) = \sum_{\pi \in F} P(\mathbf{in}_1(\pi))(\mathcal{J}in_1 \circ v_\pi) \in \mathcal{S}(\triangleleft_{A_1+A_2}; \mathbf{in}_1 M),$$

where $\mathbf{in}_1(\pi) \in \text{Paths}(\mathbf{in}_1 M, \mathbf{in}_1 V_\pi)$ is the path constructed by reducing $\mathbf{in}_1 M$ to $\mathbf{in}_1 V_\pi$, as specified by π . The membership relation is satisfied because by assumption $v_\pi \triangleleft_{A_1} V_\pi$ and then by Theorem 103 (A1). \square

Lemma 107. Let $m \overline{\triangleleft_{A_1+A_2}} M$. Next, assume that for $k \in \{1, 2\}$ we have terms $x_k : A_k \vdash N_k : B$ and morphisms $n_k : \llbracket A_k \rrbracket \rightarrow \llbracket B \rrbracket$, such that for every $v_k \triangleleft_{A_k} V_k$, it is the case that $n_k \circ v_k \overline{\triangleleft_B} N_k[V_k/x_k]$. Then

$$[n_1, n_2] \circ m \overline{\triangleleft_B} \text{case } M \text{ of } \mathbf{in}_1 x_1 \Rightarrow N_1 \mid \mathbf{in}_2 x_2 \Rightarrow N_2.$$

Proof. For brevity, let C be the term $C \stackrel{\text{def}}{=} (\text{case } M \text{ of } \mathbf{in}_1 x_1 \Rightarrow N_1 \mid \mathbf{in}_2 x_2 \Rightarrow N_2)$. Next, consider the function

$$([n_1, n_2] \circ -) : \mathbf{DCPO}_{\mathcal{M}}(1, \llbracket A_1 + A_2 \rrbracket) \rightarrow \mathbf{DCPO}_{\mathcal{M}}(1, \llbracket B \rrbracket).$$

This function is Scott continuous. By Lemma 72, to complete the proof it suffices to show that $[n_1, n_2] \circ m' \overline{\triangleleft_B} C$ for any $m' \in \mathcal{S}_0(\triangleleft_{A_1+A_2}; M)$. Towards that end, let

$$m' = \sum_{\pi \in F} P(\pi)v_\pi,$$

where F is finite and where $v_\pi \triangleleft_{A_1+A_2} V_\pi$, for each $\pi \in F$. Let $F_1 \subseteq F$ be the set of paths π such that $V_\pi = \mathbf{in}_1 V'_\pi$ for some V'_π and let $F_2 = F - F_1$. Then by Theorem 103 (A1), for each $\pi \in F_1$, it follows that $V_\pi = \mathbf{in}_1 V'_\pi$ and $v_\pi = \mathcal{J}in_1 \circ v'_\pi$ and $v'_\pi \triangleleft_{A_1} V'_\pi$. Similarly, for each $\pi \in F_2$, it follows that $V_\pi = \mathbf{in}_2 V'_\pi$ and $v_\pi = \mathcal{J}in_2 \circ v'_\pi$ and $v'_\pi \triangleleft_{A_2} V'_\pi$. Therefore, we get:

$$\begin{aligned} [n_1, n_2] \circ m' &= [n_1, n_2] \circ \left(\left(\sum_{\pi \in F_1} P(\pi)(\mathcal{J}in_1 \circ v'_\pi) \right) + \left(\sum_{\pi \in F_2} P(\pi)(\mathcal{J}in_2 \circ v'_\pi) \right) \right) \\ &= \left(\sum_{\pi \in F_1} P(\pi)(n_1 \circ v'_\pi) \right) + \left(\sum_{\pi \in F_2} P(\pi)(n_2 \circ v'_\pi) \right) \end{aligned}$$

In the above sums, by assumption, we know that $n_1 \circ v'_\pi \overline{\triangleleft}_B N_1[V'_\pi/x_1]$, for each $\pi \in F_1$ and similarly $n_2 \circ v'_\pi \overline{\triangleleft}_B N_2[V'_\pi/x_2]$, for each $\pi \in F_2$. Next, consider the function

$$\left(\left(\sum_{\pi \in F_1} P(\pi)(-) \right) + \left(\sum_{\pi \in F_2} P(\pi)(-) \right) \right) : \mathbf{DCPO}_{\mathcal{M}}(1, \llbracket B \rrbracket)^{|F_1|} \times \mathbf{DCPO}_{\mathcal{M}}(1, \llbracket B \rrbracket)^{|F_2|} \rightarrow \mathbf{DCPO}_{\mathcal{M}}(1, \llbracket B \rrbracket).$$

This function is Scott-continuous and by Lemma 72, to complete the proof it suffices to show that

$$\left(\sum_{\pi \in F_1} P(\pi)(n_1^\pi) \right) + \left(\sum_{\pi \in F_2} P(\pi)(n_2^\pi) \right) \overline{\triangleleft}_B C,$$

where $n_1^\pi \in \mathcal{S}_0(\triangleleft_B; N_1[V'_\pi/x_1])$ for $\pi \in F_1$ and $n_2^\pi \in \mathcal{S}_0(\triangleleft_B; N_2[V'_\pi/x_2])$ for $\pi \in F_2$ are taken to be arbitrary. Towards this end, let

$$\begin{aligned} n_1^\pi &= \sum_{\pi' \in F_1^\pi} P(\pi')v_{\pi'} \in \mathcal{S}_0(\triangleleft_B; N_1[V'_\pi/x_1]) \\ n_2^\pi &= \sum_{\pi' \in F_2^\pi} P(\pi')v_{\pi'} \in \mathcal{S}_0(\triangleleft_B; N_2[V'_\pi/x_2]) \end{aligned}$$

where F_k^π is finite and where $v_{\pi'} \triangleleft_B V_{\pi'}$, for every $\pi' \in F_k^\pi$ and where $k \in \{1, 2\}$. Then, we get

$$\begin{aligned} & \left(\sum_{\pi \in F_1} P(\pi)(n_1^\pi) \right) + \left(\sum_{\pi \in F_2} P(\pi)(n_2^\pi) \right) = \\ &= \left(\sum_{\pi \in F_1} \sum_{\pi' \in F_1^\pi} P(\pi)P(\pi')v_{\pi'} \right) + \left(\sum_{\pi \in F_2} \sum_{\pi' \in F_2^\pi} P(\pi)P(\pi')v_{\pi'} \right) \\ &= \left(\sum_{\pi \in F_1} \sum_{\pi' \in F_1^\pi} P(\text{case}_1(\pi, \pi'))v_{\pi'} \right) + \left(\sum_{\pi \in F_2} \sum_{\pi' \in F_2^\pi} P(\text{case}_2(\pi, \pi'))v_{\pi'} \right) \\ &\in \mathcal{S}_0(\triangleleft_B; C) \subseteq \mathcal{S}(\triangleleft_B; C), \end{aligned}$$

where $\text{case}_1(\pi, \pi') \in \text{Paths}(C, V_{\pi'})$ is the path obtained by reducing C to $C_\pi \stackrel{\text{def}}{=} (\text{case } \mathbf{in}_1 V'_\pi \text{ of } \mathbf{in}_1 x_1 \Rightarrow N_1 \mid \mathbf{in}_2 x_2 \Rightarrow N_2)$ as specified by π , then performing the beta reduction $C_\pi \xrightarrow{1} N_1[V'_\pi/x_1]$ and then reducing $N_1[V'_\pi/x_1]$ to $V_{\pi'}$ as specified by π' . Similarly for $\text{case}_2(\pi, \pi')$. The last sum is now by definition in $\mathcal{S}_0(\triangleleft_B; C)$. \square

Lemma 108. *If $m_1 \overline{\triangleleft}_{A_1} M_1$ and $m_2 \overline{\triangleleft}_{A_2} M_2$ then $\langle\langle m_1, m_2 \rangle\rangle \overline{\triangleleft}_{A_1 \times A_2} (M_1, M_2)$.*

Proof. The map $\langle\langle -, - \rangle\rangle : \mathbf{DCPO}_{\mathcal{M}}(1, \llbracket A_1 \rrbracket) \times \mathbf{DCPO}_{\mathcal{M}}(1, \llbracket A_2 \rrbracket) \rightarrow \mathbf{DCPO}_{\mathcal{M}}(1, \llbracket A_1 \times A_2 \rrbracket)$ is Scott-continuous in both arguments and therefore by Lemma 72, to complete the proof it suffices to show that $\langle\langle m'_1, m'_2 \rangle\rangle \overline{\triangleleft}_{A_1 \times A_2} (M_1, M_2)$ for any $m'_1 \in \mathcal{S}_0(\triangleleft_{A_1}; M_1)$ and $m'_2 \in \mathcal{S}_0(\triangleleft_{A_2}; M_2)$.

Now, take $m'_1 = \sum_{\pi_1 \in F_1} P(\pi_1)v_{\pi_1} \in \mathcal{S}_0(\triangleleft_{A_1}; M_1)$ and $m'_2 = \sum_{\pi_2 \in F_2} P(\pi_2)v_{\pi_2} \in \mathcal{S}_0(\triangleleft_{A_2}; M_2)$, where F_1 and F_2 are finite sets, and where $v_{\pi_1} \triangleleft_{A_1} V_{\pi_1}$ for each $\pi_1 \in F_1$ and where $v_{\pi_2} \triangleleft_{A_2} V_{\pi_2}$ for each $\pi_2 \in F_2$. We then have:

$$\langle\langle m'_1, m'_2 \rangle\rangle = \langle\langle \sum_{\pi_1 \in F_1} P(\pi_1)v_{\pi_1}, \sum_{\pi_2 \in F_2} P(\pi_2)v_{\pi_2} \rangle\rangle \quad (14)$$

$$= \sum_{\pi_1 \in F_1} \sum_{\pi_2 \in F_2} P(\pi_1)P(\pi_2)\langle\langle v_{\pi_1}, v_{\pi_2} \rangle\rangle \quad (15)$$

$$= \sum_{\pi_1 \in F_1} \sum_{\pi_2 \in F_2} P(\text{pair}(\pi_1, \pi_2))\langle\langle v_{\pi_1}, v_{\pi_2} \rangle\rangle \quad (16)$$

$$\overline{\triangleleft_{A_1 \times A_2}}(M_1, M_2). \quad (17)$$

Equation 14 holds by definition. Equation 15 is true since the function $\langle\langle -, - \rangle\rangle$ defined above is linear in each component by Proposition 39. In Equation 16 $\text{pair}(\pi_1, \pi_2) \in \text{Paths}((M_1, M_2), (V_{\pi_1}, V_{\pi_2}))$ is the path which first reduces (M_1, M_2) to (V_{π_1}, M_2) as specified by π_1 and then reduces (V_{π_1}, M_2) to (V_{π_1}, V_{π_2}) as specified by π_2 and it is easy to see that Equation 16 holds. Finally 17 holds, because $v_{\pi_1} \triangleleft_{A_1} V_{\pi_1}$ and $v_{\pi_2} \triangleleft_{A_2} V_{\pi_2}$ by assumption and then by Theorem 103 (A2) we have that $\langle\langle v_{\pi_1}, v_{\pi_2} \rangle\rangle \triangleleft_{(A_1, A_2)} (V_{\pi_1}, V_{\pi_2})$. \square

Lemma 109. *If $m \in \overline{\triangleleft_{A_1 \times A_2}} M$ then $\mathcal{J}\pi_i \circ m \in \overline{\triangleleft_{A_i}} \pi_i M$, for $i \in \{1, 2\}$.*

Proof. Without loss of generality, we will show the statement for the first projection. In order to avoid notational confusion, we will write pr_1 for π_1 for the projection on the first component in this lemma. We shall use π to range over paths, as in the other lemmas.

Using Lemma 72, to complete the proof it suffices to show that

$$\mathcal{J}\text{pr}_1 \circ m' \in \overline{\triangleleft_{A_1}} \text{pr}_1 M$$

for any $m' \in \mathcal{S}_0(\triangleleft_{A_1 \times A_2}; M)$. Towards this end, let

$$m' = \sum_{\pi \in F} P(\pi)v_{\pi} \in \mathcal{S}_0(\triangleleft_{A_1 \times A_2}; M),$$

where $F \subseteq \text{TPaths}(M)$ is finite and where $v_{\pi} \triangleleft_{A_1 \times A_2} V_{\pi}$ for every $\pi \in F$. Using Theorem 103 (A2), we see that it must be the case

$$v_{\pi} = \langle\langle v_{\pi}^1, v_{\pi}^2 \rangle\rangle \text{ and } V_{\pi} = (V_{\pi}^1, V_{\pi}^2) \text{ and } v_{\pi}^1 \triangleleft_{A_1} V_{\pi}^1 \text{ and } v_{\pi}^2 \triangleleft_{A_2} V_{\pi}^2.$$

Therefore, we have

$$\begin{aligned}
\mathcal{J}_{\text{pr}_1} \circ m' &= \mathcal{J}_{\text{pr}_1} \circ \sum_{\pi \in F} P(\pi) v_\pi \\
&= \mathcal{J}_{\text{pr}_1} \circ \sum_{\pi \in F} P(\pi) \langle\langle v_\pi^1, v_\pi^2 \rangle\rangle \\
&= \sum_{\pi \in F} P(\pi) (\mathcal{J}_{\text{pr}_1} \circ \langle\langle v_\pi^1, v_\pi^2 \rangle\rangle) \\
&= \sum_{\pi \in F} P(\pi) v_\pi^1 \\
&= \sum_{\pi \in F} P(\text{pr}_1(\pi)) v_\pi^1 \\
&\quad \overline{\triangleleft_{A_1} \text{pr}_1 M},
\end{aligned}$$

where $\text{pr}_1(\pi) \in \text{Paths}(\text{pr}_1 M, V_\pi^1)$ is the path that reduces $\text{pr}_1 M$ to $\text{pr}_1(V_\pi^1, V_\pi^2)$ as specified by π and then finally performs the reduction $\text{pr}_1(V_\pi^1, V_\pi^2) \xrightarrow{1} V_\pi^1$. \square

Lemma 110. *If $m \overline{\triangleleft_{\mu.X.A}} M$ then $\text{unfold} \circ m \overline{\triangleleft_{A[\mu.X.A/X]}} \text{unfold } M$.*

Proof. By Lemma 72, to complete the proof it suffices to show that

$$\text{unfold} \circ m' \in \mathcal{S}(\triangleleft_{A[\mu.X.A/X]}; \text{unfold } M)$$

for any $m' \in \mathcal{S}_0(\triangleleft_{\mu.X.A}; M)$. Towards this end, let

$$m' = \sum_{\pi \in F} P(\pi) v_\pi \in \mathcal{S}_0(\triangleleft_{\mu.X.A}; M)$$

for some finite $F \subseteq \text{TPaths}(M)$ and where $v_\pi \triangleleft_{\mu.X.A} V_\pi = \text{fold } V'_\pi$ for each $\pi \in F$. Then we have

$$\begin{aligned}
\text{unfold} \circ m' &= \sum_{\pi \in F} P(\pi) (\text{unfold} \circ v_\pi) \\
&= \sum_{\pi \in F} P(\text{unfold}(\pi)) (\text{unfold} \circ v_\pi) \\
&\in \mathcal{S}_0(\triangleleft_{A[\mu.X.A/X]}; \text{unfold } M),
\end{aligned}$$

where $\text{unfold}(\pi) \in \text{Paths}(\text{unfold } M, V'_\pi)$ is the path that reduces $\text{unfold } M$ to $\text{unfold fold } V'_\pi$ as specified by π and then finally performs the reduction $\text{unfold fold } V'_\pi \xrightarrow{1} V'_\pi$. This last sum satisfies the membership relation, because we know that $v_\pi \triangleleft_{\mu.X.A} V_\pi = \text{fold } V'_\pi$ and then by Theorem 103 (A4) we see that $\text{unfold} \circ v_\pi \triangleleft_{A[\mu.X.A/X]} V'_\pi$, as required. \square

Lemma 111. *If $m \overline{\triangleleft_{A[\mu.X.A/X]}} M$ then $\text{fold} \circ m \overline{\triangleleft_{\mu.X.A}} \text{fold } M$.*

Proof. The function

$$(\text{fold} \circ -): \mathbf{DCPO}_{\mathcal{M}}(1, \llbracket A[\mu X.A/X] \rrbracket) \rightarrow \mathbf{DCPO}_{\mathcal{M}}(1, \llbracket \mu X.A \rrbracket)$$

is Scott-continuous and therefore by Lemma 72, to complete the proof it suffices to show that

$$\text{fold} \circ m' \in \mathcal{S}(\triangleleft_{\mu X.A}; \mathbf{fold} M)$$

for each $m' \in \mathcal{S}_0(\triangleleft_{A[\mu X.A/X]}; M)$. Towards this end, assume that

$$m' = \sum_{\pi \in F} P(\pi)v_{\pi} \in \mathcal{S}_0(\triangleleft_{A[\mu X.A/X]}; M),$$

where $F \subseteq \text{TPaths}(M)$ is finite and for each $\pi \in F$ we have $v_{\pi} \triangleleft_{A[\mu X.A/X]} V_{\pi}$. Therefore, by Theorem 103 (A4) we conclude that $\text{fold} \circ v_{\pi} \triangleleft_{\mu X.A} \mathbf{fold} V_{\pi}$, for each $\pi \in F$. Now we finish the proof with the following derivation:

$$\begin{aligned} \text{fold} \circ m' &= \text{fold} \circ \sum_{\pi \in F} P(\pi)v_{\pi} \\ &= \sum_{\pi \in F} P(\pi)(\text{fold} \circ v_{\pi}) \\ &= \sum_{\pi \in F} P(\mathbf{fold}(\pi))(\text{fold} \circ v_{\pi}) \\ &\in \mathcal{S}_0(\triangleleft_{\mu X.A}; \mathbf{fold} M) \subseteq \mathcal{S}(\triangleleft_{\mu X.A}; \mathbf{fold} M), \end{aligned}$$

where $\mathbf{fold}(\pi) \in \text{Paths}(\mathbf{fold} M, \mathbf{fold} V_{\pi})$ is the path that reduces $\mathbf{fold} M$ to $\mathbf{fold} V_{\pi}$ as specified by π . \square

Lemma 112. *If $m \overline{\triangleleft}_{A \rightarrow B} M$ and $n \overline{\triangleleft}_A N$, then $m[n] \overline{\triangleleft}_B MN$.*

Proof. Consider the function $g: \mathbf{DCPO}_{\mathcal{M}}(1, \llbracket A \rightarrow B \rrbracket) \times \mathbf{DCPO}_{\mathcal{M}}(1, \llbracket A \rrbracket) \rightarrow \mathbf{DCPO}_{\mathcal{M}}(1, \llbracket B \rrbracket)$ defined by $g(x, y) = x[y]$ (see Notation 85). This function is Scott continuous and linear in both arguments. By Lemma 72, to complete the proof it suffices to show that $m'[n'] \overline{\triangleleft}_B MN$ for any $m' \in \mathcal{S}_0(\triangleleft_{A \rightarrow B}; M)$ and $n' \in \mathcal{S}_0(\triangleleft_A; N)$. Towards that end, let

$$\begin{aligned} m' &= \sum_{\pi \in F} P(\pi)v_{\pi} \in \mathcal{S}_0(\triangleleft_{A \rightarrow B}; M) \\ n' &= \sum_{\pi' \in F'} P(\pi')v_{\pi'} \in \mathcal{S}_0(\triangleleft_A; N) \end{aligned}$$

with $v_{\pi} \triangleleft_{A \rightarrow B} V_{\pi}$ and $v_{\pi'} \triangleleft_A V_{\pi'}$. Then by Theorem 103 (A3) we have that $v_{\pi}[v_{\pi'}] \overline{\triangleleft}_B V_{\pi}V_{\pi'}$ and

$$\begin{aligned} m'[n'] &= \sum_{\pi \in F} \sum_{\pi' \in F'} (P(\pi) \cdot P(\pi')) v_{\pi}[v_{\pi'}] \\ &= \sum_{(\pi, \pi') \in F \times F'} P(\text{app}(\pi, \pi')) v_{\pi}[v_{\pi'}] \\ &\overline{\triangleleft}_B MN \tag{Lemma 104} \end{aligned}$$

where $\text{app}(\pi, \pi') \in \text{Paths}(MN, V_\pi V_{\pi'})$ is the path where we first reduce MN to $V_\pi N$ in the same way as in π and then we reduce $V_\pi N$ to $V_\pi V_{\pi'}$ in the same way as in π' . Note: in the above sum $V_\pi V_{\pi'}$ is not a value, so Lemma 104 is crucial. \square

6.7. Fundamental Lemma and Strong Adequacy

We may now prove the Fundamental Lemma which then easily implies our adequacy result.

Lemma 113 (Fundamental). *Let $x_1 : A_1, \dots, x_n : A_n \vdash M : B$ be a term. Assume further we are given a collection of morphisms v_i and values V_i , such that $v_i \triangleleft_{A_i} V_i$ for $i \in \{1, \dots, n\}$. Then:*

$$\llbracket M \rrbracket \circ \langle \langle \vec{v} \rangle \rangle \overline{\triangleleft}_B M[\vec{V}/\vec{x}].$$

Proof. By induction on the derivation of the term M .

For the case of lambda abstractions, we reason as follows. Let us assume that the term of the induction hypothesis is

$$x_1 : A_1, \dots, x_n : A_n, y : A \vdash M : B.$$

Let us write $l \stackrel{\text{def}}{=} \llbracket \lambda y. M \rrbracket \circ \langle \langle \vec{v} \rangle \rangle$ and $R \stackrel{\text{def}}{=} \lambda y. M[\vec{V}/\vec{x}]$. Observe that $l \in \mathbf{TD}$ and therefore by Theorem 103 (C5), we may equivalently show that

$$l \triangleleft_{A \rightarrow B} R.$$

By Theorem 103 (A3), this is in turn equivalent to showing that

$$l \leq \llbracket R \rrbracket \text{ and } \forall (w \triangleleft_A W). l[w] \overline{\triangleleft}_B RW.$$

The inequality is satisfied, because

$$\begin{aligned} l &= \llbracket \lambda y. M \rrbracket \circ \langle \langle \vec{v} \rangle \rangle \\ &\leq \llbracket \lambda y. M \rrbracket \circ \langle \langle \vec{V} \rangle \rangle && \text{(Theorem 103 (C1))} \\ &= \llbracket R \rrbracket. && \text{(Lemma 59)} \end{aligned}$$

For the other requirement, assuming that $w \triangleleft_A W$, we reason as follows

$$\begin{aligned} l[w] &= (\llbracket \lambda y. M \rrbracket \circ \langle \langle \vec{v} \rangle \rangle)[w] && \text{(Definition)} \\ &= \epsilon \circ (\llbracket \lambda y. M \rrbracket \times \mathbf{id}) \circ \langle \langle \vec{v}, w \rangle \rangle \\ &= \epsilon \circ (\mathcal{J}\lambda(\llbracket M \rrbracket) \times \mathbf{id}) \circ \langle \langle \vec{v}, w \rangle \rangle && \text{(Definition)} \\ &= \lambda^{-1}(\lambda(\llbracket M \rrbracket)) \circ \langle \langle \vec{v}, w \rangle \rangle && \text{(Property of adjunction (5))} \\ &= \llbracket M \rrbracket \circ \langle \langle \vec{v}, w \rangle \rangle \\ &\overline{\triangleleft}_B M[\vec{V}/\vec{x}, W/y]. && \text{(Induction Hypothesis)} \end{aligned}$$

Finally, observe that $RW = (\lambda y.M[\vec{V}/\vec{x}])W \xrightarrow{1} M[\vec{V}/\vec{x}, W/y]$, i.e. RW beta-reduces to $M[\vec{V}/\vec{x}, W/y]$. Therefore by Lemma 104 it follows that

$$l[w] \overleftarrow{B} RW,$$

as required.

The case for variables follows immediately by expanding definitions and Theorem 103 (C5).

All other cases follow by straightforward induction using closure Lemmas 105 – 112. \square

Adequacy now follows as a corollary of this lemma.

Theorem 114 (Strong Adequacy). *For any closed term $\cdot \vdash M : A$, we have*

$$\llbracket M \rrbracket = \sum_{V \in \text{Val}(M)} P(M \rightarrow_* V) \llbracket V \rrbracket.$$

Proof. Let

$$u \stackrel{\text{def}}{=} \sum_{V \in \text{Val}(M)} P(M \rightarrow_* V) \llbracket V \rrbracket.$$

From Corollary 63, we know that $\llbracket M \rrbracket \geq u$. To finish the proof, we have to show the converse inequality. Next, observe that $\mathcal{S}_0(\triangleleft_A; M) \subseteq \downarrow u$, which follows from Theorem 103 (C1). To see this, we reason as follows. Taking an arbitrary element of $\mathcal{S}_0(\triangleleft_A; M)$ as in Theorem 103 (B):

$$\begin{aligned} \sum_{\pi \in F} P(\pi) v_\pi &\leq \sum_{\pi \in F} P(\pi) \llbracket V_\pi \rrbracket && \text{(Theorem 103 (C1))} \\ &= \sum_{V \in \mathcal{U}\{V_\pi \mid \pi \in F\}} \left(\sum_{\substack{\pi \in F \\ V_\pi = V}} P(\pi) \right) \llbracket V \rrbracket \\ &\leq \sum_{V \in \mathcal{U}\{V_\pi \mid \pi \in F\}} \left(\sum_{\pi \in \text{Paths}(M, V)} P(\pi) \right) \llbracket V \rrbracket \\ &= \sum_{V \in \mathcal{U}\{V_\pi \mid \pi \in F\}} P(M \rightarrow_* V) \llbracket V \rrbracket \\ &\leq \sum_{V \in \text{Val}(M)} P(M \rightarrow_* V) \llbracket V \rrbracket. \end{aligned}$$

The set $\downarrow u$ is Scott-closed and therefore $\mathcal{S}(\triangleleft_A; M) \subseteq \downarrow u$. By Lemma 113, we know that $\llbracket M \rrbracket \overleftarrow{A} M$. By definition of \overleftarrow{A} it follows $\llbracket M \rrbracket \in \mathcal{S}(\triangleleft_A; M)$ and therefore $\llbracket M \rrbracket \leq u$, thus finishing the proof. \square

7. Conclusion and Future work

We constructed three commutative valuations monads on **DCPO** that contain the simple valuations, and showed how to use any of them to give purely domain-theoretic models for PFPC that are sound and adequate in a strong sense. Our construction using topological methods can be applied to any K-category **K** with $\mathbf{K} \subseteq \mathbf{D}$, offering the possibility of further such monads. We also proved the most general Fubini theorem for dcpo's yet available. Furthermore, we showed that the Eilenberg-Moore algebras of each monad are Kegelspitzen. In the special case of domains, we showed our three monads all coincide with \mathcal{V} on **DOM**, and we characterized the Eilenberg-Moore algebras of all three over **DOM** as precisely the continuous Kegelspitzen. Characterising the Eilenberg-Moore algebras of our monads and \mathcal{V} over **DCPO** remains an open problem.

In recent work [53] we applied our results reported here to construct a model of a programming language for hybrid classical-quantum computation (also known as variational quantum programming [54, 55]). The language under consideration in [53] allows for the manipulation of quantum resources and it also supports classical probabilistic effects, which are induced by quantum measurements. The interpretation of the quantum fragment is done within a category of von Neumann algebras and the interpretation of classical probabilistic effects is achieved using the monad \mathcal{M} we described in this paper. Switching between the quantum and classical modes of operation in [53] is achieved by describing novel semantic methods that rely on Kegelspitzen as a unifying theory. In our opinion, these results showcase an interesting relationship between different kinds of semantic models where probability arises in very different ways. We plan on studying this relationship further and extending it to include *continuous* probabilistic choice.

Towards this end, we would like to identify a commutative monad of valuations which is suitable for *continuous* probabilistic choice. In this paper we showed that our three monads \mathcal{M}, \mathcal{W} and \mathcal{P} are suitable for *discrete* probabilistic choice, but it remains to be seen to what extent they can be used to model continuous probabilistic choice. It is an open question whether these monads are large enough to contain the required valuations for continuous probabilistic choice. In recent work [56], we considered an alternative approach to finding commutative valuations submonads. In particular, in [56], we use *algebraic methods* to construct a commutative monad \mathcal{Z} , called the *central valuations* monad. In fact, \mathcal{Z} is the largest commutative submonad of \mathcal{V} known so far. Specifically, for any dcpo D , we have the following relationship between spaces of valuations:

$$SD \subseteq MD \subseteq WD \subseteq PD \subseteq ZD \subseteq VD.$$

Determining whether any of these monads is suitable for continuous probabilistic choice is left for future work.

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