Is ZX complete for Clifford+T? Nobody knows (yet)

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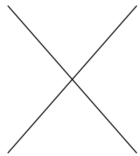
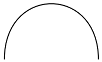
- Diagramatic logical calculus for studying quantum information processing
- Can be used as an alternative to traditional Hilbert space formalism
- Has been used to study:
  - Quantum algorithms
  - Quantum security protocols
  - Quantum error-correcting codes
  - and other problems involving quantum information

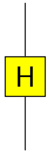
Atomic Diagrams (1)

$[\![$ $]\!] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$

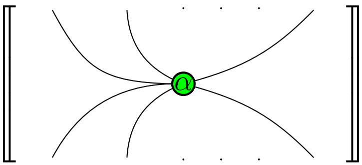
$[\![$  $]\!] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \sigma$

$[\![$  $]\!] = \langle 00| + \langle 11|$

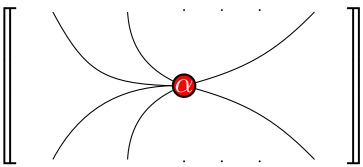
$[\![$  $]\!] = |00\rangle + |11\rangle$

$[\![$  $]\!] = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = H$

Atomic Diagrams (2)



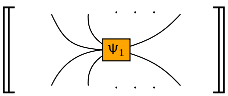
$$= \begin{cases} |0^m\rangle & \mapsto |0^n\rangle \\ |1^m\rangle & \mapsto e^{i\alpha}\,|1^n\rangle \\ \text{rest} & \mapsto 0 \end{cases}$$



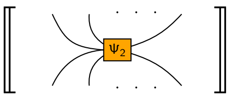
$$= \begin{cases} |+^m\rangle & \mapsto |+^n\rangle \\ |-^m\rangle & \mapsto e^{i\alpha}\,|-^n\rangle \\ \text{rest} & \mapsto 0 \end{cases}$$

where $\alpha \in [0, 2\pi)$

Compound Diagrams

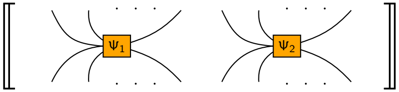


$= D_1$ and



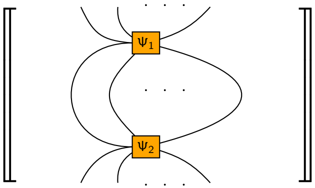
$= D_2$

then



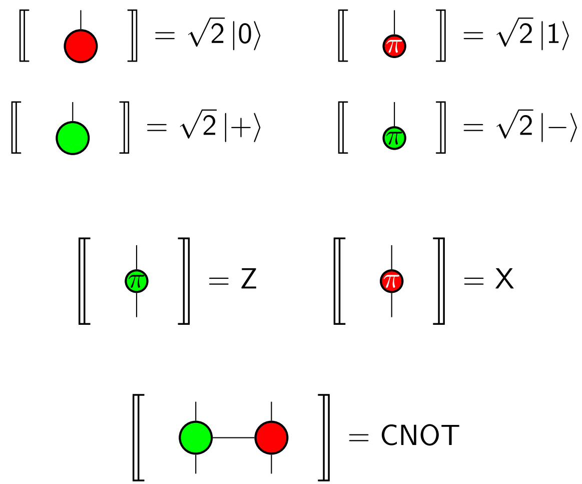
$= D_1 \otimes D_2$

and



$= D_1 \circ D_2$

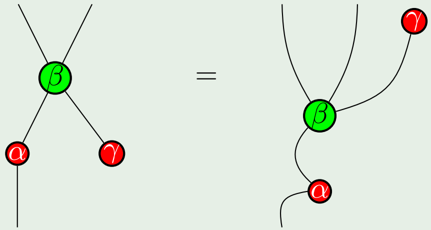
Examples


$= \sqrt{2}\,|0\rangle$
π
$= \sqrt{2}\,|1\rangle$
$= \sqrt{2}\,|+\rangle$
π
$= \sqrt{2}\,|-\rangle$
π
$= Z$
π
$= X$
$= \mathrm{CNOT}$

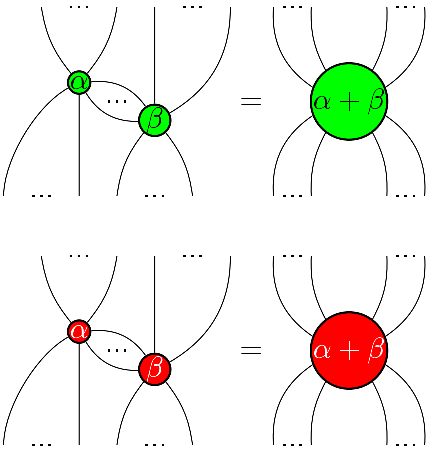
Axioms (1)

"Only the topology matters"

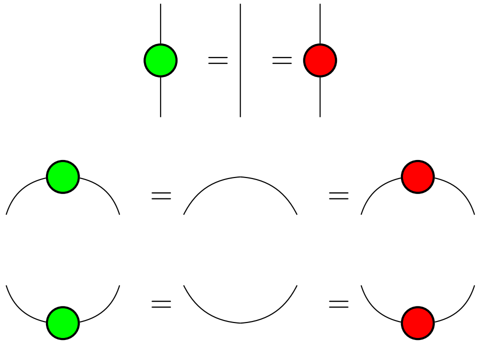
Example



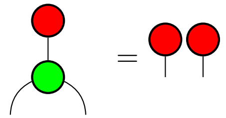
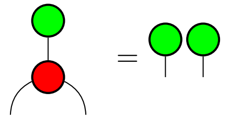
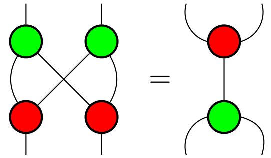
Axioms (2)



Axioms (3)



Axioms (4)

- 
- 
- 

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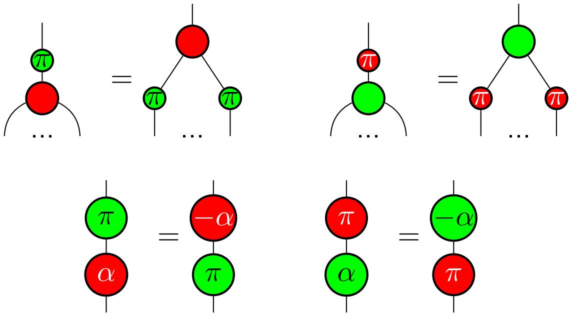

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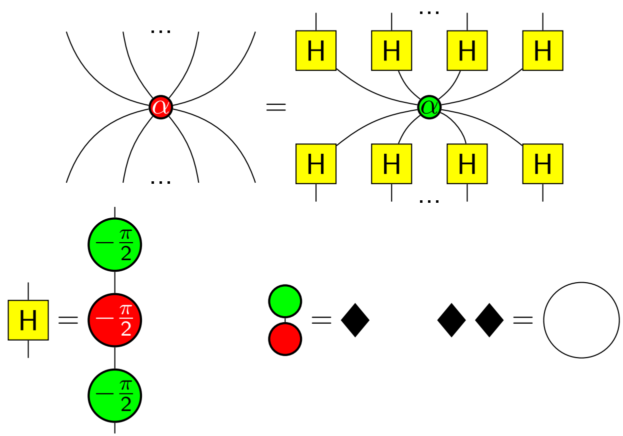
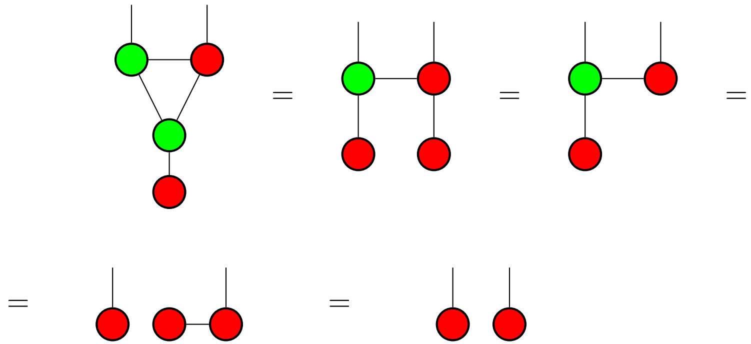


Example derivation



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  - If D_1 and D_2 are single qubit Clifford+T ZX-diagrams and $[\![D_1]\!] = [\![D_2]\!]$ then $ZX \vdash D_1 = D_2$
- It's not known if it is complete for Clifford+T in general

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  - completeness is unknown in general

Clifford+T

- From now on, we restrict ourselves to the Clifford+T segment of QM
- We discuss two (failed) attempts of showing incompleteness for this segment of QM.
- The first one is based on an invariant for the axioms of ZX.
- The second one makes use of alternative interpretations of ZX-diagrams

Invariant for ZX

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In terms of circuits, two Clifford+T circuits are not equal under the axioms of the ZX-calculus if their T-count modulo 2 is different.

Example

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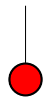
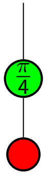
$D_1 :=$ 

$D_2 :=$ 

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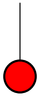
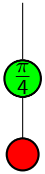
$D_1 :=$  $D_2 :=$ 

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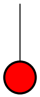
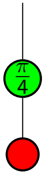
$D_1 :=$ 

$D_2 :=$ 

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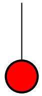
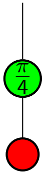
$D_1 :=$ 
$D_2 :=$ 

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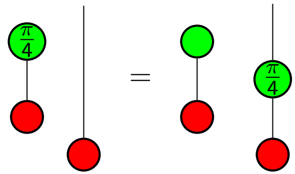
So, we have an example of two ZX-diagrams which are equal in Hilbert space, but which are not equal under the axioms of the ZX-calculus.

Why it doesn't work – phases

Recall, that:

$$ZX \vdash D_1 = D_2 \Longrightarrow [\![D_1]\!] = e^{i\phi}[\![D_2]\!]$$

Indeed, the same equality holds up to scalars in ZX:

$ZX \vdash$ 

For this reason, this approach won't work for any other pairs of diagrams, because we can always introduce a global phase on one side of the equation. We need a stronger invariant.

Alternative Models

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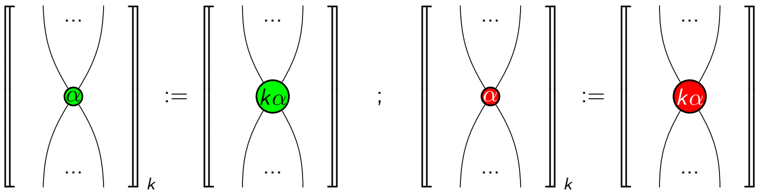

$$[\![\cdot]\!]_k := [\![\cdot]\!] \text{ , otherwise}$$

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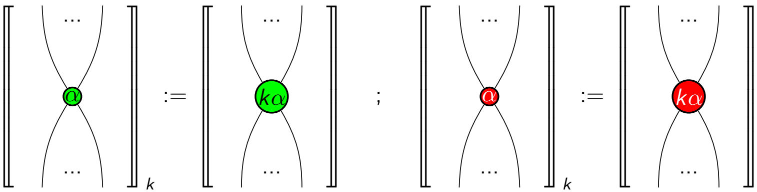

$$[\![\cdot]\!]_k := [\![\cdot]\!] \text{ , otherwise}$$

These models are sound when $k = 4p + 1$ for $p \in \mathbb{Z}$. That is:

$$ZX \vdash D_1 = D_2 \Longrightarrow [\![D_1]\!]_5 = [\![D_2]\!]_5$$

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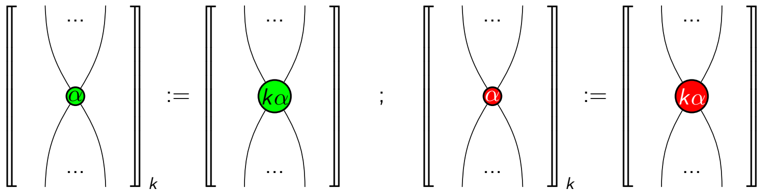

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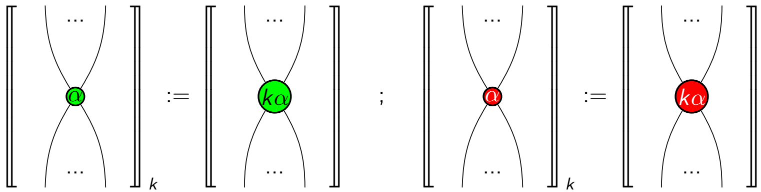

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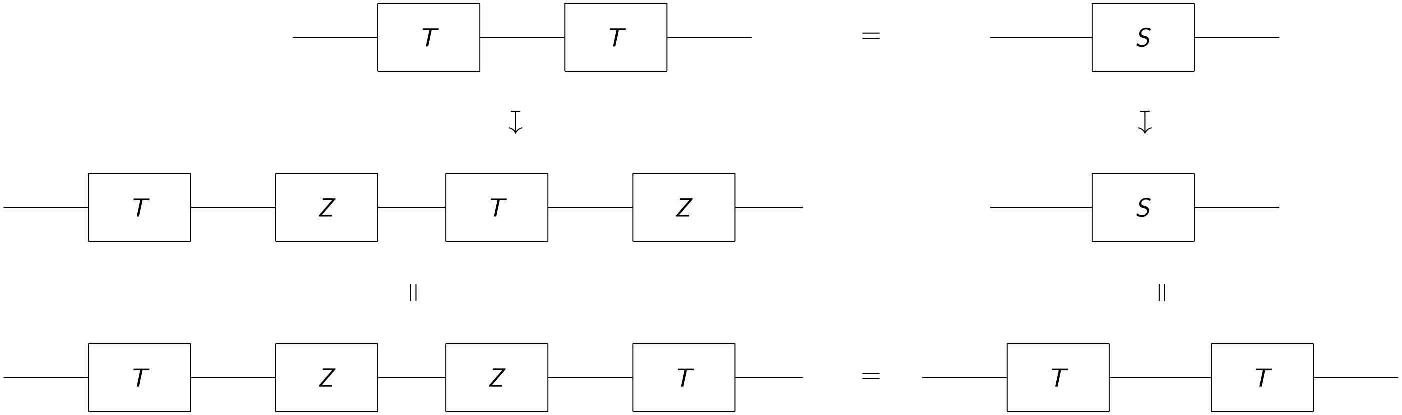
In terms of Clifford+T circuits, this means the following: Find two Clifford+T circuits which are equal, such that when in each of them we replace all T and $T^\dagger$ gates with $T \circ Z$ and $T^\dagger \circ Z$ gates then equality doesn't hold anymore (even up to a scalar). If we can find such a pair of Clifford+T circuits, then the ZX-calculus is incomplete. Note, that this can be established outside of ZX.

Example

An example of this argument in action:



So, this example doesn't demonstrate incompleteness.

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- Maybe this is always the case?
- Need to consider more equalities or do numerical experiments

Conclusion

- We don't know if ZX is complete for Clifford+T
- My inutition is that it is incomplete
- The invariant approach won't work unless the invariant is significantly strengthened
- The alternative model approach might work, but we need to consider further equalities or get a corpus of circuit equalities and run an algorithm on them to check for incompleteness

That's it!

Thank you for your attention!