Higher-order rewriting of String Diagrams

Vladimir Zamdzhiev

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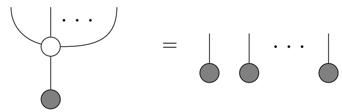
Families of string diagrams

- String diagrams can be used to establish equalities between pairs of objects, one at a time.
- Proving infinitely many equalities simultaneously is only possible using metalogical arguments.

Example



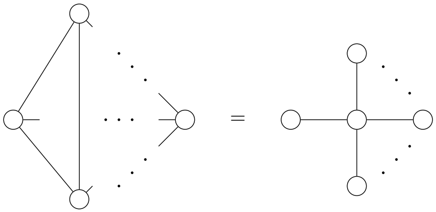
- However, this is imprecise and implementing software support for it would be very difficult.

Motivation

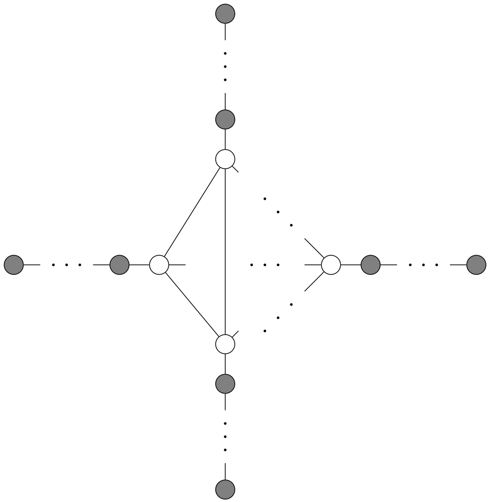
- Given an equational schema between two families of string diagrams, how can we apply it to a target family of string diagrams and obtain a new equational schema?

Example

Equational schema between complete graphs on n vertices and star graphs on n vertices:

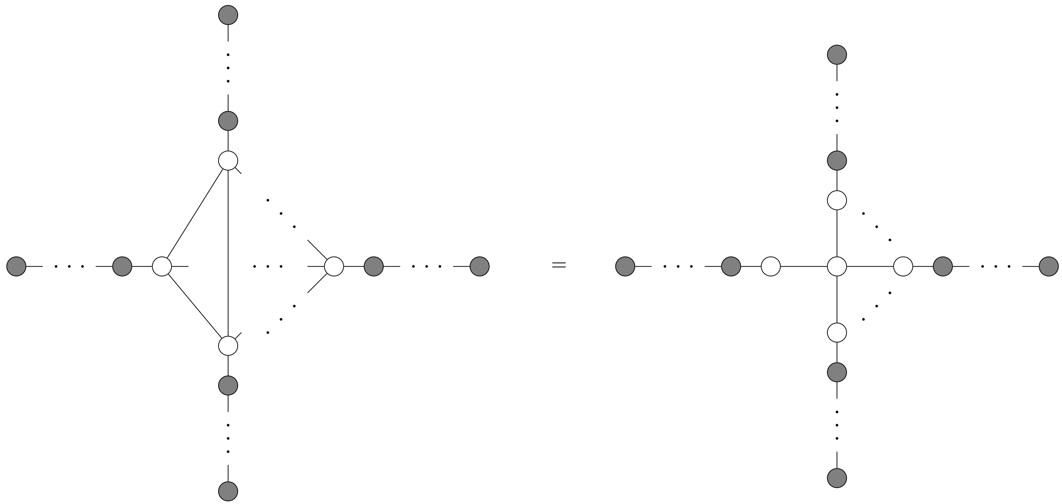


Then, we can apply this schema to the following family of graphs:



Motivation

and we obtain a new equational schema:

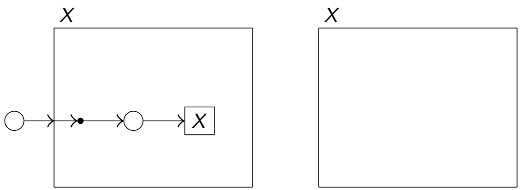


The main ideas are:

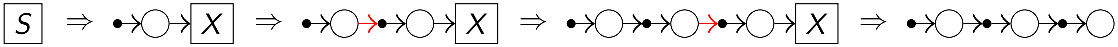
- Context-free graph grammars represent families of graphs
- "Grammar" DPO rewrite rules represent equational schemas
- "Grammar" DPO rewriting represents equational reasoning on families of graphs
- "Grammar" DPO rewriting is admissible (or correct) w.r.t. concrete instantiations

edNCE grammar example

The following grammar generates the set of all chains of node vertices with an input and no outputs:



A derivation in the above grammar of the string graph with three node vertices:

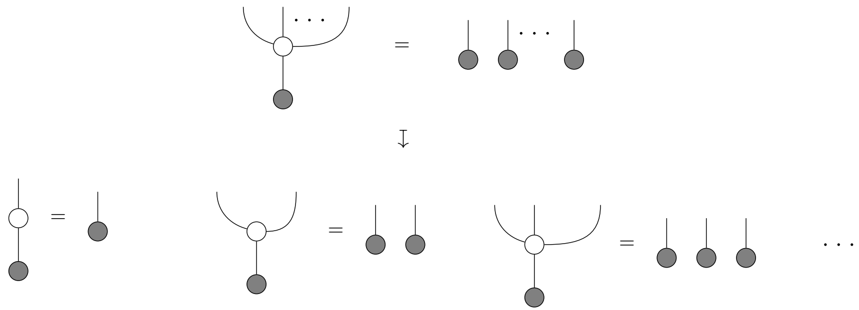


where we color the newly established edges in red.

- An edNCE grammar is a graph-like structure – essentially it is a partition of graphs equipped with connection instructions

Quantification over equalities

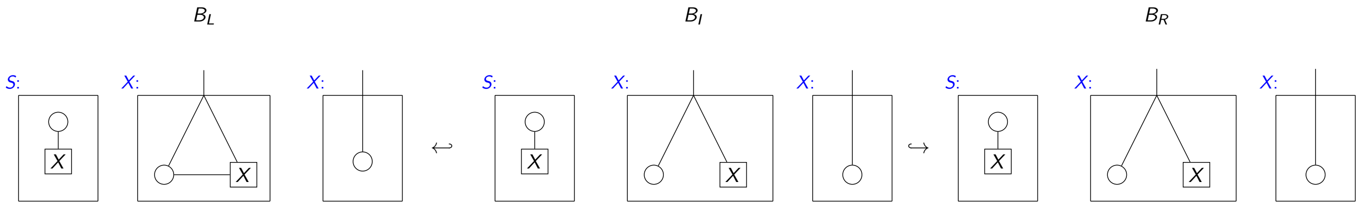
- an equational schema between two families of string diagrams establishes infinitely many equalities:



- How do we model this using edNCE grammars?
- Idea: DPO rewrite rule in **GGram**, where productions are in 1-1 correspondance

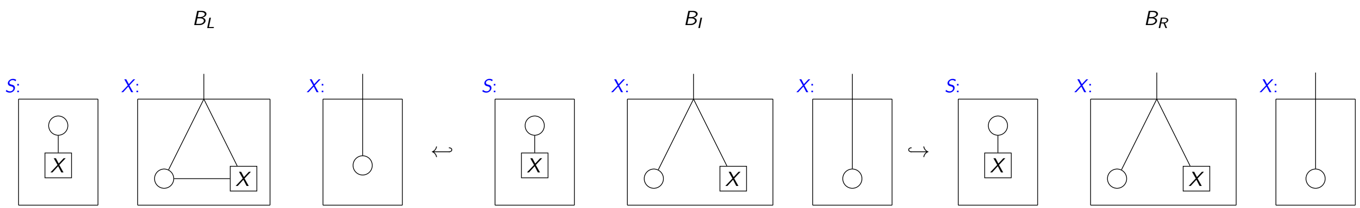
Grammar rewrite pattern

Example



Grammar rewrite pattern

Example



- Instantiation :

S

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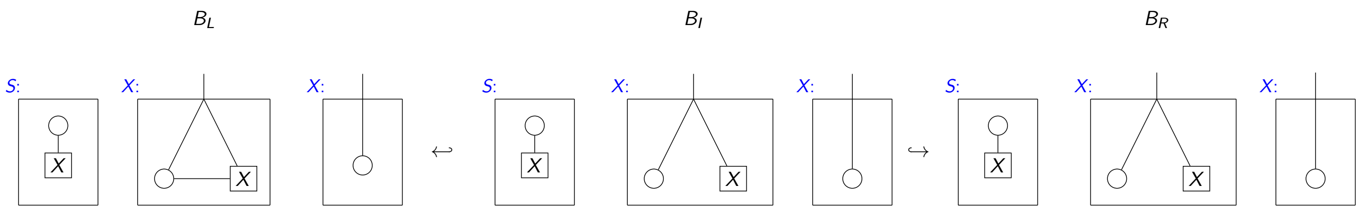
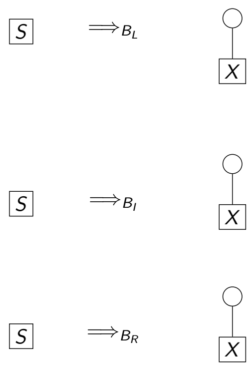
Grammar rewrite pattern

Example

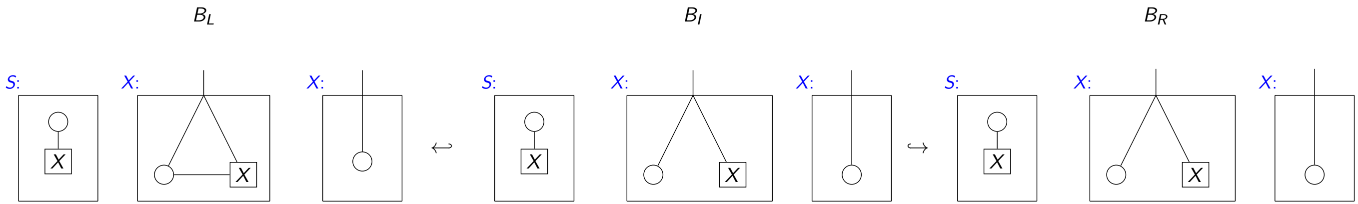


- Instantiation :

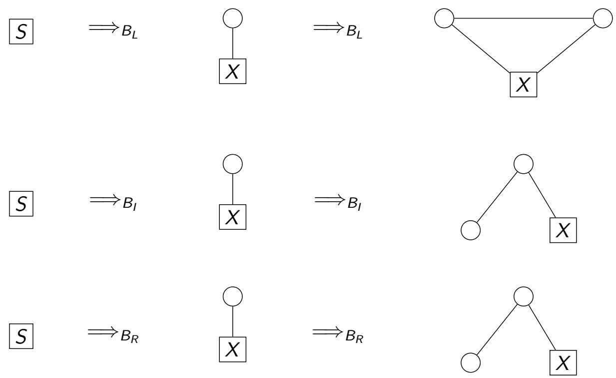


Grammar rewrite pattern

Example

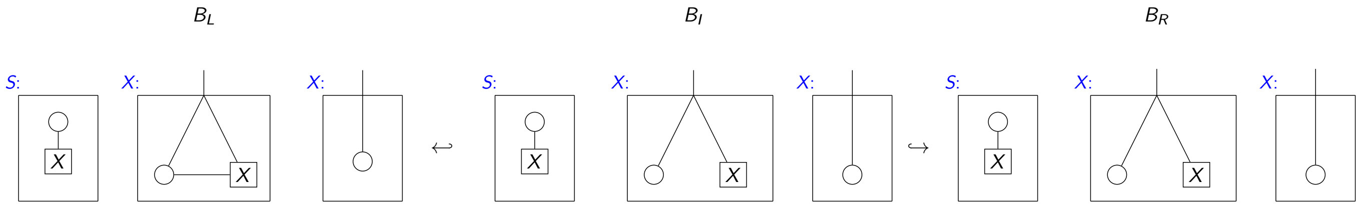


- Instantiation :

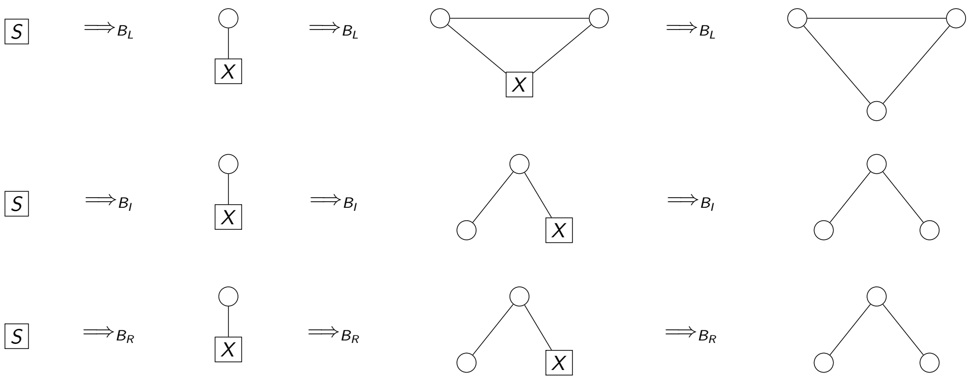


Grammar rewrite pattern

Example

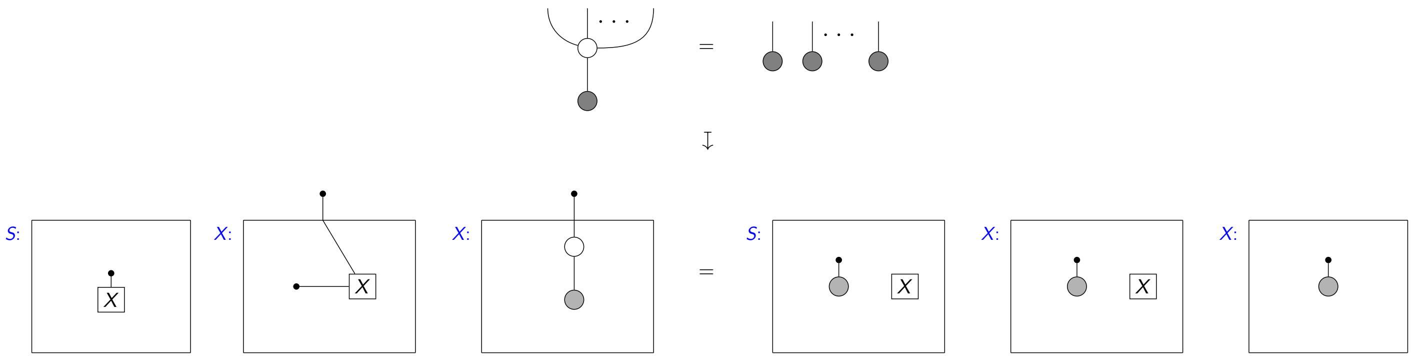


- Instantiation :



Obtaining new equalities

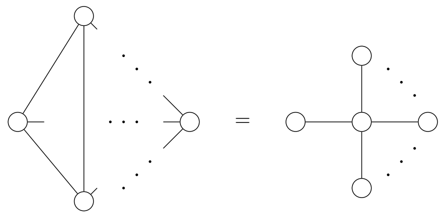
- We can encode infinitely many equalities between string diagrams by using grammar rewrite patterns



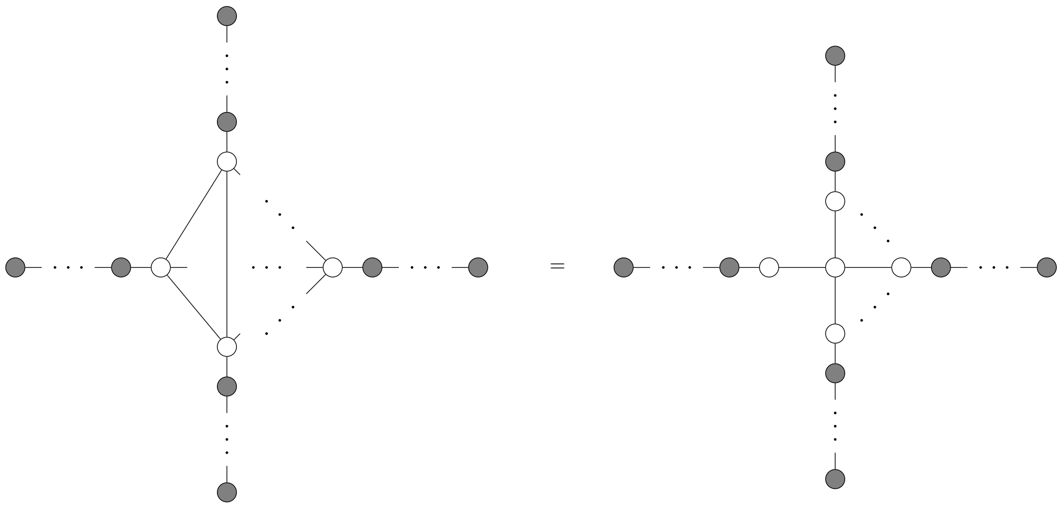
- Next, we show how to rewrite a family of diagrams using an equational schema in an admissible way

Example

Given an equational schema:

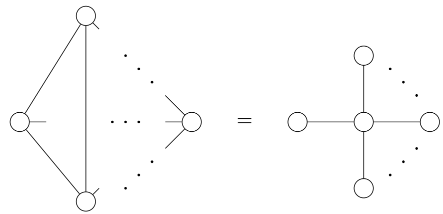


how do we apply it to a target family of string diagrams (left) and get the resulting family (right):

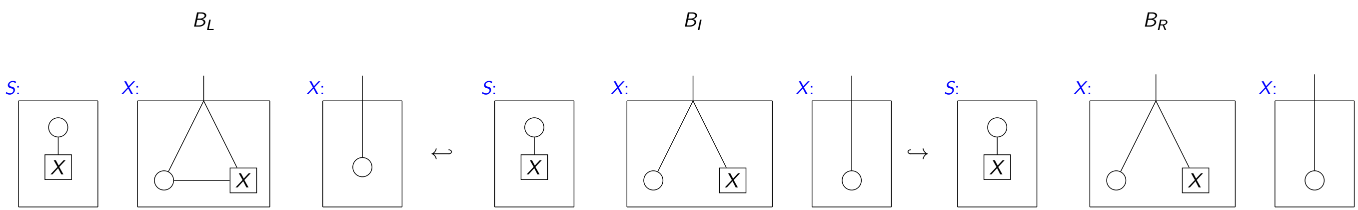


Step one

Encode equational schema as a grammar rewrite pattern.
This:



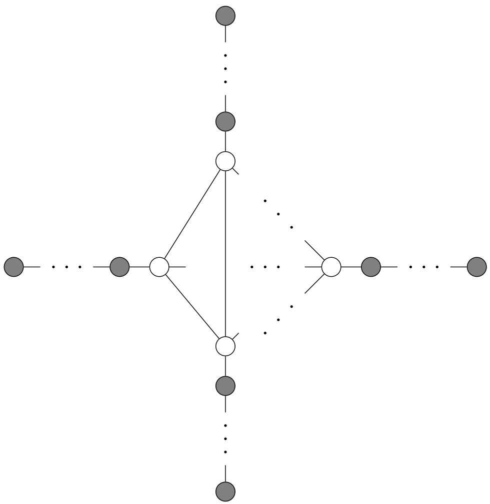
becomes this:



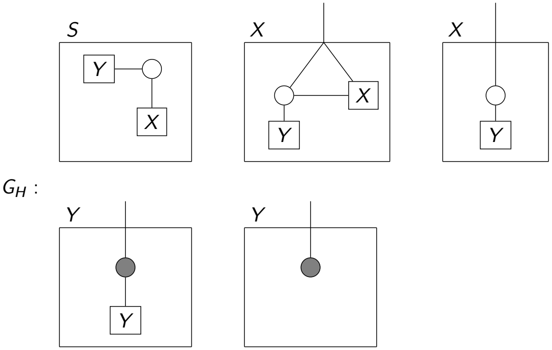
Step two

Encode the target family of string diagrams using a grammar

This:



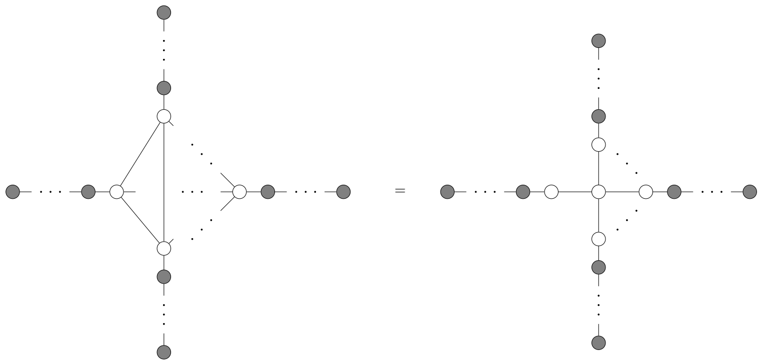
becomes this:



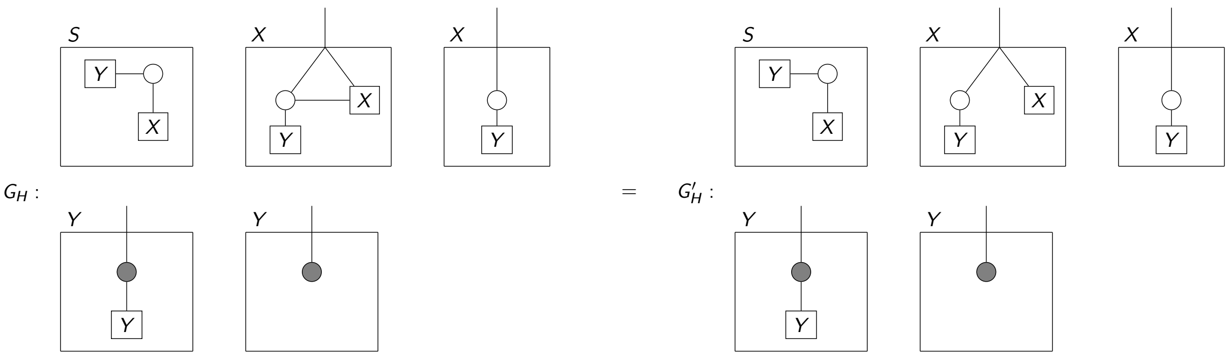
Step three

- Match the grammar rewrite rule into the target grammar and perform DPO rewrite (in **GGram**)
- Note, both the rewrite rules and the matchings are more restricted than what is required by adhesivity in order to ensure admissibility

This:



is then given by:



Conclusion and Future Work

- Basis for formalized equational reasoning for context-free families of string diagrams.
  - Framework can handle equational schemas and it can apply them to equationally reason about families of string diagrams
- Identify more general conditions for grammar rewriting such that the desired theorems and decidability properties hold
- Implementation in software (e.g. Quantomatic proof assistant)
- Once implemented, software tools can be used for automated reasoing for quantum computation, petri nets, etc.

Thank you for your attention!