Categorical models of circuit description languages

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Proto-Quipper-M

- We will consider several variants of a functional programming language called *Proto-Quipper-M*.
- Language and model developed by Francisco Rios and Peter Selinger.
- Language is equipped with formal denotational and operational semantics.
- Primary application is in quantum computing, but the language can describe arbitrary string diagrams.
- Their model supports primitive recursion, but does not support general recursion.

Circuit Model

Proto-Quipper-M is used to describe *families* of morphisms of an arbitrary, but fixed, symmetric monoidal category, which we denote **M**.

Example

If **M** = **FdCStar**, the category of finite-dimensional C^*-algebras and completely positive maps, then a program in our language is a family of quantum circuits.

Example

Shor's algorithm for integer factorization may be seen as an infinite family of quantum circuits – each circuit is a procedure for factorizing an $n-$bit integer, for a fixed n.

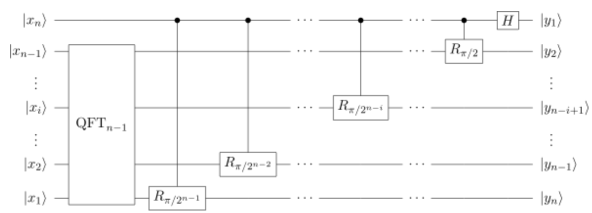


Figure: Quantum Fourier Transform on n qubits (subroutine in Shor's algorithm).[1]

[1]Figure source: https://commons.wikimedia.org/w/index.php?curid=14545612

Syntax of Proto-Quipper-M

The type system is given by:

Types	A, B	$::=$	$\alpha \mid 0 \mid A + B \mid I \mid A \otimes B \mid A \multimap B \mid !A \mid \mathbf{Circ(T, U)}$
Parameter types	P, R	$::=$	$\alpha \mid 0 \mid P + R \mid I \mid P \otimes R \mid !A \mid \mathbf{Circ(T, U)}$
M-types	T, U	$::=$	$\alpha \mid I \mid T \otimes U$

The term language is given by:

$$
\begin{aligned}
\text{Terms} \quad M, N \quad ::= \quad & x \mid l \mid c \mid \text{let } x = M \text{ in } N \\
& \mid \Box_A M \mid \text{left}_{A,B} M \mid \text{right}_{A,B} M \mid \text{case } M \text{ of } \{\text{left } x \to N \mid \text{right } y \to P\} \\
& \mid * \mid M; N \mid \langle M, N \rangle \mid \text{let } \langle x, y \rangle = M \text{ in } N \mid \lambda x^A.M \mid MN \\
& \mid \text{lift } M \mid \text{force } M \mid \mathbf{box_T M} \mid \mathbf{apply(M, N)} \mid \mathbf{(\tilde{l}, C, \tilde{l}')}
\end{aligned}
$$

Families Construction

The following construction is well-known.

Definition

Given a category $\mathbf{C}$, we define a new category $\mathbf{Fam}[\mathbf{C}]$:

- Objects are pairs (X, A) where X is a discrete category and $A : X \to \mathbf{C}$ is a functor.
- A morphism $(X, A) \to (Y, B)$ is a pair (f, ϕ) where $f : X \to Y$ is a functor and $\phi : A \to B \circ f$ is a natural transformation.
- Composition of morphisms is given by: $(g, \psi) \circ (f, \phi) = (g \circ f, \psi f \circ \phi)$.

Remark

$\mathbf{Fam}[\mathbf{C}]$ *is the free coproduct completion of* $\mathbf{C}$ *and as a result has all small coproducts.*

Proposition

If $\mathbf{C}$ *is a symmetric monoidal closed and product-complete category, then* $\mathbf{Fam}[\mathbf{C}]$ *is a symmetric monoidal closed category.*

Categorical Model

Definition

- A symmetric monoidal closed and product-complete category $\overline{\mathbf{M}}$.
- A fully faithful strong monoidal embedding $\mathbf{M} \to \overline{\mathbf{M}}$.
- A symmetric monoidal closed category $\mathbf{Fam}[\overline{\mathbf{M}}]$ which we will refer to as $\mathbf{Fam}$.
- A symmetric monoidal adjunction:

$$\mathbf{Set} \underset{\mathbf{Fam}(I,-)}{\overset{F}{\underset{\bot}{\rightleftarrows}}} \mathbf{Fam}$$

where

$$F(X) = (X, I_X), \qquad \text{where } I_X(x) = I$$
$$F(f) = (f, \iota), \qquad \text{where } \iota_x = \mathrm{id}_I.$$

Remark

For any symmetric monoidal category $\mathbf{M}$, *we can set* $\overline{\mathbf{M}} := [\mathbf{M}^{op}, \mathbf{Set}]$ *and then the Yoneda embedding, together with the Day tensor product, satisfy the first two requirements.*

Categorical Model

Theorem (Rios & Selinger 2017)

Every categorical model of Proto-Quipper-M is computationally sound and adequate with respect to its operational semantics.

Question

Sam Staton: Why do you need the **Fam** *construction for this?*

Open Problem

Find a categorical model of Proto-Quipper-M which supports general recursion.

Our approach

- Consider an *abstract* categorical model for the same language.
- Describe a *candidate* categorical model for each of the following language variants:
  - The original Proto-Quipper-M language (`base`).
  - Proto-Quipper-M extended with general recursion (`base+rec`).
  - Proto-Quipper-M extended with dependent types (`base+dep`).
  - Proto-Quipper-M extended with dependent types and recursion (`base+dep+rec`).

Related work: Rennela and Staton describe a different circuit description language where they also use enriched category theory.

Commercial break

- Everybody is advertising books, so I have to do it as well.

Models of Intuitionistic Linear Logic

A model of Intuitionistic Linear Logic (ILL) as described by Benton is given by the following data:

- A cartesian closed category **V**.
- A symmetric monoidal closed category **L**.
- A symmetric monoidal adjunction:

$$\mathbf{V} \underset{G}{\overset{F}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathbf{L}$$

Nick Benton. *A mixed linear and non-linear logic: Proofs, terms and models.* CSL'94

Models of the Enriched Effect Calculus

A model of the Enriched Effect Calculus (EEC) is given by the following data:

- A cartesian closed category **V**, enriched over itself.
- A **V**-enriched category **L** with powers, copowers, finite products and finite coproducts.
- A **V**-enriched adjunction:

$$\mathbf{V} \underset{G}{\overset{F}{\rightleftarrows}} \mathbf{L} \qquad (F \dashv G)$$

Theorem

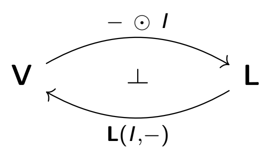
Every model of ILL with additives determines an EEC model.

Egger, Møgelberg, Simpson. *The enriched effect calculus: syntax and semantics*. Journal of Logic and Computation 2012

An abstract model of the base language

A model of the base language is given by the following data:

1. A cartesian closed category $\mathbf{V}$ (the category of parameter values) enriched over itself such that:
   - $\mathbf{V}_0$ has finite coproducts.
   - $\mathbf{V}_0$ has colimits of initial sequences.
2. A $\mathbf{V}$-enriched symmetric monoidal category $\mathbf{M}$ which describes the circuit model.
3. A $\mathbf{V}$-enriched symmetric monoidal closed category $\mathbf{L}$ (the category of (linear) higher-order circuits) such that:
   - $\mathbf{L}$ has $\mathbf{V}$-copowers.
   - $\mathbf{L}_0$ has finite coproducts.
   - $\mathbf{L}_0$ has colimits of initial sequences.
4. A $\mathbf{V}$-enriched fully faithful strong symmetric monoidal embedding $E : \mathbf{M} \to \mathbf{L}$.
5. A $\mathbf{V}$-enriched symmetric monoidal adjunction:



Less formally, a model of Proto-Quipper-M is given by an **enriched** model of ILL.

Concrete models of the base language

Fix an arbitrary symmetric monoidal category $\mathbf{M}$.
The original Proto-Quipper-M model is given by the model of ILL

$$\mathbf{Set} \underset{\mathbf{Fam}[\overline{\mathbf{M}}](I,-)}{\overset{-\odot I}{\rightleftarrows}} \mathbf{Fam}[\overline{\mathbf{M}}] \quad (\bot)$$

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A simpler model for the same language is given by the model of ILL:

$$\mathbf{Set} \underset{\overline{\mathbf{M}}(I,-)}{\overset{-\odot I}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \overline{\mathbf{M}}$$

where in both cases $\overline{\mathbf{M}} = [\mathbf{M}^{\mathrm{op}}, \mathbf{Set}]$.

Remark

When $\mathbf{M} = \mathbf{1}$, *the latter model degenerates to* $\mathbf{Set}$ *which is a model of a simply-typed (non-linear) lambda calculus.*

Concrete models of the base language (contd.)

Fix an arbitrary symmetric monoidal category $\mathbf{M}$.
Equipping $\mathbf{M}$ with the free $\mathbf{DCPO}$-enrichment yields another concrete (order-enriched) Proto-Quipper-M model:

$$\mathbf{DCPO} \underset{\overline{\mathbf{M}}(I,-)}{\overset{-\odot I}{\rightleftarrows}} \overline{\mathbf{M}} \qquad (\bot)$$

where $\overline{\mathbf{M}} = [\mathbf{M}^{\mathrm{op}}, \mathbf{DCPO}]$.

Remark

The three concrete models of Proto-Quipper-M are EEC models whose underlying (unenriched) structure is a model of ILL.

Original model revisited

Fix an arbitrary symmetric monoidal category $\mathbf{M}$.
Original Proto-Quipper-M model:

$$\mathbf{Set} \underset{\mathbf{Fam}[\overline{\mathbf{M}}](I,-)}{\overset{-\odot I}{\underset{\bot}{\rightleftarrows}}} \mathbf{Fam}[\overline{\mathbf{M}}]$$

Simpler model:

$$\mathbf{Set} \underset{\overline{\mathbf{M}}(I,-)}{\overset{-\odot I}{\underset{\bot}{\rightleftarrows}}} \overline{\mathbf{M}}$$

Question: What does the extra layer of abstraction provide?
Answer: A model of the language extended with dependent types.

Linear dependent types

Theorem

The category $\mathbf{Fam}[\overline{\mathbf{M}}]$ *is a model of dependently typed intuitionistic linear logic (type dependence is allowed only on intuitionistic terms)* [2].

Conjecture

The symmetric monoidal adjunction:

$$\mathbf{Set} \underset{\mathbf{Fam}[\overline{\mathbf{M}}](I,-)}{\overset{-\odot I}{\rightleftarrows}} \mathbf{Fam}[\overline{\mathbf{M}}] \qquad (\perp)$$

is a model of Proto-Quipper-M extended with dependent types.

Remark

If $\mathbf{M} = \mathbf{1}$, *the above model degenerates to* $\mathbf{Fam}[\overline{\mathbf{M}}] = \mathbf{Fam}[\mathbf{M}^{op}, \mathbf{Set}] \cong \mathbf{Fam}[\mathbf{Set}] \simeq [\mathbf{2}^{op}, \mathbf{Set}]$, *which is a closed comprehension category and thus a model of intuitionistic dependent type theory*[3].

[2] Matthijs Vákár. *In Search of Effectful Dependent Types*. PhD thesis, University of Oxford.

[3] Bart Jacobs. *Categorical Logic and Type Theory*. 1999.

Abstract model with dependent types?

Theorem

A model of dependently typed intuitionistic linear logic is given by an indexed monoidal category with some additional structure (comprehension, strictness, ...) [4]*.*

Conjecture

An abstract model of Proto-Quipper-M extended with dependent types is given by an ***enriched*** *indexed monoidal category* [5] *with some additional structure (comprehension, strictness, ...).*

[4] Matthijs Vákár. *In Search of Effectful Dependent Types*. PhD thesis, University of Oxford.

[5] Michael Shulman. *Enriched Indexed Categories*. Theory and Application of Categories, 2013.

What about recursion?

- Forget about dependent types for now.
- Consider the base Proto-Quipper-M language.
- How can we model general recursion?

What about recursion?

- Forget about dependent types for now.
- Consider the base Proto-Quipper-M language.
- How can we model general recursion?
  - We already have a concrete order-enriched model:

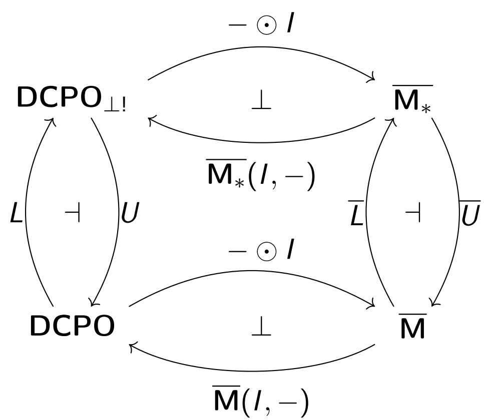
$$\mathbf{DCPO} \underset{\overline{\mathbf{M}}(I,-)}{\overset{-\odot I}{\rightleftarrows}} \overline{\mathbf{M}} \qquad (\bot)$$

    where $\overline{\mathbf{M}} = [\mathbf{M}^{\mathrm{op}}, \mathbf{DCPO}]$.
  - Thus, we add partiality to the above model:

$$\mathbf{DCPO}_{\perp!} \underset{\overline{\mathbf{M}_*}(I,-)}{\overset{-\odot I}{\rightleftarrows}} \overline{\mathbf{M}_*} \qquad (\bot)$$

    where $\mathbf{M}_*$ is the $\mathbf{DCPO}_{\perp!}$-category obtained by freely adding a zero object to $\mathbf{M}$ and $\overline{\mathbf{M}_*} = [\mathbf{M}_*^{\mathrm{op}}, \mathbf{DCPO}_{\perp!}]$ is the associated enriched functor category.

Concrete model of Proto-Quipper-M extended with recursion



Remark

If $\mathbf{M} = \mathbf{1}$, then the above model degenerates to the left vertical adjunction, which is a model of a simply-typed lambda calculus with term-level recursion.

Abstract model with recursion?

Intuitionistic linear logics correspond to linear/non-linear lambda calculi under the Curry-Howard isomorphism.

Theorem

A categorical model of a linear/non-linear lambda calculus extended with recursion is given by a model of ILL:

$$\mathbf{V} \underset{G}{\overset{F}{\underset{\perp}{\rightleftarrows}}} \mathbf{L}$$

where FG (or equivalently GF) is algebraically compact [6]*.*

[6] Benton & Wadler. *Linear logic, monads and the lambda calculus.* LiCS'96.

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Theorem

A categorical model of a linear/non-linear lambda calculus extended with recursion is given by a model of ILL:

$$\mathbf{V} \underset{G}{\overset{F}{\rightleftarrows}} \mathbf{L} \quad (F \dashv G)$$

where FG (or equivalently GF) is algebraically compact [6].

Definition

An abstract model of Proto-Quipper-M extended with recursion is given by a model of Proto-Quipper-M:

$$\mathbf{V} \underset{\mathbf{L}(I,-)}{\overset{- \odot I}{\rightleftarrows}} \mathbf{L} \quad (- \odot I \dashv \mathbf{L}(I,-))$$

where the underlying induced (co)monad endofunctors are algebraically compact.

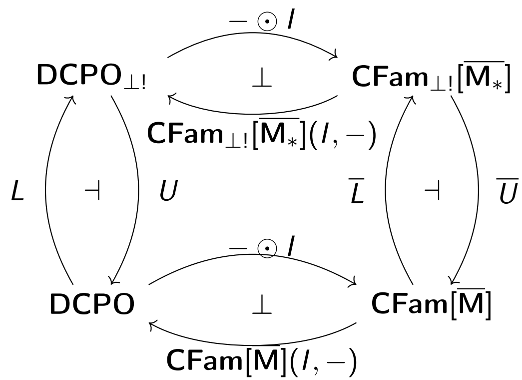
Remark

The above definition is not the whole picture, but it describes the essential idea.

[6] Benton & Wadler. *Linear logic, monads and the lambda calculus.* LiCS'96.

What about recursion and dependent types simultaneously?

- This is the most complicated case by far.



Remark

If $\mathbf{M} = \mathbf{1}$*, then the model collapses to a model which is very similar to Palmgren and Stoltenberg-Hansen's model of partial intuitionistic dependent type theory* [7]*.*

[7] Erik Palmgren & Viggo Stoltenberg-Hansen. *Domain interpretations of Martin-Löf's partial type theory*. Annals of Pure and Applied Logic 1990.

Abstract model with recursion and dependent types?

Conjecture

An abstract model of Proto-Quipper-M extended with recursion and dependent types is given by an ***enriched*** *indexed monoidal category with some additional structure (comprehension, strictness, ...) and suitable algebraic compactness conditions on the underlying adjoint functors.*

Conclusion

- You can cheese yourself into a model of circuit description languages by categorically enriching certain denotational models.
- We have identified different *candidate* models for Proto-Quipper-M depending on the feature set.
- Systematic construction for concrete models that works for any circuit (string diagram) model described by a symmetric monoidal category.
- Plenty of work (and verification) remains to be done...

Thank you for your attention and happy birthday Dusko!