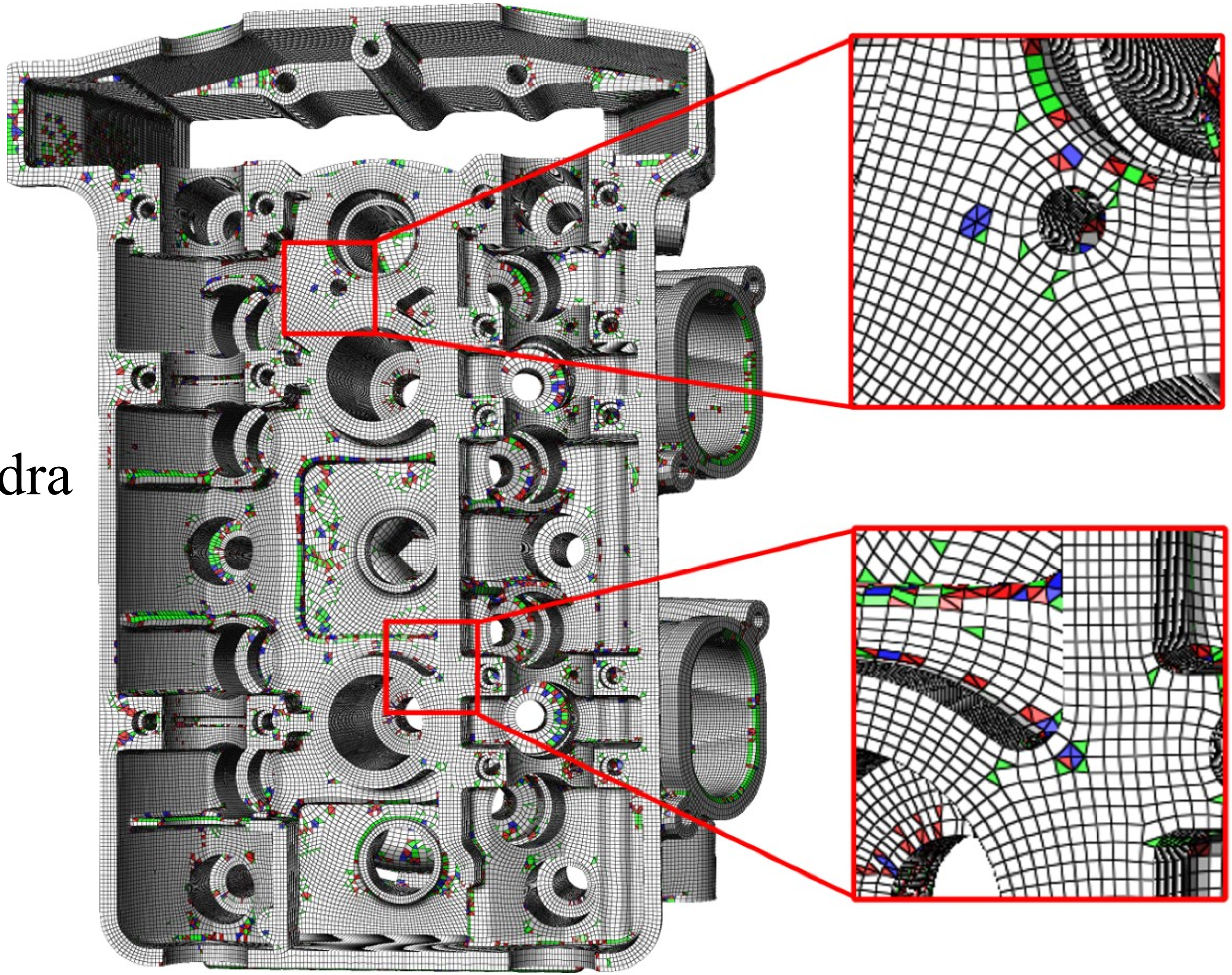
# Hexahedral-Dominant Meshing

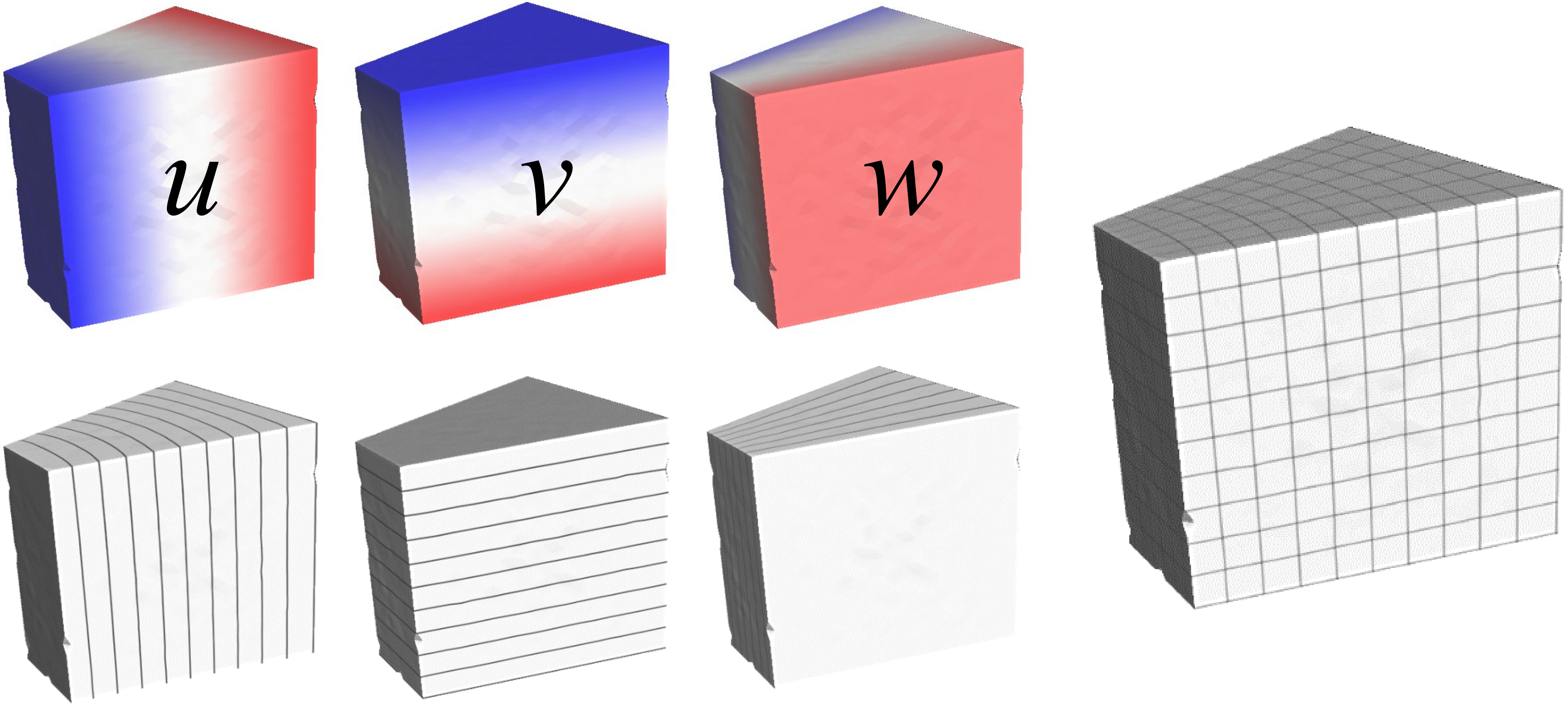
Dmitry Sokolov, Nicolas Ray,
Lionel Untereiner and Bruno Lévy

# Teaser!

Remesh $10^7$ tetrahedra models in several minutes on a laptop

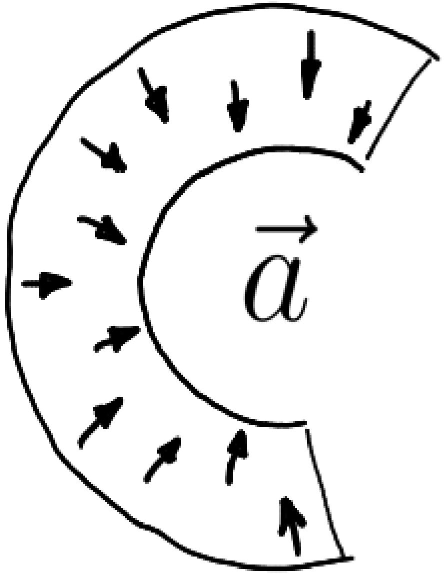


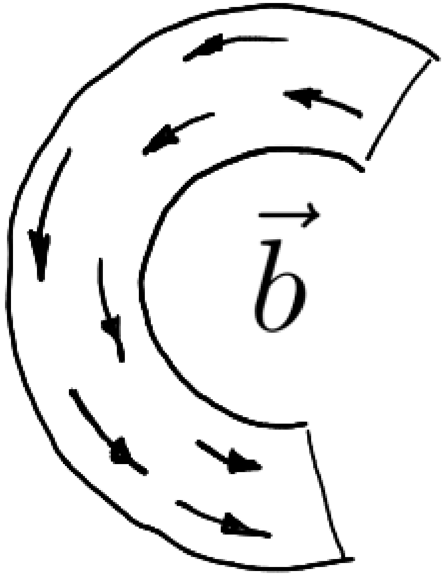
# Main idea: compute 3 scalar fields and cut along integer isovalues




in other words, compute texture coordinates *(u,v,w)* and apply a grid texture

# Introductory example 1 out of 3




Directional Field Synthesis, Design, and Processing
SIGGRAPH 2017 course

A. Vaxman, M. Campen, O. Diamanti, D. Panozzo,
B. D. Bommes, K. Hildebrandt, M. Ben-Chen

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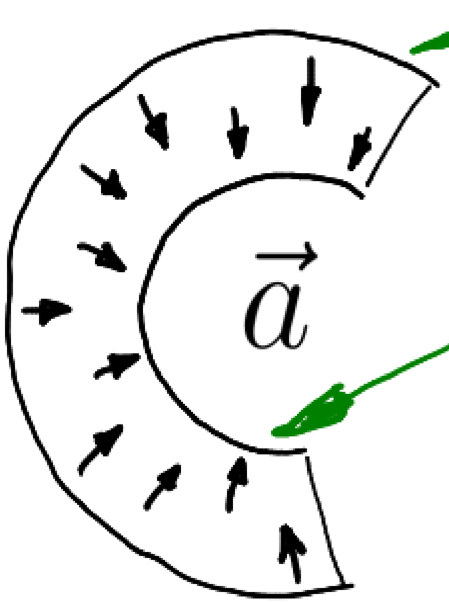
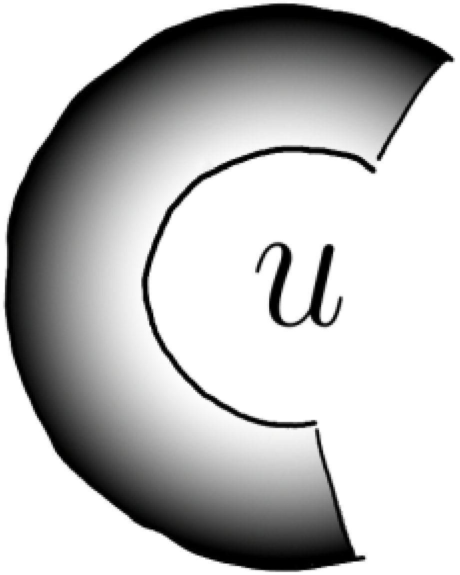
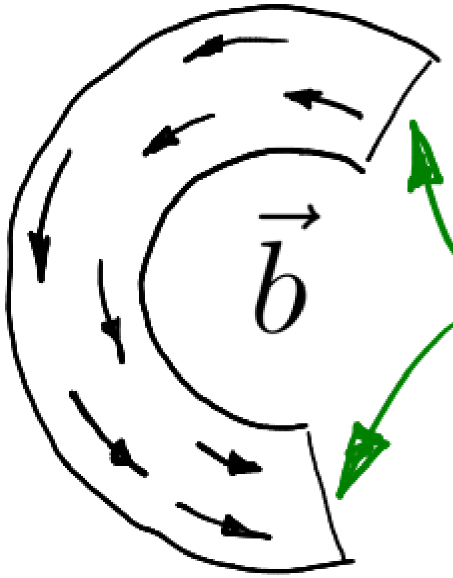
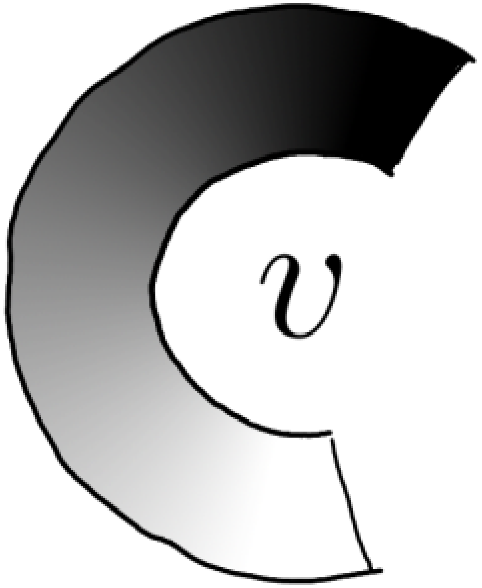



find $u$ such that:
$u$ is integer here
and minimizes $\int |\nabla u - \vec{a}|^2$




find $v$ such that:
$v$ is integer here
and minimizes $\int |\nabla v - \vec{b}|^2$



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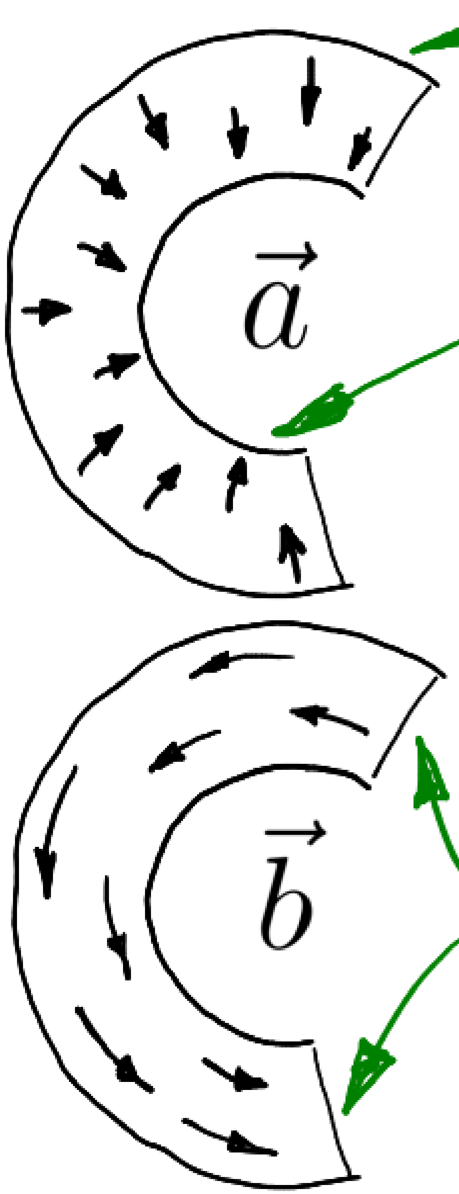


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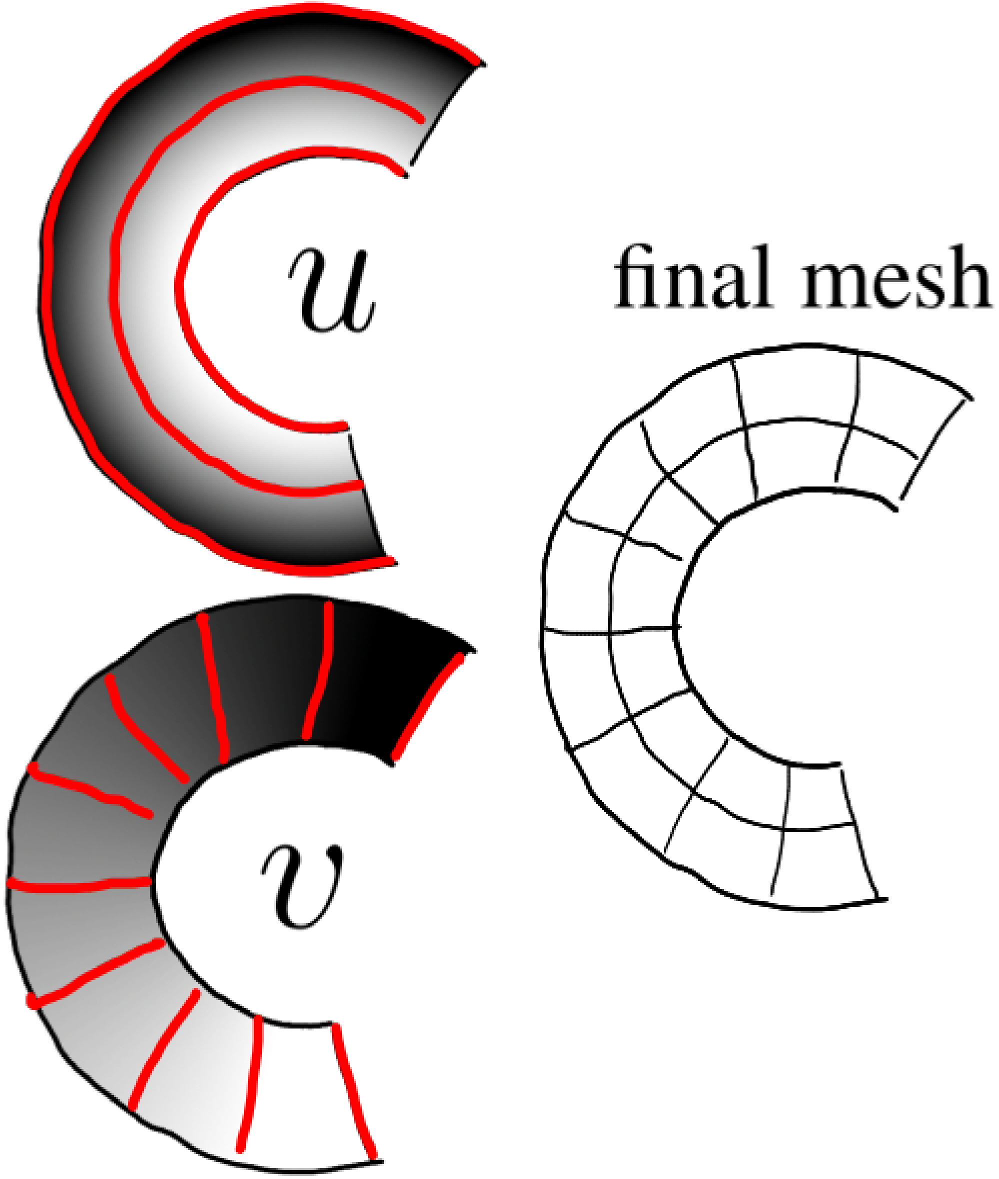
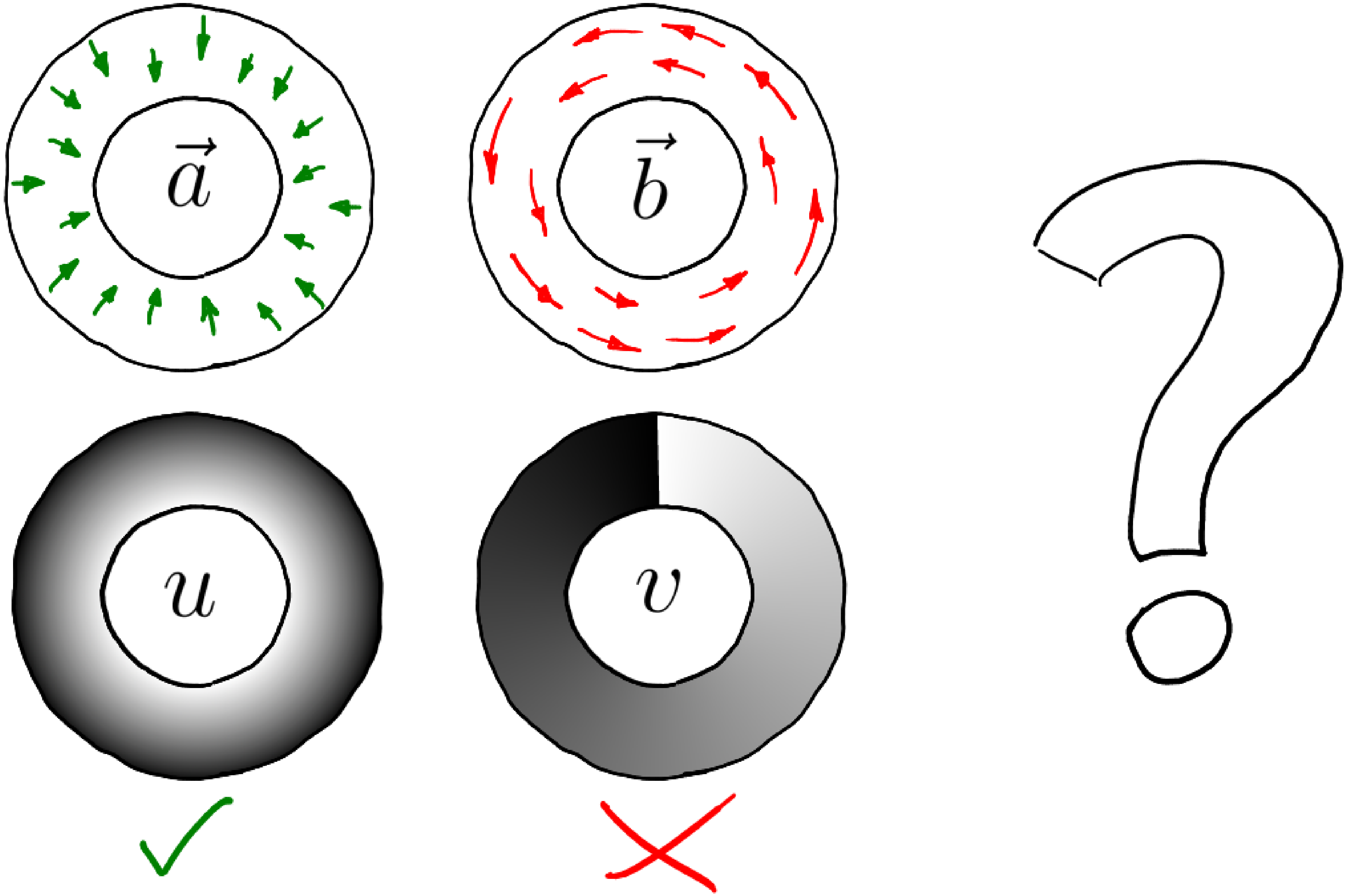


# Introductory example 2 out of 3



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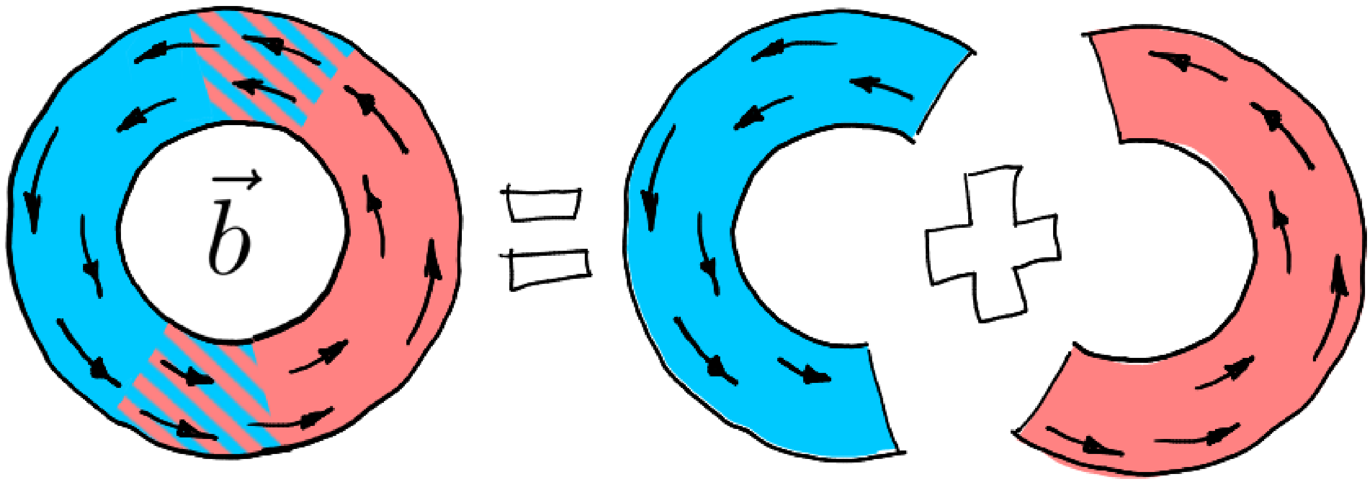

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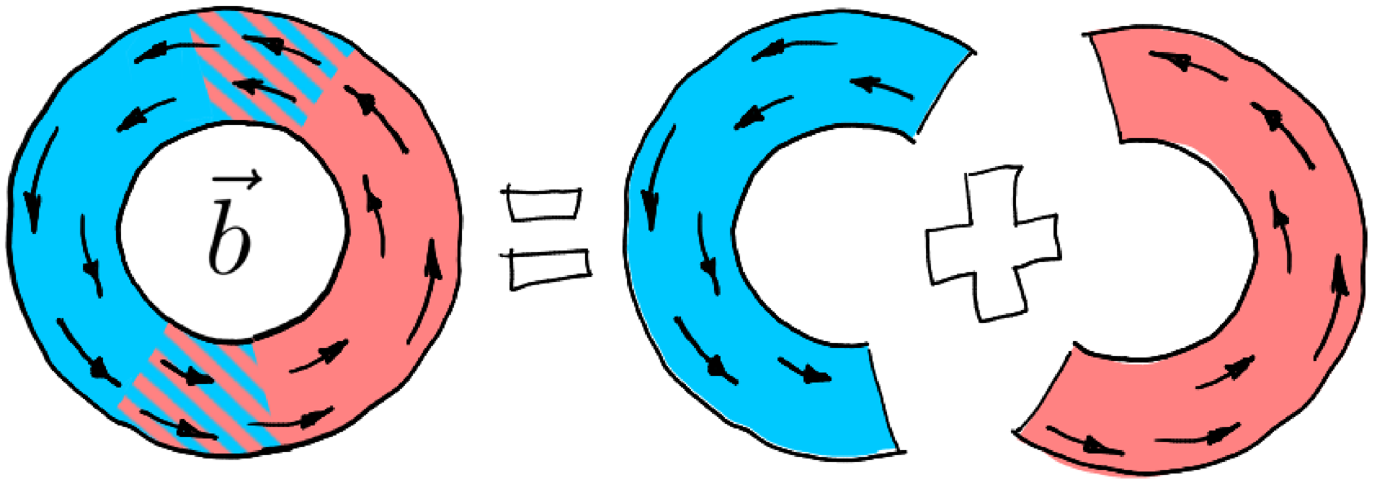
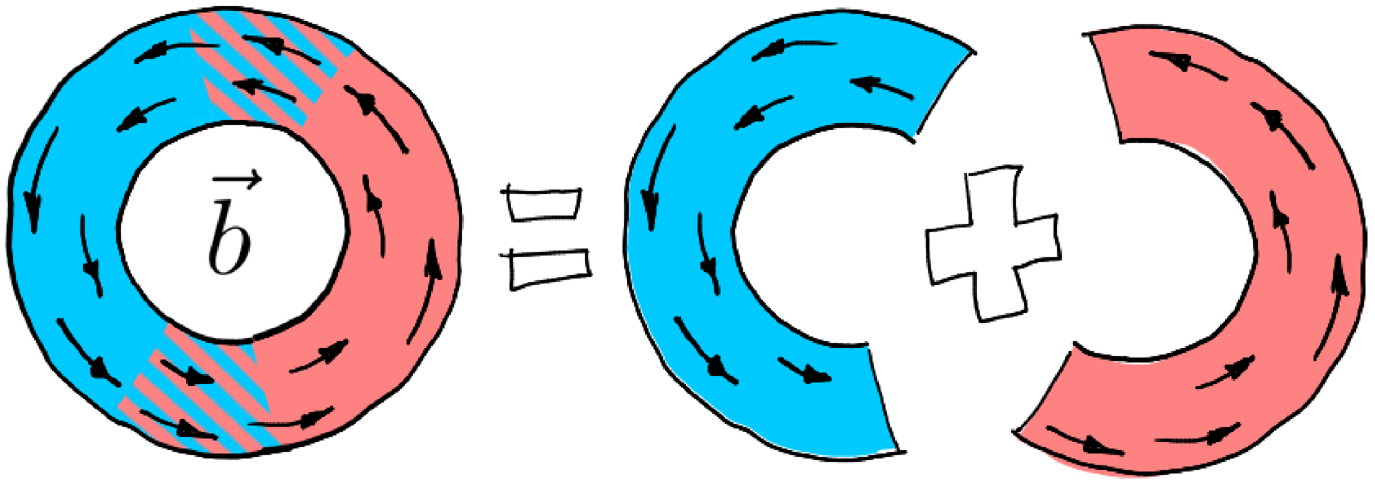



find $v_1$ and $v_2$ that minimize

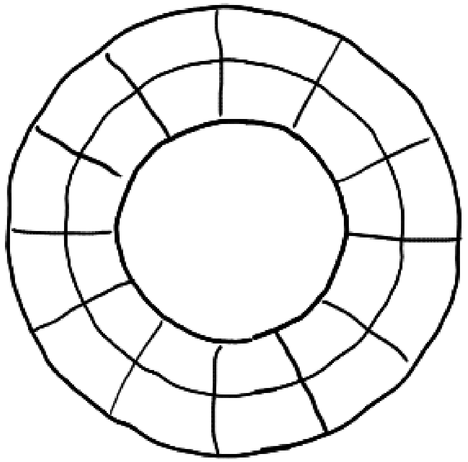
$\int |\nabla v_1 - \vec{b}|^2 + \int |\nabla v_2 - \vec{b}|^2$

$v_1 - v_2$ integer where defined

# Introductory example 2 out of 3




final mesh



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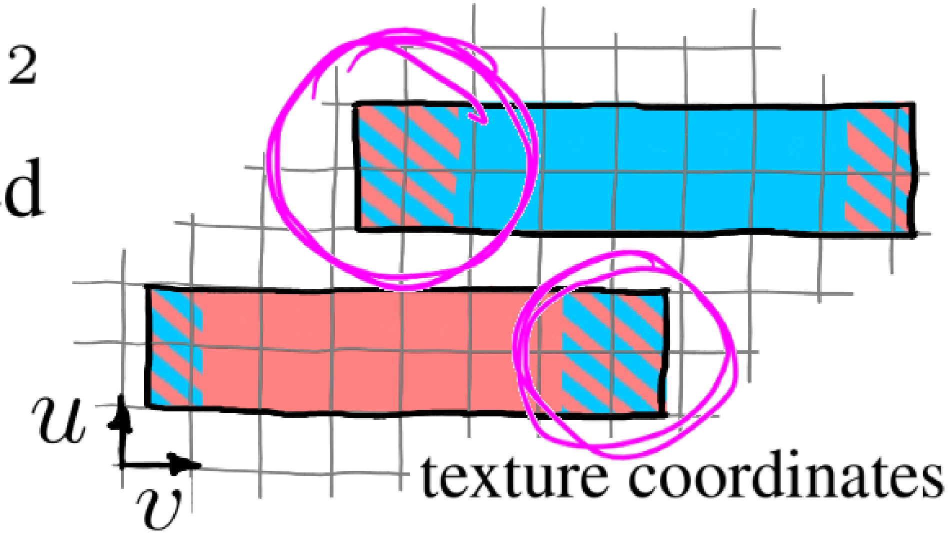
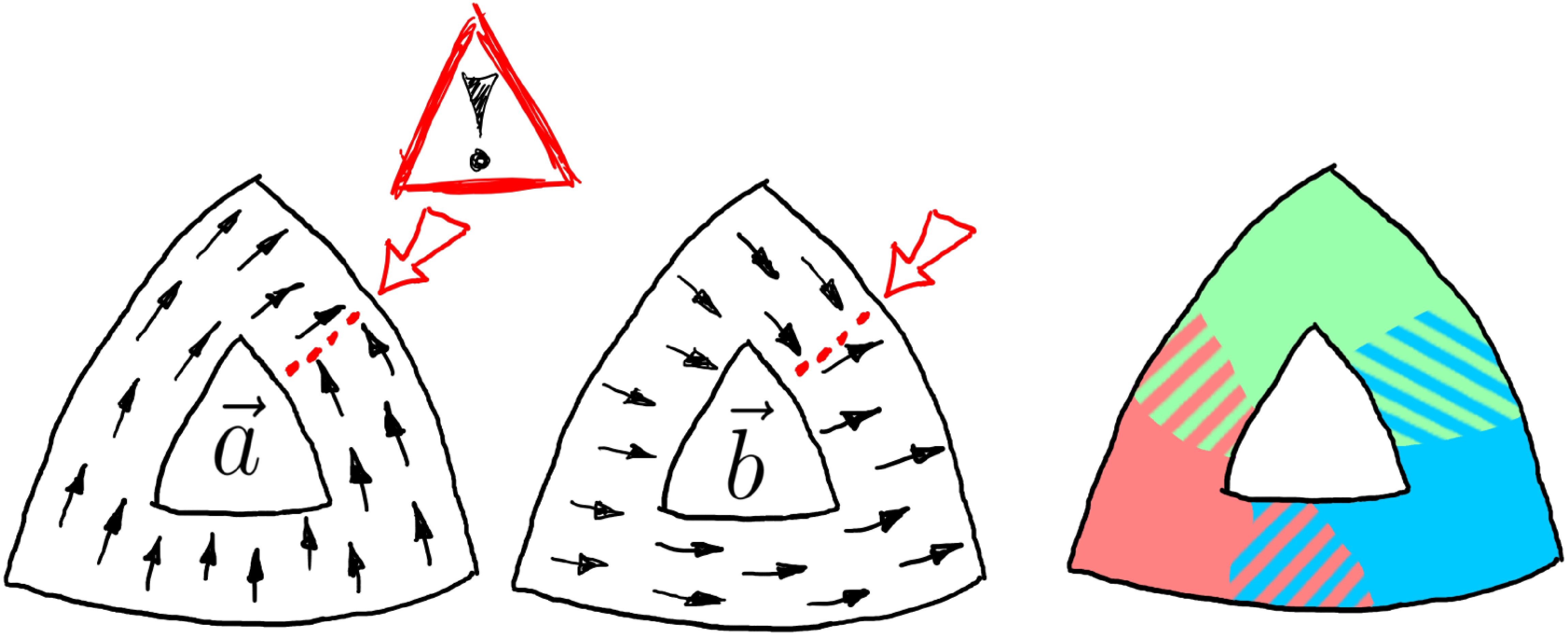


# Introductory example 3 out of 3



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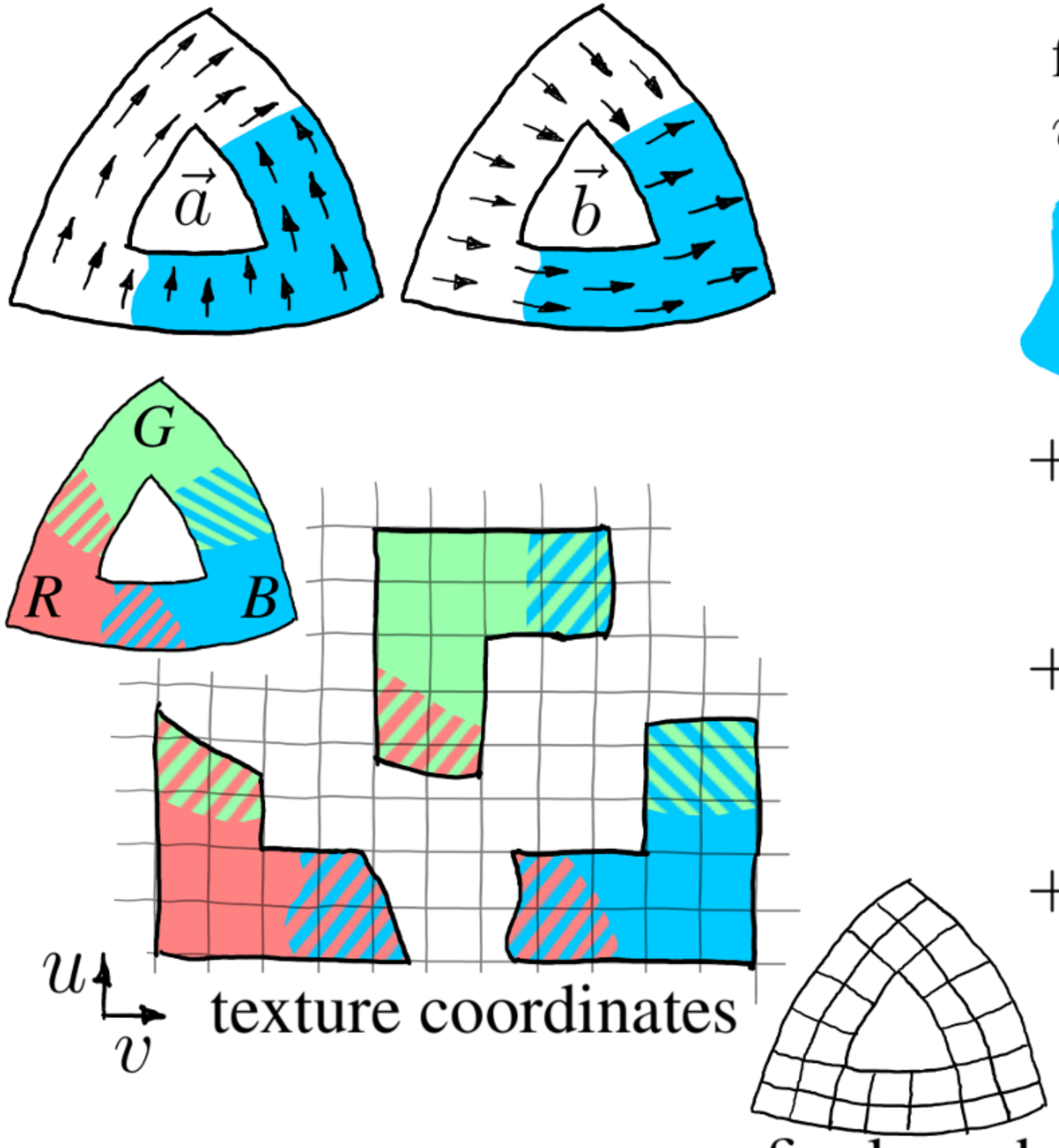



find scalar functions $u_B, v_B, u_R, v_R, u_G, v_G$ that minimize

$$\int_B |\nabla u_B - \vec{a}|^2 + \int_B |\nabla v_B - \vec{b}|^2$$

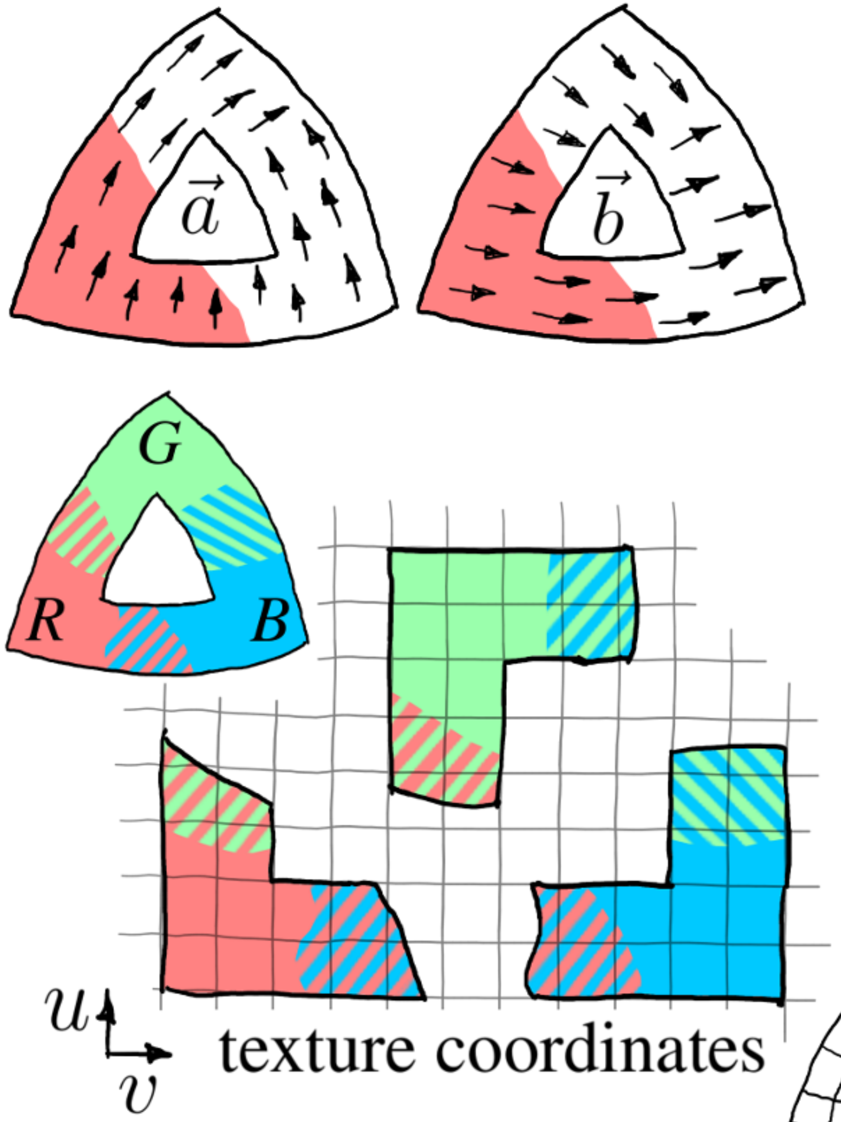
$$+ \int_R |\nabla u_R - \vec{a}|^2 + \int_R |\nabla v_R - \vec{b}|^2$$

$$+ \int_{G \setminus B} |\nabla u_G - \vec{a}|^2 + \int_{G \setminus B} |\nabla v_G - \vec{b}|^2$$

$$+ \int_{G \cap B} |\nabla u_G - \vec{b}|^2 + \int_{G \cap B} |\nabla v_G + \vec{a}|^2$$

$u_B - u_R, v_B - v_R, \; u_G - u_R, v_G - v_R,$
$u_G - v_B, v_G + u_B$ are integer

# Introductory example 3 out of 3




find scalar functions $u_B, v_B, u_R, v_R,$ $u_G, v_G$ that minimize

$$\int_B |\nabla u_B - \vec{a}|^2 + \int_B |\nabla v_B - \vec{b}|^2$$
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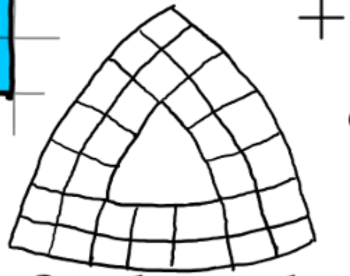


$u_B - u_R, v_B - v_R,\ u_G - u_R, v_G - v_R,$ $u_G - v_B, v_G + u_B$ are integer

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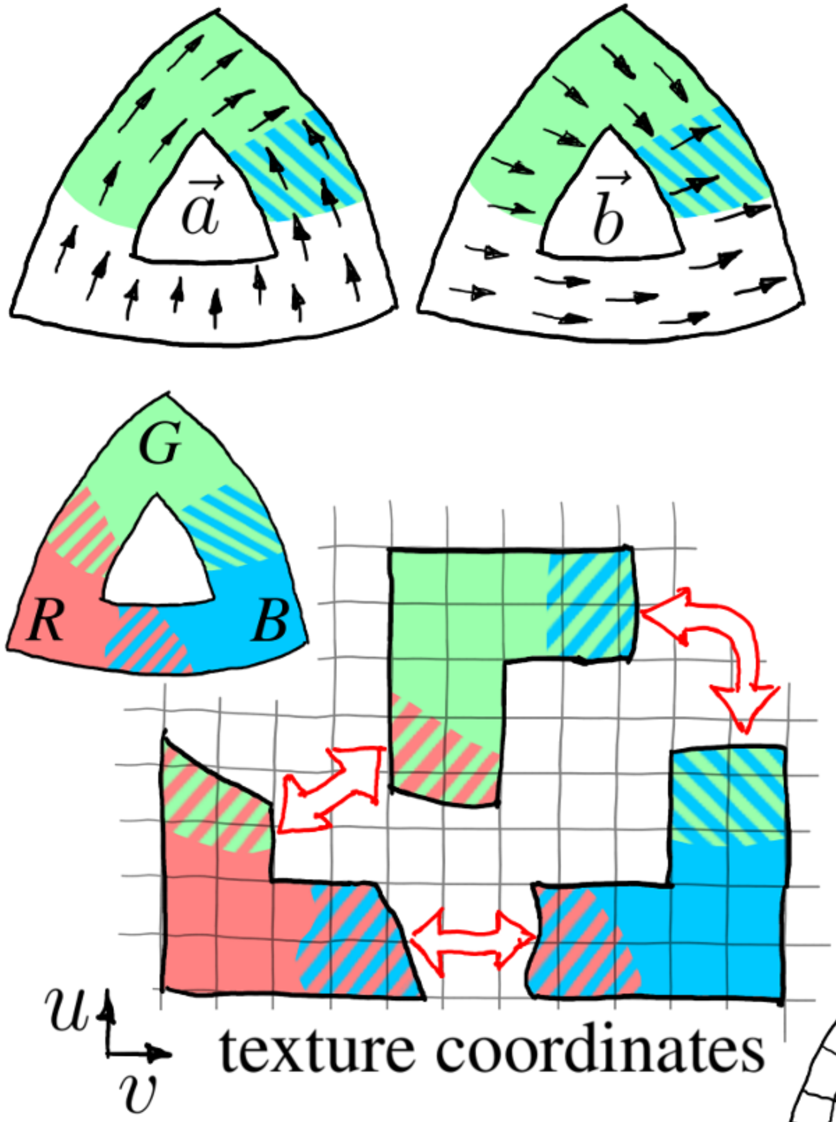


find scalar functions $u_B, v_B, u_R, v_R,$ $u_G, v_G$ that minimize

$$\int_B |\nabla u_B - \vec{a}|^2 + \int_B |\nabla v_B - \vec{b}|^2$$

$$+ \int_R |\nabla u_R - \vec{a}|^2 + \int_R |\nabla v_R - \vec{b}|^2$$

$$+ \int_{G \setminus B} |\nabla u_G - \vec{a}|^2 + \int_{G \setminus B} |\nabla v_G - \vec{b}|^2$$

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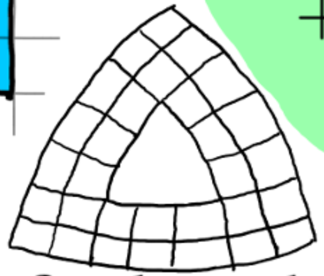
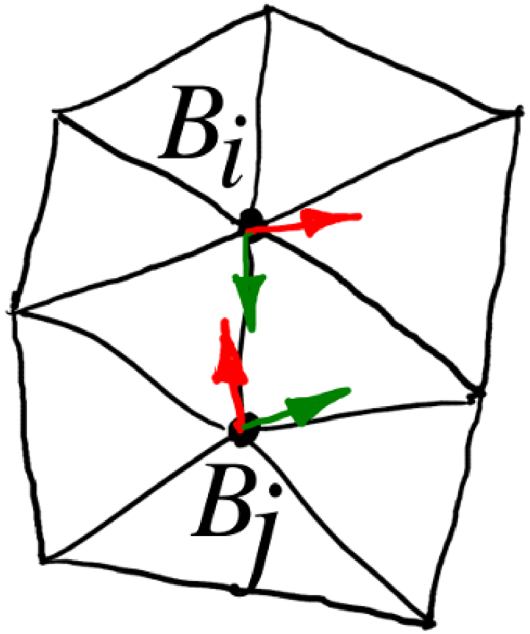


final mesh

$u_B - u_R, v_B - v_R,\ u_G - u_R, v_G - v_R,$ $u_G - v_B, v_G + u_B$ are integer

# Global parameterization: problem statement

Input: one basis per vertex and
one of 24 (4 in 2d) matrices per edge

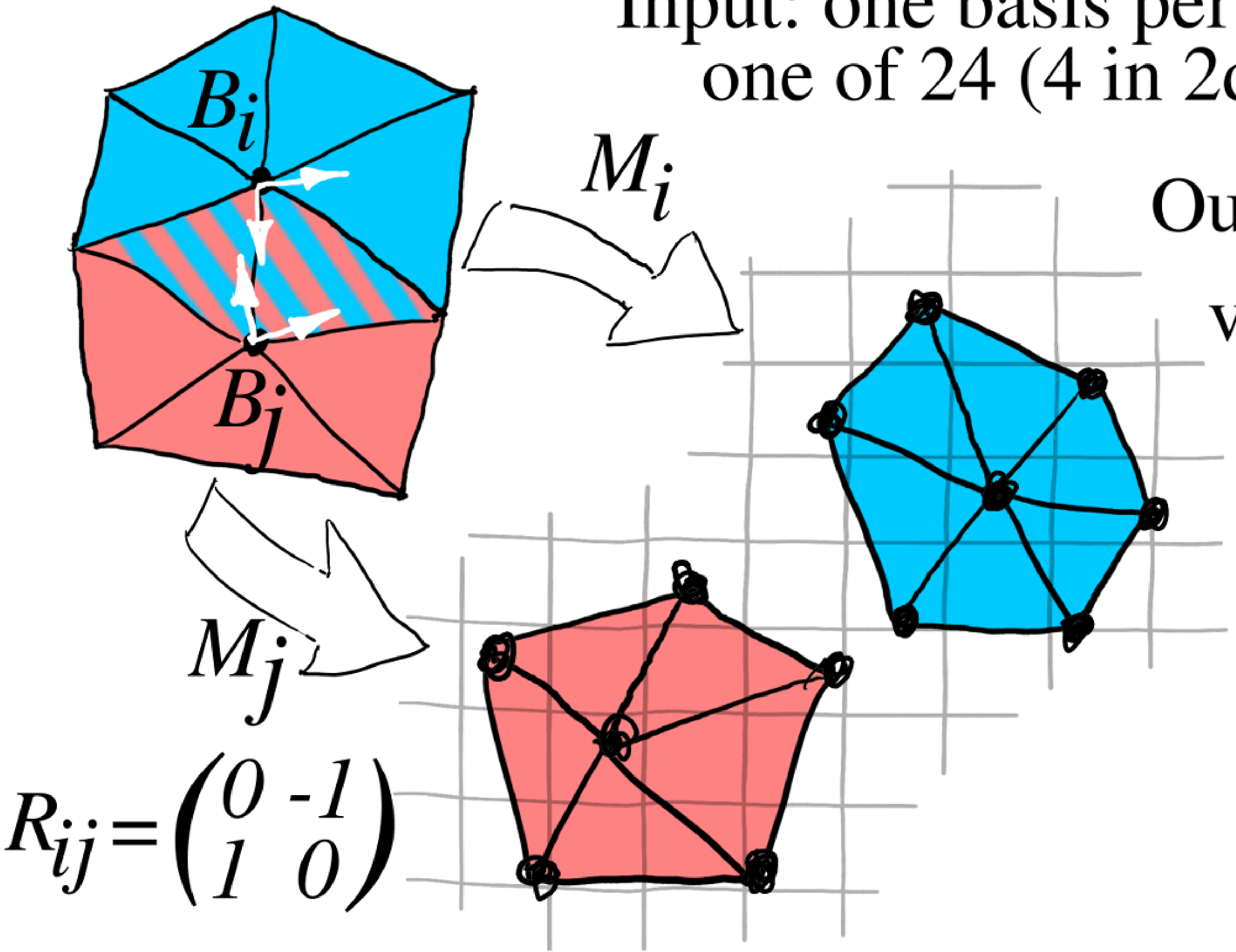



$$R_{ij} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

# Global parameterization: problem statement

Input: one basis per vertex and one of 24 (4 in 2d) matrices per edge

Output: one map per vertex star $J(M_i^{-1}) \approx B_i$



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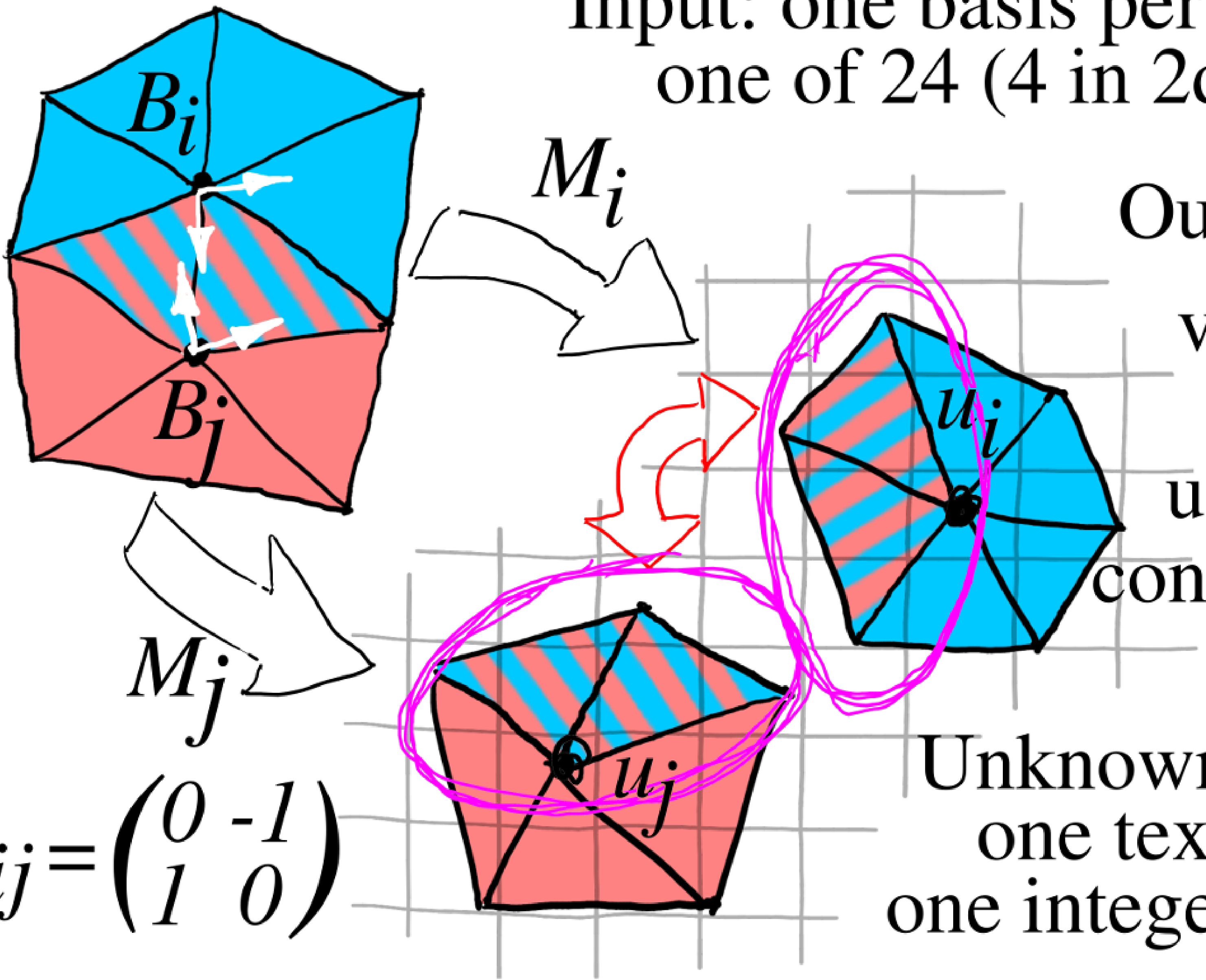



Output: one map per vertex star $J(M_i^{-1}) \approx B_i$

under grid equivalence constraint $M_j = R_{ij} M_i + t_{ij}$

$$R_{ij} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Unknowns:
one tex coord per vertex $(u_i)$
one integer vector per edge $(t_{ij})$

# Global parameterization: problem statement

The problem is to find an atlas of grid-equivalent maps;

each map is linear per tet, thus 4 points suffice to enforce the gr. eq. constraint:

for a tet $(i,j,k,l)$, maps $M_i$ and $M_j$ must satisfy:

$$\left\{\begin{array}{lcl} M_i(\mathbf{x}_i) & = & R_{ij}M_j(\mathbf{x}_i)+\mathbf{t}_{ij} \\ M_i(\mathbf{x}_j) & = & R_{ij}M_j(\mathbf{x}_j)+\mathbf{t}_{ij} \\ M_i(\mathbf{x}_k) & = & R_{ij}M_j(\mathbf{x}_k)+\mathbf{t}_{ij} \\ M_i(\mathbf{x}_l) & = & R_{ij}M_j(\mathbf{x}_l)+\mathbf{t}_{ij} \end{array}\right.$$

2 out of 4 constraints are naturally satisfied with our choice of the variables

# Global parameterization: problem statement

## Least-squares problem

INPUT

$$\forall i < j, \quad \vec{u}_i - R_{ij}\vec{u}_j - \vec{t}_{ij} + \vec{g}_{ij} = \vec{0} \quad \text{with } \vec{g}_{ij} = \frac{\left(B_i^{-1} + R_{ij}B_j^{-1}\right)(\vec{x}_j - \vec{x}_i)}{2}$$

unknowns

## under constraints:

per edge $(i,j)$ $\quad \vec{t}_{ij} \in \mathbb{Z}^3$ and

REDUNDANT

per tet $(i,j,k,l)$

$$\left\{\begin{array}{lcllr} \cancel{M_i(\vec{x}_i)} & \cancel{=} & \cancel{R_{ij}M_j(\vec{x}_i)} & \cancel{+\ \vec{t}_{ij}} & \cancel{(1)} \\ \cancel{M_i(\vec{x}_j)} & \cancel{=} & \cancel{R_{ij}M_j(\vec{x}_j)} & \cancel{+\ \vec{t}_{ij}} & \cancel{(2)} \\ M_i(\vec{x}_k) & = & R_{ij}M_j(\vec{x}_k) & +\ \vec{t}_{ij} & (3) \\ M_i(\vec{x}_l) & = & R_{ij}M_j(\vec{x}_l) & +\ \vec{t}_{ij} & (4) \end{array}\right.$$

# Global parameterization: problem statement

Least-squares problem

INPUT

$$\forall i < j, \quad \vec{u}_i - R_{ij}\vec{u}_j - \vec{t}_{ij} + \vec{g}_{ij} = \vec{0} \quad \text{with } \vec{g}_{ij} = \frac{\left(B_i^{-1} + R_{ij}B_j^{-1}\right)(\vec{x}_j - \vec{x}_i)}{2}$$

unknowns

under constraints:

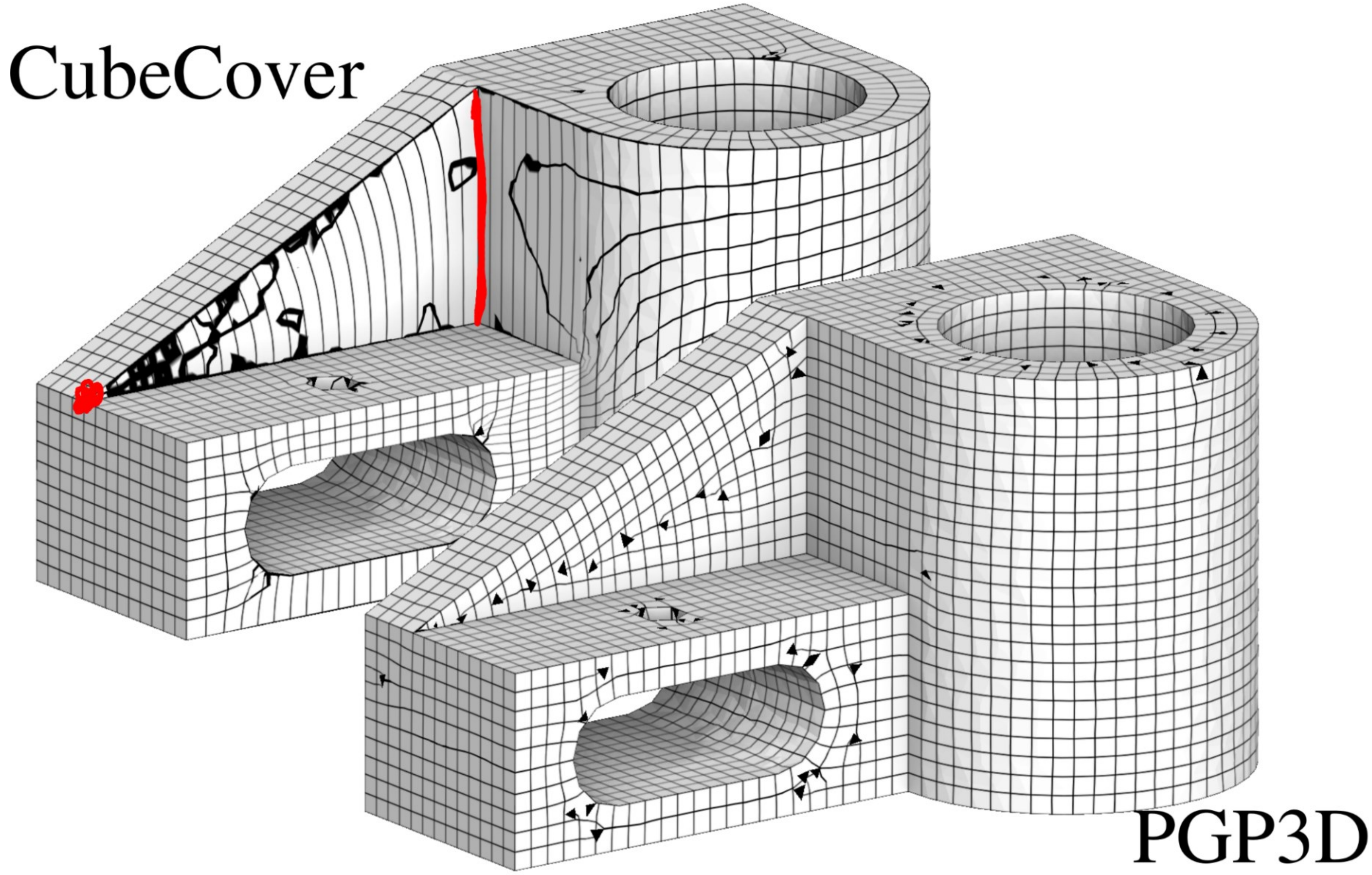
per edge $(i,j)$ $\quad \vec{t}_{ij} \in \mathbb{Z}^3$ and

per tet $(i,j,k,l)$

$$\begin{cases} R_{ik}(\vec{u}_k - \vec{t}_{ki}) & = & R_{ij}R_{jk}(\vec{u}_k - \vec{t}_{kj}) \\ R_{il}(\vec{u}_l - \vec{t}_{li}) & = & R_{ij}R_{jl}(\vec{u}_l - \vec{t}_{lj}) \end{cases}$$

# Global parameterization: solution mechanism

CubeCover [Nieser et al, 2011] enforces all the constraints, whereas the we ignore half of grid equivalence constraints (it is a feature, not a bug).



# Periodic global parameterization

as $\vec{t}_{ij} \in \mathbb{Z}^3$ and appears in exactly one line of the system

$$\forall i < j, \quad \vec{u}_i - R_{ij}\vec{u}_j - \vec{t}_{ij} + \vec{g}_{ij} = \vec{0},$$

we can solve first

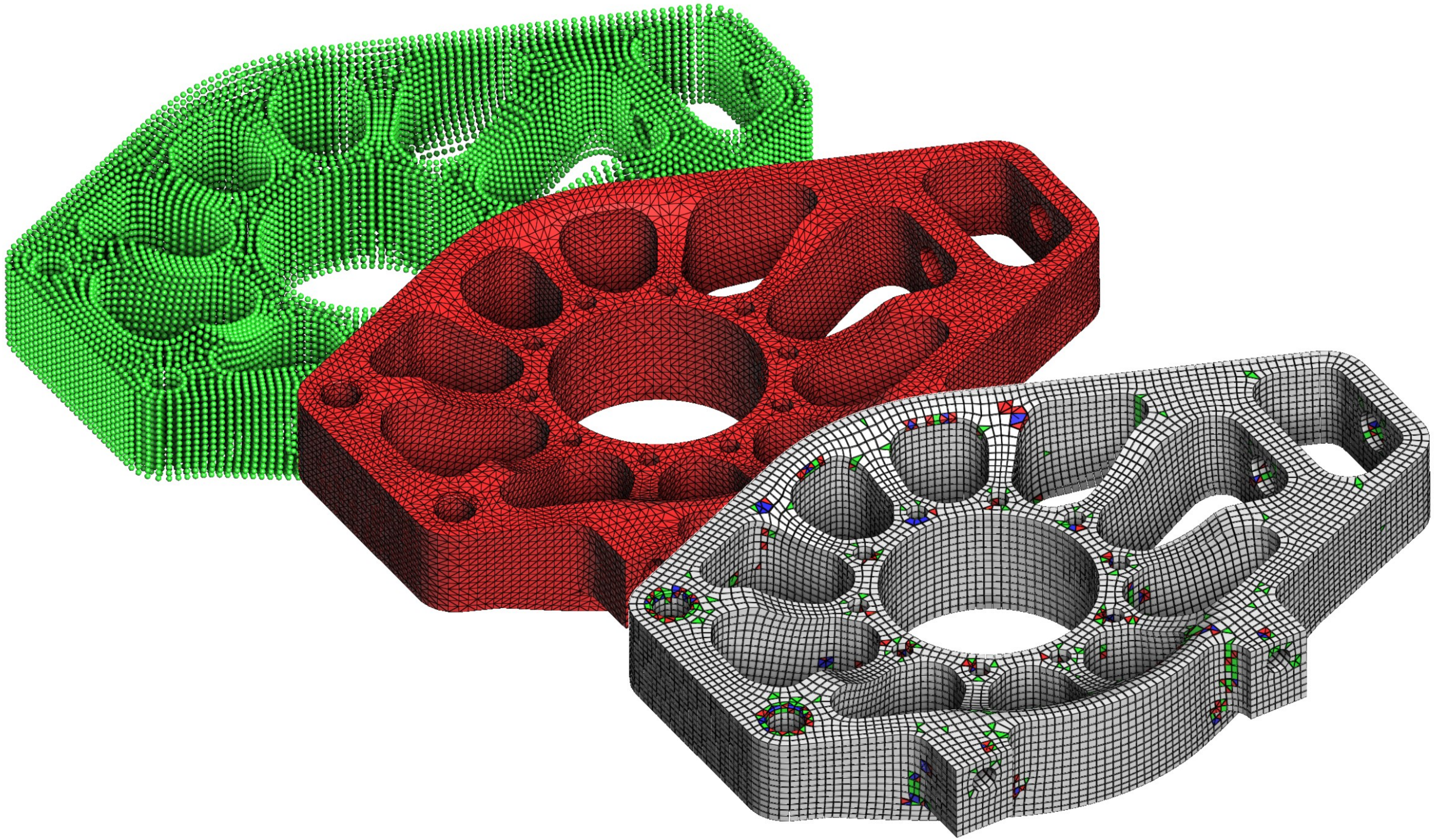
$$\forall i < j, \quad \vec{u}_i + \vec{g}_{ij} = R_{ij}\vec{u}_j \mod 1,$$

then deduce the value of $\vec{t}_{ij}$

$$\forall i < j, \forall d \in \{0,1,2\} \left\{ \begin{array}{c} \cos(2\pi(\vec{u}_i + \vec{g}_{ij})[d]) = \cos(2\pi(R_{ij}\vec{u}_j)[d]) \\ \sin(2\pi(\vec{u}_i + \vec{g}_{ij})[d]) = \sin(2\pi(R_{ij}\vec{u}_j)[d]) \end{array} \right.$$

$$\vec{a}_i[d] = \cos(2\pi\vec{u}_i[d]) \quad \text{and} \quad \vec{b}_i[d] = \sin(2\pi\vec{u}_i[d])$$

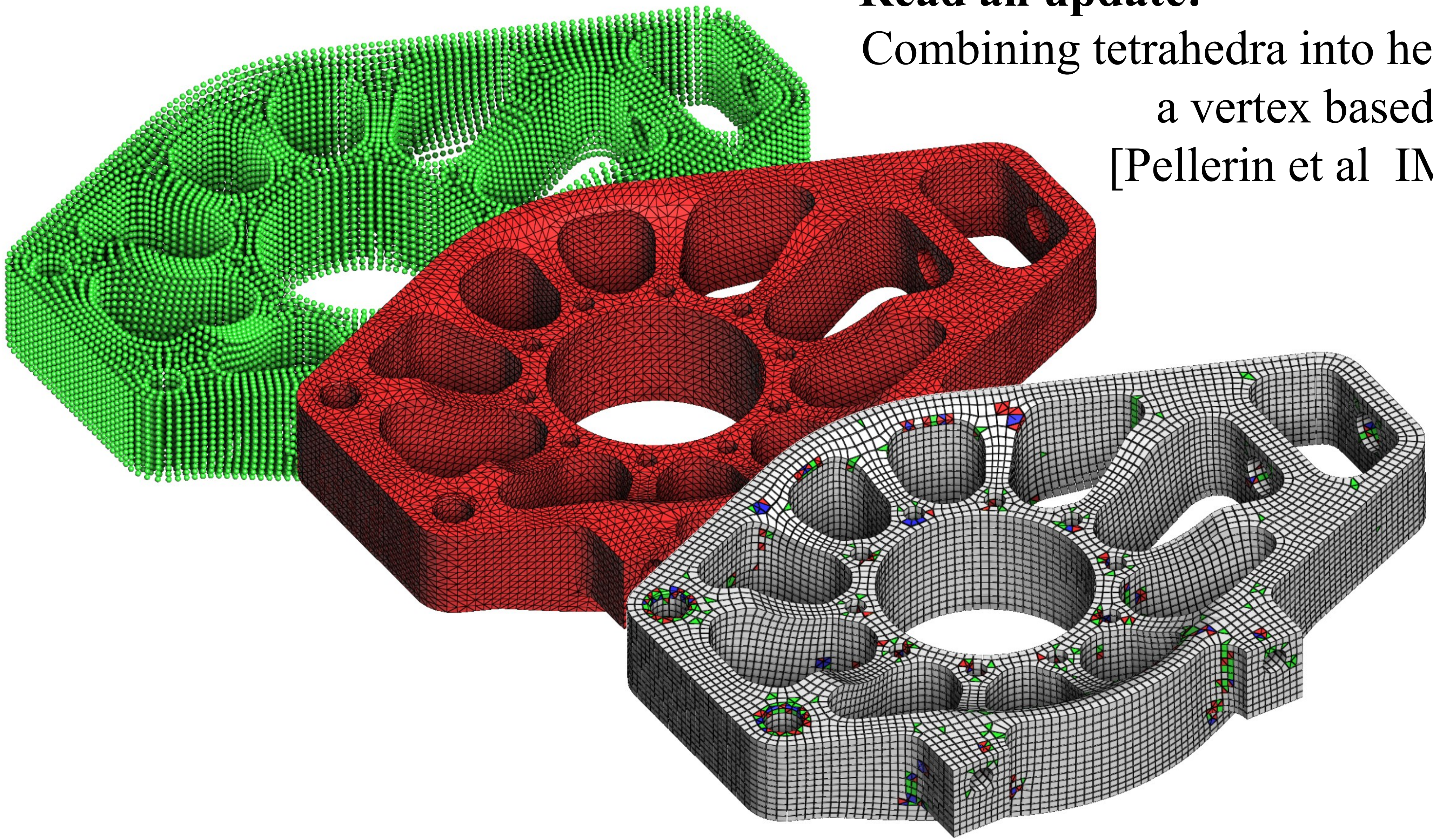
# Extract the hex-dominant mesh



# Extract the hex-dominant mesh

**Read an update:**

Combining tetrahedra into hexahedra:
a vertex based strategy
[Pellerin et al  IMR2017]



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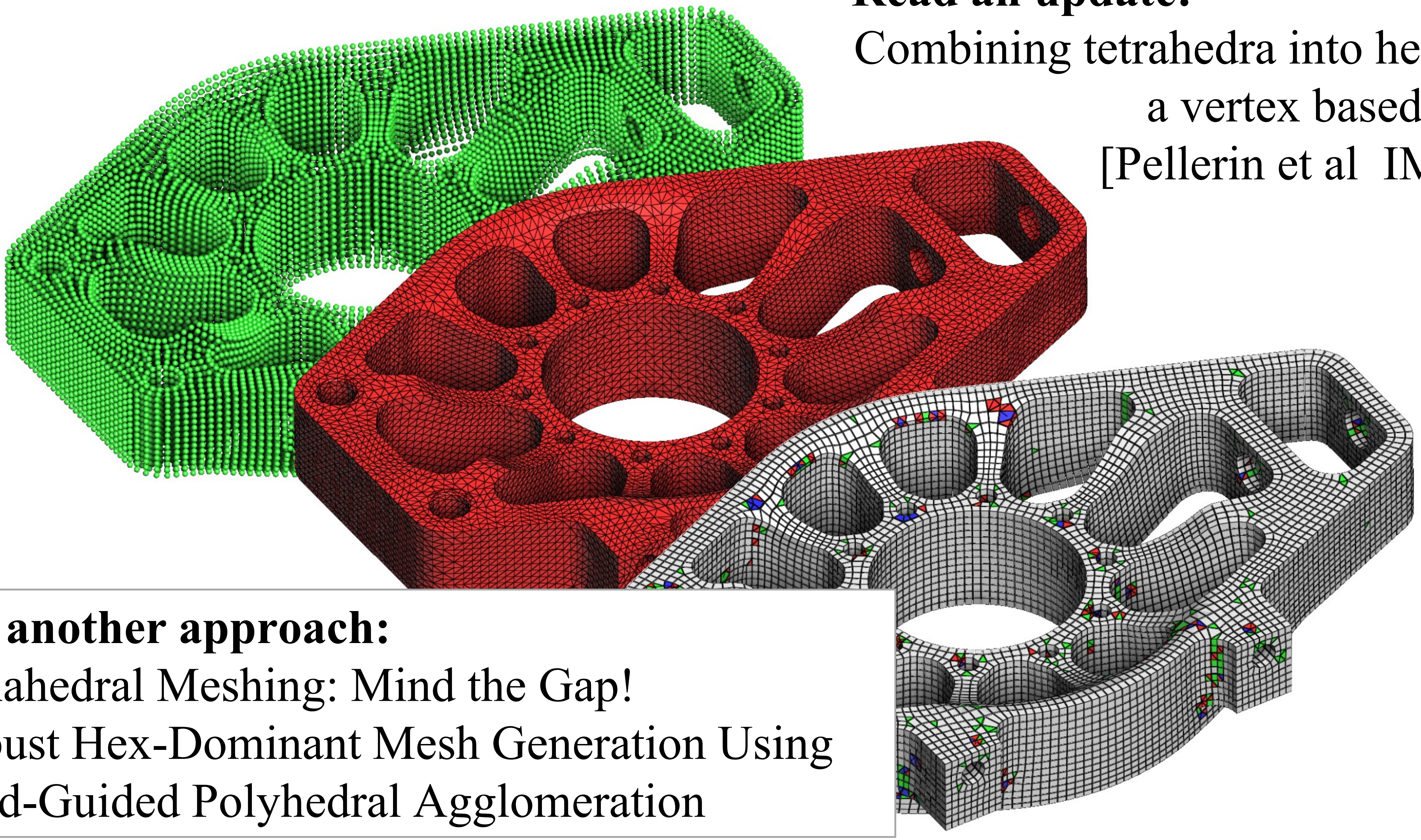

**Or use another approach:**
- Hexahedral Meshing: Mind the Gap!
- Robust Hex-Dominant Mesh Generation Using Field-Guided Polyhedral Agglomeration

# Thank you for your attention!