An introduction to discrete & computational geometry

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**Computational geometry** studies the algorithmic foundations of geometric computing.

“Builds general tools – analytic and computational – to satisfy the algorithmic needs of geometric computing.”

[CG impact task force, '96]

*Design of algorithms, analysis of their resource consumption.*
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Algorithms are described and analyzed in a computational model.

Définition: the operations allowed and their costs.

Usual model: Real RAM (exact arithmetic over $\mathbb{R}$ in constant time).

Complexity measured as a function of the input size.
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Complexity measured as a function of the input size.

Interested in the asymptotic behaviour.

Compare algorithms independently of implementation or technology.
70’s - beginnings

segments intersection
Voronoi diagrams
nearest neighbours
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80’s - exact solutions, lower bounds

range searching
convex hulls
arrangements
triangulations
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90’s - approximation, probabilistic methods

ε-nets, cuttings...
robustness, CGAL...
applications: GIS, CAD...
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\( \epsilon \)-nets, cuttings…
robustness, CGAL…
applications: GIS, CAD...

00’s - other geometries, large amount of data

metric spaces, embeddings…
computational topology
external memory algorithms
Computational geometry makes heavy use of **discrete** properties of geometric objects…

… which are the specialty of **discrete geometry**.

- Packing & covering
- Polytopes
- Space partitions
- Points configurations
- Geometric (hyper)graphs

These two communities merged during the 80’s.
Some classics
Arrangements
Delaunay triangulations & Voronoi diagrams

New development: algebraic/polynomial methods
Incidence geometry, around the Szemeredi-Trotter theorem
Solution to the joint problem by algebraic arguments

New developments: topological methods
Topological combinatorics and inclusion-exclusion formulas
Persistent homology and topological inference
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(Geometric) arrangements

The arrangement of a family of subsets of $\mathbb{R}^2$
is the subdivision of $\mathbb{R}^2$ induced by the boundaries of these objects.

Embedded combinatorial structure.

The boundaries of any two objects intersect in a finite number of points.
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Various interesting sub-structures and refinements.

An arrangement of 6 segments.
Their lower envelope.
Their trapezoidal decomposition.
The zone of a 7th segment.
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The piano mover’s problem.
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$\mathbb{R}^2$ for translations in the plane.

$\mathbb{R}^3 \times SO(3)$ for rigid motions in space.

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Estimate the **asymptotic complexity** of various sub-structures of arrangements.
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Notion of **complexity** (of a sub-structure) of an arrangement.

The number of **elements**, of all dimensions.

\[ \# \text{vertices} + \# \text{edges} + \# \text{faces}. \]

Expressed as a function of the number \( n \) of objects.

In the **worst-case** position.

Possibly for a **restricted class** of objects.

*Line, segments, circles, squares, rectangles...*

*In arbitrary position, congruents under translation...*
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**Question:** What is the complexity of the lower envelope of \( n \) segments in the plane?
Lower envelope of $n$ segments $\rightarrow$ word on $\{1, 2, \ldots n\}$.

*Number the segments (arbitrarily).*

*Read off the sequence of segments appearing on the lower envelope.*
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This word obeys (at least) two rules:

1. Consecutive letters are distinct.

2. For any letter \( a, b \) there is no sub-word \( a \ldots b \ldots a \ldots b \ldots a \).
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Define $\lambda_3(n)$ as the maximum length of a word on $\{1, 2, \ldots, n\}$ obeying these two rules.
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**Theorem.** [Hart-Sharir, 1986] \( \lambda_3(n) = \Theta(n\alpha(n)) \).

\( \alpha(n) \) is the inverse of the Ackermann function \( n \mapsto A_n(n) \) defined by \( A_2(n) = 2^n \) and \( A_k(n) = A_{k-1}(A_k(n-1)) \).

\( \alpha(n) \leq 4 \) for \( n \leq 2^{2^{\ldots^2}} \) (tower of 65536 exponentials).

Related to path compressions on trees and inspired by union-find structures [Tarjan 1975].
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**Theorem.** The lower envelope of $n$ segments in the plane has complexity $\Theta(n \alpha(n))$. 

Lower envelope of \( n \) segments \( \rightarrow \) word on \( \{1, 2, \ldots n\} \).

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One of many applications of Davenport-Schinzel sequences.
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The number of triangulations grows exponentially with the number $n$ of points.

$$\frac{1}{n-1} \binom{2n-4}{n-2} \sim \frac{4^n}{n\sqrt{\pi n}}$$ triangulations for $n + 2$ points in convex position in $\mathbb{R}^2$.

Any planar set of $n$ points has $\Omega(2.43^n)$ and $O(30^n)$ triangulations and there exist examples with $\Omega(8.65^n)$ triangulations.
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*Exception: cospherical $(d + 2)$-tuples of points need to be handled separately.*

Nice properties, eg. maximizes the smallest angle.

$\text{Triangulation} \rightarrow \text{vector of its angles in increasing order}.$

$\text{Triangulation de Delaunay} \rightarrow \text{lexicographically maximal vector}.$
$S = \{p_1, p_2 \ldots p_n\}$ a set of points in $\mathbb{R}^2$.

The **Voronoi diagram** of $S$ is the partition of $\mathbb{R}^2$ into regions $R_1, \ldots R_n$ where

$R_i = \{x \in \mathbb{R}^2 | \forall j \in \{1, 2, \ldots, n\} \setminus \{i\}, \|xp_j\| \geq \|xp_i\|\}$
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 Captures growth phenomena.
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*Also known as Dirichlet tesselations.*

*Used in meteorology under the name of Thiessen polygons.*

*Used in chemistry under the name of Wigner-Seitz cell.*

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Generalizes to arbitrary families of subsets of a **metric space**.
Dualities between Delaunay triangulation and Voronoi diagram.
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Combinatorial

Voronoi vertex
    $= \text{center of a ball circumscribed to a Delaunay simplex.}$
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Quadratic form (here for $d = 2$)

Let $\Gamma$ be the paraboloid in $\mathbb{R}^3$ with equation $z = x^2 + y^2$.
We “lift” $q = (x, y)$ to $q' = (x, y, x^2 + y^2) \in \Gamma$.
The dual of $q'$ is $h(q')$, the tangent plane to $\Gamma$ in $q'$.
**Dualities** between Delaunay triangulation and Voronoi diagram.

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The Voronoi diagram of $p_1, p_2, \ldots, p_n$  
$\simeq$ upper enveloppe of $h(p_1'), h(p_2'), \ldots, h(p_n')$.

Delaunay triangulation of $p_1, p_2, \ldots, p_n$  
$\simeq$ convex hull of $p_1', p_2', \ldots, p'_n$.

Let $q_1, q_2 \in \mathbb{R}^2$ and let $q_2''$ be the lift of $q_2$ on $h(q_1')$.

$h(q_1') : 2x_1(x - x_1) + 2y_1(y - y_1) - z + x_1^2 + y_1^2 = 0$

height of $q_2'' : (2x_1(x_2 - x_1) + 2y_1(y_2 - y_1) + x_1^2 + y_1^2)$

height of $q_2' : x_2^2 + y_2^2$.

$\Rightarrow ||q_1q_2||^2 = ||q_2q_2''||$. 
**Dualities** between Delaunay triangulation and Voronoi diagram.

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\(\simeq\) upper envelope of \(h(p'_1), h(p'_2), \ldots, h(p'_n)\).

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\(\simeq\) convex hull of \(p'_1, p'_2, \ldots, p'_n\).

**Topological**

Delaunay = Nerve(Voronoi). cf. “topological methods”

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height of \(q'_2\) : \(x_2^2 + y_2^2\).

\(\Rightarrow \|q_1q_2\|^2 = \|q_2q'_2\|\).
Incremental algorithm to construct Delaunay triangulations.

\[
T \leftarrow \{p_1p_2p_3\} \quad // \text{We assume } p_1p_2p_3 \text{ contains all the points}
\]

For \( i = 4 \ldots n \)

1. Find the triangle \( pqr \) in \( T \) that contains \( p_i \).
2. \( T \leftarrow T \setminus \{pqr\} \cup \{p_ipq, p_iqr, p_irp\} \)
3. Make \( T \) Delaunay by flips.

Localization of point \( p_i \)
Incremental algorithm to construct Delaunay triangulations.

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Localization

Subdivision

Correction
Incremental algorithm to construct Delaunay triangulations.

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T \leftarrow T \setminus \{pqr\} \cup \{p_iq,p_iq,p_irp\}
\]
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Incremental algorithm to construct Delaunay triangulations.

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**Analysis** of the correction phase:
\[ d_i = \text{degree of } p_i \text{ in } \text{Del}(\{p_1, p_2, \ldots, p_i\}). \]
At most \( d_i \) flips.
Backward analysis for a **random** insertion order.
Efficient **localization** uses (random) walks.

Start from a known triangle.
Walk in the triangulation
(triangles have pointers to adjacent triangles).
Various walking strategies exist.

Example of **open question**.

Start from a **known** triangle.
Choose one of its edges whose supporting line separates the target point from the interior of the current triangle.
Cross this edge to find the new current triangle.
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**Conjecture:** Let $P$ be a set of $n$ random points chosen uniformly in $[0, 1]^2$. The expectation of the maximum number of steps in a visibility walk is $O(\sqrt{n})$. 
Efficient **localization** uses (random) walks.

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Other open problem: probabilistic complexity, in particular **smoothed complexity**.

[Spielman-Tang'04]
Some classics
Arrangements
Delaunay triangulations & Voronoi diagrams

New development: algebraic/polynomial methods
Incidence geometry, around the Szemeredi-Trotter theorem
Solution to the joint problem by algebraic arguments

New developments: topological methods
Topological combinatorics and inclusion-exclusion formulas
Persistent homology and topological inference
An incidence between a set $C$ of curves and a set $P$ of points is a pair $(c, p) \in C \times P$ such that $c \in p$.

Problem: estimate the maximum number of incidences given $|C|$, $|P|$ and the nature of the curves in $C$. 
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**Problem:** estimate the maximum number of incidences given $|C|$, $|P|$ and the nature of the curves in $C$.

**Theorem.** [Szemeredi-Trotter, 1983] The number of incidences between $n$ points and $\ell$ lines in the plane is $O(n + \ell + n^{2/3} \ell^{2/3})$. 
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Application:

**Sum-product conjecture** [Erdös-Szemeredi, 1983]: For any $\epsilon > 0$ there exists $C' \in \mathbb{R}$ such that for any $A \subset \mathbb{N}$ finite and large enough,

$$\max\{|A + A|, |A \ast A|\} \geq C'|A|^{2-\epsilon}.$$ 

$$A + A = \{x + y \mid x, y \in A\} \text{ and } A \ast A = \{x \ast y \mid x, y \in A\}$$
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**Application:**

**Sum-product conjecture** [Erdös-Szemeredi, 1983]: For any \( \epsilon > 0 \) there exists \( C \in \mathbb{R} \) such that for any \( A \subset \mathbb{N} \) finite and large enough, \( \max\{|A + A|, |A \cdot A|\} \geq C |A|^{2-\epsilon} \).

\[
A + A = \{x + y \mid x, y \in A\} \text{ and } A \cdot A = \{x \cdot y \mid x, y \in A\}
\]

Associate to \( A \) the points \( P = (A + A) \times (A \cdot A) \) and lines \( L = \{y = a(x - b) : a, b \in A\} \).

The number of incidence is **at least** \( |A||L| \).

*any line \( y = a(x - b)\) contains every point \((c + b, a \cdot c)\) for \( c \in A\).*

\[
\text{Szemeredi-Trotter } \Rightarrow |A||L| = O(|P|^{2/3}|L|^{2/3}) \Rightarrow |P| = \Omega(|A|^{3/2}|L|^{1/2}) = \Omega(|A|^{5/2}).
\]

\[
|P| = |A + A||A \cdot A| \text{ implies } \max\{|A + A|, |A \cdot A|\} = \Omega(|A|^{5/4}).
\]

[Elekes, 1997]
Preparation for the proof of the Szemeredi-Trotter theorem.

**Crossing lemma.** Let $G$ be a graph with $v$ vertices and $e \geq 4v$ edges. In any drawing of $G$ in the plane, at least $\frac{e^3}{64v^2}$ pairs of edges cross.

[Leighton, 1983] [Ajtai-Chvátal-Newborn-Szemeredi, 1982]

Conjectured by [Erdős-Guy, 1972]

edges = Jordan arcs
crossing = intersection other than a common endpoint.
Preparation for the proof of the Szemeredi-Trotter theorem.

**Crossing lemma.** Let $G$ be a graph with $v$ vertices and $e \geq 4v$ edges. In any drawing of $G$ in the plane, at least $\frac{e^3}{64v^2}$ pairs of edges cross.

Proof: $\chi$ a drawing of $G$ and $Cr(\chi) = \text{its number of crossings.}$

\[ (*) \quad Cr(\chi) \geq e - 3v. \]

Any planar graph has at most $3v - 3$ edges (Euler relation + double counting).

Fix $p \in [0, 1]$ and delete every vertex of $G$ with probability $1 - p$ (independently).

We obtain a random graph $G_p$ and a drawing $\chi_p$ of $G_p$ induced by $\chi$.

\[ (*) \Rightarrow p^4Cr(\chi) = E[Cr(\chi_p)] \geq p^2e - 3pv \quad \text{so} \quad Cr(\chi) \geq \frac{e}{p^2} - 3\frac{v}{p^3} \]

Setting $p = \frac{4v}{e}$ yields $Cr(\chi) \geq \frac{e^3}{16v^2} - 3\frac{e^3}{64v^2} = \frac{e^3}{64v^2}$. \[ \square \]
**Theorem.** [Szemeredi-Trotter, 1983] The number of incidences between \( n \) points and \( \ell \) lines in the plane is \( O(n + \ell + n^{2/3} \ell^{2/3}) \).

**Proof:**

Start with a family of \( n \) points and \( \ell \) lines. Let \( k \) be the number of point/line incidences.

Consider the graph with vertices the \( n \) points and edges the pairs of points **incident** to a common line and **consecutive** on that line.

This graph has at most \( k \) edges.
**Theorem.** [Szemerédi-Trotter, 1983] The number of incidences between $n$ points and $\ell$ lines in the plane is $O(n + \ell + n^{2/3}\ell^{2/3})$.

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In the straight-line drawing of this graph there are at most $\binom{\ell}{2}$ crossings.

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Start with a family of $n$ points and $\ell$ lines.

Let $k$ be the number of point/line incidences.

Consider the graph with vertices the $n$ points and edges the pairs of points *incident* to a common line and *consecutive* on that line.

This graph has at most $k$ edges.

In the straight-line drawing of this graph there are at most $\binom{\ell}{2}$ crossings.

By the **crossing lemma**, any drawing of a graph with $v$ vertices and $e \geq 4v$ edges has at least $\frac{e^3}{64v^2}$ crossings.

$$k < 4n \text{ or } \frac{k^3}{64n^2} \leq \binom{\ell}{2} \quad \Rightarrow \quad k = O(n + \ell + n^{2/3} \ell^{2/3}).$$
A **joint** in a set of lines is a point incident to 3 non-coplanar lines.

**Question:** what is the maximum number of joints in a set of $n$ lines in $\mathbb{R}^3$?

**Motivation:** understand **cycles** in **depth orders**.
A joint in a set of lines is a point incident to 3 non-coplanar lines.

Question: what is the maximum number of joints in a set of \( n \) lines in \( \mathbb{R}^3 \)?

Motivation: understand cycles in depth orders.

Lower bound: \( \Omega(n\sqrt{n}) \).

\[ \sqrt{n} \left\{ \begin{array}{c} \text{\sqrt{n}} \\ \text{\sqrt{n}} \end{array} \right\} \left\{ \begin{array}{c} \text{\sqrt{n}} \end{array} \right\} \]

\[ 3n \text{ lines} \]

\[ n\sqrt{n} \text{ joints} \]
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- **Lower bound:** \( \Omega(n\sqrt{n}) \).

- **Upper bounds:**
  
  \[
  O(n^{7/4}) \rightarrow O(n^{23/14} \log^{31/14} n) \rightarrow O(n\sqrt{n})
  \]

[Chazelle et al.’92] [Sharir’94] [Guth-Katz’08]

*Uses an idea from Dvir’s solution to the discrete Kakeya problem.*
A Kakeya set is a subset of $\mathbb{R}^d$ containing a unit-length segment in every direction.
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**Kakeya conjecture.** Any Kakeya set in $\mathbb{R}^d$ has dimension $d$.

Need clarifications: which classes of sets? Which dimension?...

Proven for $d = 2$ and open for any $d \geq 3$. 
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Discrete analogue of the Kakeya conjecture:

Fix a field $\mathbb{F}$ with $q$ elements.

A discrete Kakeya set is a subset $K \subseteq \mathbb{F}^d$ satisfying: for any $x \in \mathbb{F}^d$ there exists $y \in \mathbb{F}^d$ such that $\{y + ax : a \in \mathbb{F}\} \subseteq K$.

How large must a discrete Kakeya set be?
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How large must a discrete Kakeya set be?

**Lower bounds:** $\Omega(q^{\frac{d+2}{2}}) \rightarrow \Omega(q^{\frac{4}{7}d}) \rightarrow \Omega(q^{d-1})$

[Wolff’99] [Bourgain-Katz-Tao’04] [Mockenhaupt-Tao’04][Dvir’09]

Dvir’s idea: a non-zero polynomial of deg. $b$ has at most $bq^{d-1}$ zeros. [Zippel’79][Shwartz’80]
**Theorem.** [Guth-Katz, 2008] Any set of $n$ lines in $\mathbb{R}^3$ has $O(n\sqrt{n})$ joints.

**Proof:** Let $L$ be a set of $n$ lines in $\mathbb{R}^3$ with $j$ joints.

We can assume that every line of $L$ contains at least $\frac{j}{2n}$ joints.

Eliminate any line with less than $\frac{j}{2n}$ joints and any joint on less than 3 remaining lines.

If $\alpha n$ lines were eliminated then at most $\frac{\alpha}{2} j$ joints were eliminated.

$$(1 - \frac{\alpha}{2})j = O((1 - \alpha)^{3/2} n^{3/2}) \Rightarrow j = O(n^{3/2}).$$
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Choose a polynomial $P(x, y, z) \neq 0$ vanishing in every joints and with minimal total degree $b$.

Every joint defines a linear constraint in the $\binom{b+3}{3}$ coefficients of $P$ so $b = O(j^{1/3})$. 

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Minimality imposes $b \geq \frac{j}{2n}$.

Else every line of $L$ is contained in $P(x, y, z) = 0$ (Bezout).

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Choose a polynomial \(P(x, y, z) \neq 0\) vanishing in every joints and with minimal total degree \(b\).

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Thus, \(\frac{j}{2n} = O(j^{1/3})\) and we have \(j = O(n^{3/2})\).

\(\square\)
Stimulated a re-examination of old questions through a polynomial lens.

Erdős’ **distinct distances** problem.

Question: how few distinct distances are determined by $n$ points in the plane?
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**Erdős’ distinct distances problem.**

**Question:** how few distinct distances are determined by \( n \) points in the plane?

\( O(n/\sqrt{\log n}) \) for regular grids, conjectured to be minimal.

Slow progress on lower bounds of the form \( \Omega(n^c) \).

\[
c = \frac{1}{2} \rightarrow \frac{2}{3} \rightarrow \frac{5}{7} \rightarrow \frac{4}{5} \rightarrow \frac{6}{7} \rightarrow \frac{4e}{5e-1} - \epsilon \rightarrow \frac{48-14e}{55-16e} - \epsilon
\]

[Erdős'46][Moser’52][Chung’84][Chung-Szemerédi-Trotter’92][Szekely’93][Solymosi-Tóth’01][Tardos’03][Katz-Tardos’04]
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\( \Omega(n/\log n) \) by incidence geometry + polynomial method.

\[\text{[Guth-Katz’15] using [Elekes-Sharir’10]}\]
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**Partitioning** theorems.

Partition a set of \( n \) points into \( r \) balanced subsets so that any line intersects \( O(\sqrt{r}) \) of the convex hulls of the subsets.

*Cuttings, simplicial partition theorem...* [Chazelle-Friedman’90]

*Algorithmic applications (eg. range searching) by divide-and-conquer.*

[Chazelle’93][Matoušek & Chazelle’88–’93]

*Complicated proofs, not effective.*
Stimulated a re-examination of old questions through a polynomial lens.

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[Chazelle'93][Matoušek & Chazelle'88-'93]

*Complicated proofs, not effective.*
Stimulated a re-examination of old questions through a polynomial lens.

**Erdős’ distinct distances** problem.

**Question:** how few distinct distances are determined by $n$ points in the plane?

$O(n/\sqrt{\log n})$ for regular grids, conjectured to be minimal.

Slow progress on lower bounds of the form $\Omega(n^c)$.

$$c = \frac{1}{2} \rightarrow \frac{2}{3} \rightarrow \frac{5}{7} \rightarrow \frac{4}{5} \rightarrow \frac{6}{7} \rightarrow \frac{4e}{5e-1} - \epsilon \rightarrow \frac{48-14e}{55-16e} - \epsilon$$

[Chazelle-Friedman’90] [Chazelle’93] [Matoušek & Chazelle’88-’93]

$\Omega(n/\log n)$ by incidence geometry + polynomial method.


**Partitioning** theorems.

Partition a set of $n$ points into $r$ balanced subsets so that any line intersects $O(\sqrt{r})$ of the convex hulls of the subsets.

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Algorithmic applications (eg. range searching) by divide-and-conquer.

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Complicated proofs, not effective.

Simplified and strengthened using the polynomial ham-sandwich theorem.
Some classics
  Arrangements
  Delaunay triangulations & Voronoi diagrams

New development: algebraic/polynomial methods
  Incidence geometry, around the Szemeredi-Trotter theorem
  Solution to the joint problem by algebraic arguments

New developments: topological methods
  Topological combinatorics and inclusion-exclusion formulas
  Persistent homology and topological inference
Geometric graph theory studies graphs through their embedding properties.

Rich theory of **planar graphs**.

- \# edges \leq 3\# vertices - 3
- Circle packing theorem [Koebe][Thurston][Andreev]
- Structural properties (eg. planar separator theorem).
- Characterization by excluded minors.
  ...

Generalizes to the theory of graphs embedding on a **compact 2-manifold**.

- Euler characteristic, characterization by excluded minors...
- Heawood inequality: if $K_n$ embeds into $M$ then
  \[ (n - 3)(n - 4) \leq 6b_1(M) = 12 - 6\chi(M). \]
  
  \[ M \text{ is a compact 2-manifold and } b_i(M) \text{ is its } i\text{th Betti number.} \]
Generalization of graphs: (uniform) **hypergraphs** model $r$-wise interactions for $r > 2$.

*Hypergraph with vertex set $V = \text{set of subsets of } V$. Uniform if all subsets have the same size.*
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Generalization of **embedded** graphs: **simplicial complexes**.

abstract simplicial complex

“Collection of sets that is closed under taking subsets.”

\{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{1, 3\}, \{1, 4\}\}

general geometric simplicial complex

“Collection of geometric simplices in $\mathbb{R}^d$ s.t. any two are disjoint or intersect in a common face.”
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Generalization of embedded graphs: simplicial complexes.

- **Abstract simplicial complex**
  - "Collection of sets that is closed under taking subsets."
  - \{∅, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{1, 3\}, \{1, 4\}\}

- **Geometric simplicial complex**
  - "Collection of geometric simplices in $\mathbb{R}^d$ s.t. any two are disjoint or intersect in a common face."

- **Geometric realization**: map singletons to points in general position in $\mathbb{R}^d$, $d$ large enough, take convex hulls of points corresponding to abstract simplices.
Conjecture [Grünbaum][Sarkaria][Kalai][Dey] For any $d$ there exists $C_d$ such that $f_d(K) \leq C_d f_{d-1}(K)$ for any finite simplicial complex $K$ embedding in $\mathbb{R}^d$.

$f_i(K)$ is the number of faces of dimension $i$ of $K$.

Known for $d = 2, 3$ and open for any $d \geq 4$. 
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We can study these **combinatorial** objects through their associated **topological** space.

If \( K \) is a simplicial complex and \( |K| \) is its realization then

\[
\chi(K) = \sum_{i \geq 0} (-1)^i f_i(K) = \sum_{i \geq 0} (-1)^i \beta_i(|K|)
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Nerves are simplicial complexes.

**Nerve Theorem.**[Borsuk’48][Leray] If $\mathcal{F}$ is a good cover in $\mathbb{R}^d$ then $\mathcal{N}(\mathcal{F}) \simeq \bigcup \mathcal{F}$.

good cover = any subfamily has empty or contractible intersection.
We can use simplicial complexes to simplify **inclusion-exclusion formulas**.

For any family $\mathcal{F} = \{A_1, A_2, \ldots, A_n\}$ of sets $1 \bigcup_{i=1}^{n} A_i = \sum_{\emptyset \neq S \subseteq [n]} (-1)^{|S|+1} 1 \bigcap_{i \in S} A_i$.

$1_X$ is the indicator function of $X$.

**Question:** can we express $1_{\bigcup}$ using fewer $1_{\bigcap}$'s?

*Effective volume computation, inclusion-exclusion algorithms*
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*Effective volume computation, inclusion-exclusion algorithms*

Not in general...

\[
1_{A \cup B \cup C} = \lambda_a 1_A + \lambda_b 1_B + \lambda_c 1_C + \lambda_{ab} 1_{A \cap B} + \lambda_{ac} 1_{A \cap C} + \lambda_{bc} 1_{B \cap C} + \lambda_{abc} 1_{A \cap B \cap C}
\]

\( \Rightarrow \lambda_a = 1 \quad \Rightarrow \lambda_b = 1 \quad \Rightarrow \lambda_a + \lambda_b + \lambda_{ab} = 1 \Rightarrow \lambda_{ab} = -1 \)

*Set systems with complete Venn diagram have only one inclusion-exclusion formula.*
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1_{A \cup B \cup C} = 1_A + 1_B + 1_C - 1_{A \cap B} - 1_{A \cap C} - 1_{B \cap C} + 1_{A \cap B \cap C} \text{ since } A \cap C = A \cap B \cap C.
\]

*Any set system has a inclusion-exclusion formula quasi-polynomial in the number of sets and the size of the Venn diagram.*

[Goaoc-Matoušek-Paták-Safernova-Tancer’15]
Let $\mathcal{F} = \{B_1, B_2, \ldots, B_n\}$ be a family of unit balls in $\mathbb{R}^d$.

Write $p_i$ for the center of $B_i$.

$K$ the simplicial complex encoding the Delaunay triangulation of $\{p_1, p_2, \ldots, p_n\}$.

$\sigma \in K \iff \{p_i : i \in \sigma\}$ has no $p_j$ in the interior of its circumscribed ball.

**Theorem** [Naiman-Wynn’92’97] \[ \bigcup_{i=1}^{n} B_i = \sum_{\sigma \in K} (-1)^{\dim \sigma} 1_{\cap_{i \in \sigma} B_i}. \]

For $d = 2$, $6n - 11$ terms, each of size at most 3, and computable in $O(n \log n)$ time.

For $d \geq 3$, $O(n^{\lceil d/2 \rceil})$ terms, each of size at most $d$, and computable in $O(n^{\lceil d/2 \rceil})$ time.

Idem for balls of different radii but uses the power diagram.

“A correct inclusion-exclusion formula for a family of unit radius balls in $\mathbb{R}^d$ is given by the Delaunay triangulation of their centers.”
Let $\mathcal{F} = \{B_1, B_2, \ldots, B_n\}$ be a family of unit balls in $\mathbb{R}^d$.

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“A correct inclusion-exclusion formula for a family of unit radius balls in $\mathbb{R}^d$ is given by the Delaunay triangulation of their centers.”

It suffices to prove that for any $x \in \bigcup_{i=1}^n B_i$

$$\sum_{\sigma \in K} (-1)^{\text{dim } \sigma} \mathbf{1}_{\bigcap_{i \in \sigma} B_i}(x) = 1.$$
Claim. For any $x \in \bigcup_{i=1}^{n} B_i$, \( \sum_{\sigma \in K} (-1)^{\dim \sigma} 1_{\cap_{i \in \sigma} B_i}(x) = 1 \).

$B_1, B_2, \ldots, B_n$ unit balls in $\mathbb{R}^d$ with $B_i$ centered in $p_i$.

$R_i = \{ y \in \mathbb{R}^d \mid \forall j \in \{1, 2, \ldots, n\} \setminus \{i\}, \|yp_j\| \geq \|yp_i\| \}$

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$\sigma \in K \Leftrightarrow \{p_i : i \in \sigma\}$ has no $p_j$ in the interior of its circumscribed ball.

$\mathbf{Claim.}$ For any $x \in \bigcup_{i=1}^{n} B_i$, \( \sum_{\sigma \in K} (-1)^{\dim \sigma} 1_{\cap_{i \in \sigma} B_i}(x) = 1 \).
Claim. For any $x \in \bigcup_{i=1}^{n} B_i$, $\sum_{\sigma \in K} (-1)^{\dim \sigma} 1_{\cap_i \in \sigma} B_i(x) = 1$.

Proof: Define $F_x = \{i : x \in B_i\}$. It suffices to show that $K[F_x]$ is contractible:

$$\sum_{\sigma \in K} (-1)^{\dim \sigma} 1_{\cap_i \in \sigma} B_i(x) = \sum_{\sigma \in K; \sigma \subseteq F_x} (-1)^{\dim \sigma} = \chi(K[F_x]).$$

$K[F_x]$ is the subcomplex of $K$ induced by $F_x = \text{all simplices contained in } F_x$. 

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**Claim.** For any $x \in \bigcup_{i=1}^{n} B_i$, $\sum_{\sigma \in K} (-1)^{\dim \sigma} 1_{\cap_{i \in \sigma} B_i}(x) = 1$.

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Delaunay = Nerve(Voronoi) $\Rightarrow$ $K[\mathcal{F}_x]$ is the nerve of $\{R_i : x \in B_i\}$.

Nerve theorem $\Rightarrow$ it suffices to argue that $\bigcup_{i : x \in B_i} R_i$ is contractible.
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$B_1, B_2, \ldots, B_n$ unit balls in $\mathbb{R}^d$ with $B_i$ centered in $p_i$.

$R_i = \{ y \in \mathbb{R}^d \mid \forall j \in \{1, 2, \ldots, n\} \setminus \{i\}, \|yp_j\| \geq \|yp_i\| \}$

$K$ the simplicial complex encoding the Delaunay triangulation of $\{p_1, p_2, \ldots, p_n\}$.

Claim. For any $x \in \bigcup_{i=1}^n B_i$, $\sum_{\sigma \in K} (-1)^{\dim \sigma} 1_{\cap_i \in \sigma B_i}(x) = 1$.

Proof: Define $F_x = \{ i : x \in B_i \}$. It suffices to show that $K[F_x]$ is contractible:

$$\sum_{\sigma \in K} (-1)^{\dim \sigma} 1_{\cap_i \in \sigma B_i}(x) = \sum_{\sigma \in K; \sigma \subset F_x} (-1)^{\dim \sigma} = \chi(K[F_x]).$$

Delaunay = Nerve(Voronoi) $\Rightarrow$ $K[F_x]$ is the nerve of $\{R_i : x \in B_i\}$.

Nerve theorem $\Rightarrow$ it suffices to argue that $\bigcup_{i : x \in B_i} R_i$ is contractible.

Renumber the $p_i$ so that $\|xp_1\| \leq \|xp_2\| \leq \ldots \leq \|xp_n\|$.

Define $R'_i = \{ y \in \mathbb{R}^d \mid \forall j \in \{i + 1, i + 2, \ldots, n\}, \|yp_j\| \geq \|yp_i\| \}$

$$\bigcup_{i : x \in B_i} R_i = \bigcup_{i : x \in B_i} R'_i$$ which is star-shaped wrt $x$. \qed
Another kind of use of topological methods: **topological data analysis**

Data is commonly obtained by **sampling**.
Distance/similarity notion influenced by the underlying topology/geometry.
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*Better understand the observed phenomena.*

*Improve the treatment of this data (dimension reduction, parameterization...)*

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> Applications: clustering, matching, classification, visualization, learning, ...

Need for intermediate constructions (simplicial complexes).
Distinguish "signal" from "topological noise", multi-scale information.
Exemple: **Betti numbers** inference.

$\beta_i(X)$ is the rank of the $i$th homology group $H_i(X)$.

$\beta_0 = \text{number of connected components}$.

$\beta_i$ formalizes the “number of independent holes of dimension $i$”.
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*The geometry/topology of \( X \) is not known (ex: all configurations of a given molecule).*

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We use the **reach** of \( X \).

*Distance from \( X \) to its medial axis.*
* = locus of the points with more than one nearest neighbour on \( X \).*
*positive if \( X \) is smooth.*
Approach: union of balls.

For $\varepsilon > 0$ let $S^{(\varepsilon)} = \bigcup_{p \in S} B(p, \varepsilon)$. 
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**Theorem.** [Nyogi-Smale-Weinberger’04] Let $X \subset \mathbb{R}^d$ be a smooth, compact manifold and $S \subset \mathbb{R}^d$ finite. For any $\varepsilon \in [d_H(P, X), (3 - \sqrt{8})r_X]$, 

$$\forall i \in \mathbb{N}, \quad \beta_i \left( S^{((2+\sqrt{2})\varepsilon)} \right) = \beta_i (X)$$

$r_X$: reach of $X$.

$$d_H(P, X) = \max\{\sup_{p \in P} \inf_{x \in X} \|px\|, \sup_{x \in X} \inf_{p \in P} \|px\|\} \ (\text{Hausdorff distance})$. 
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Intuition: take \( \varepsilon \) large enough to fill the holes in the sampling, small enough not to fill the holes in \( X \).
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**Algorithm:**

- **Input:** a sample $S$ of a space $X$.
- Compute a radius $\varepsilon$ such that $\beta_i \left( S^{(\varepsilon)} \right) = \beta_i (X)$.
- Compute the nerve $\mathcal{N}$ of the balls of radius $\varepsilon$ centered in $S$.
- Compute the Betti numbers of $\mathcal{N}$ using simplicial homology.
This algorithm can be improved using **Vietoris-Rips complexes**.

\[ S = \{p_1, p_2, \ldots, p_n\} \subset \mathbb{R}^d \text{ and } S_t = \{B(p_1, t), B(p_2, t), \ldots, B(p_n, t)\}. \]

The **Vietoris-Rips complex** \( R_t(S) \) of \( S \) with parameter \( t \) is the clique complex of the intersection graph of \( S_t \).

\[ R_t(S) = \{I \subset \{1, 2, \ldots, n\} \mid \forall i, j \in I, \|p_ip_j\| \leq 2t\}. \]
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\( R_t(S) \) is easier to compute than the nerve of \( S_t \).

*Only needs the 1-skeleton, makes for a greater numerical stability.*

*Critical because one extra or fewer simplex changes the \( \beta_i \).*

**Theorem.** [Attali-Lieutier-Salinas’10] For any smooth, compact manifold \( X \subset \mathbb{R}^d \), any finite subset \( S \subset \mathbb{R}^d \), any \( \varepsilon \in [d_H(P, X), 0.034r_X] \),

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\forall i \in \mathbb{N}, \quad \beta_i \left( R_{7.22\varepsilon}(S) \right) = \beta_i(X)
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*Proof: collapse Vietoris-Rips complexes onto nerves.*
Another improvement: consider the sequence of $S^{(r)}$ as $r$ ranges in $[0, +\infty]$. We obtain a family of nerves filtered by the radius of the balls.
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We can follow the “birth” and “death” of the generators of homology groups,

For $x < y$ the injection $S^{(x)} \hookrightarrow S^{(y)}$ induces a morphism $H_k(S^{(x)}) \to H_k(S^{(y)})$.

to obtain a persistence diagram (or a topological bar code).

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Effective computation based on Rips-Vietoris / nerve interleavings.

This persistence diagram is continuous in the space that is sampled

(If the metrics are chosen adequately.)

and provides a multiscale topological signature of the sampled space.
And the story could continue...
Discrepancy

[Weil][Roth][Beck][Spencer][Matoušek][Chazelle]… [Bansal]…

VC-dimension, sampling

[Vapnick-Chervonenkis][Alon][Frankl][Clarkson][Haussler-Welzl]…

Convex/combinatorial geometry, convex optimization, combinatorial LP…

[Lovasz][Kannan][Barvinok][Vempala][Kalai][Clarkson][Matoušek]…

Geometric Ramsey theory…

[Erdős][Szemeredi][Gowers][Tao][Pach][Fox]…

Embedding & dimension reduction, computational geometry on GPU, algebraic (hyper)graphs, matroids, optimal transport…
Thank you for your attention
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Further reading...
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And for your next linear algebra class check out