Coverability as local rule

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Workshop on aperiodicity and hierarchical structures in tilings
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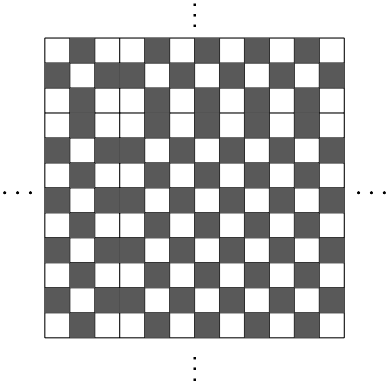
Introduction

- Σ an alphabet, e.g. $\{\square, \blacksquare\}$
- **Colorings** of groups
  - In my case, $\mathbb{Z}$ and $\mathbb{Z}^2$



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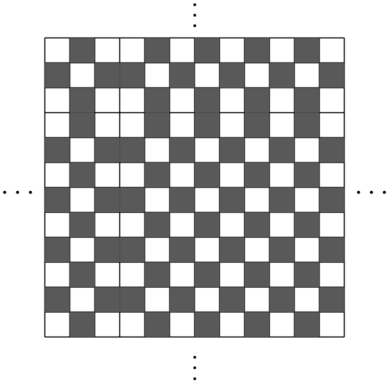
- **Local rules**
  - Wang tiles
  - Forbidden patterns

$\cdots \square\blacksquare\square\square\blacksquare\square\blacksquare\square\square\blacksquare\square\square\blacksquare\square \cdots$



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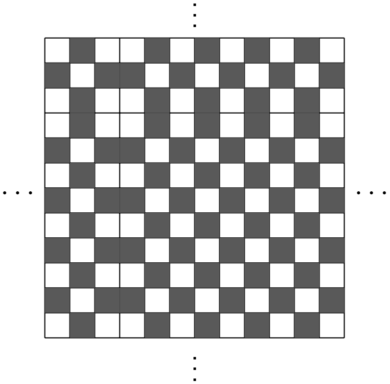
- Notions of **regularity**
  - Periodicity
  - Repetitivity
  - Existence of frequencies
  - Entropy



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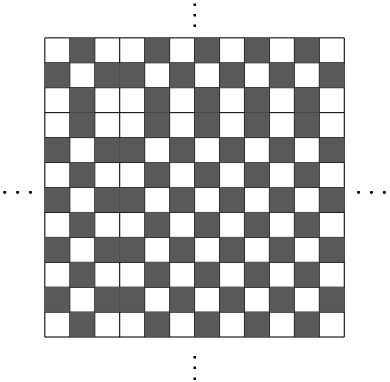


- Notions of **regularity**
  - Periodicity
  - Repetitivity
  - Existence of frequencies
  - Entropy
- **Coverability**

"Coverable" vs. "Quasiperiodic"

Warning

Quasiperiodic has different meanings in different communities.

Combinatorics on words:	quasiperiodic = coverable
Tilings and dynamics:	quasiperiodic = repetitive

I coined the term "coverable" to resolve this ambiguity.

But it is not standard in the literature.

Plan

1. Introduction
2. Coverability in $\mathbb{Z}$
3. Coverability in $\mathbb{Z}^2$
4. Forcing entropy with covers
5. Multi-scale coverability

Plan

1 Introduction

2 Coverability in $\mathbb{Z}$

3 Coverability in $\mathbb{Z}^2$

4 Forcing entropy with covers

5 Multi-scale coverability

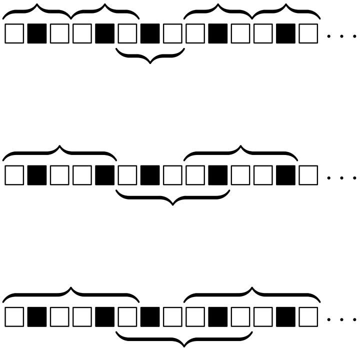
Coverability

Let w, q be words (q is finite).

Definition

The word q is a **cover** of w if w is covered with copies of q.

- w finite or infinite
- $q \neq w$
- q prefix of w



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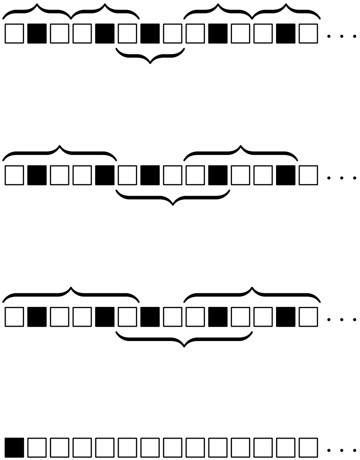
Definition

Coverable $=$ has a cover
Superprimitive $=$ no covers



Previous work on coverability

Text algorithms (1990's)

- Definition
- Detection algorithms
- Normal form

Infinite words (2000's)

- Definition, questions
- Independence results
- Multi-scale case

Characterization of covers...

- ... of Sturmian words
- ... of Episturmian words

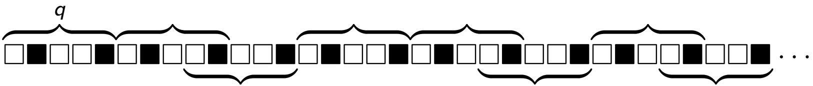
Combinatorics (2016)

- Tools to determine covers
- Characterize periodic words...
- ...and standard Sturmian words

On $\mathbb{Z}^2$ (2015, 2017)

- Knowing "trivial" covers
- Independence results
- Multi-scale case

Normal form of coverable words



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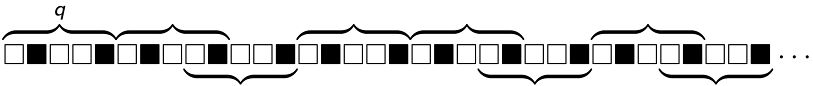
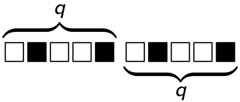
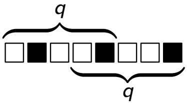


Two possibilities:

1. 

2. 

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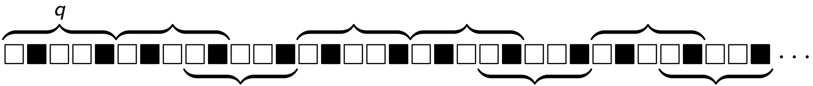

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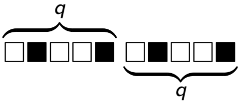
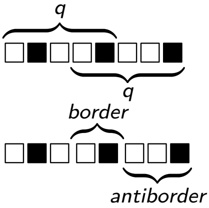
1. 

2. 

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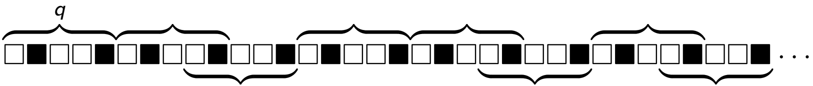

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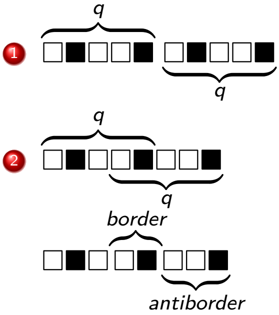

Theorem (Mouchard, 2000)

A word is q-coverable iff
it is a concatenation of q-antiborders,
starting with q.

- **Border:** prefix + suffix
- **Antiborder:** right complement of a border

Substitutions from covers

Fix a word q, say with n antiborders.

Definition

$\mu_q(i)$ is the i^{th} antiborder of q
(by decreasing size)

Example

$$q = \square\blacksquare\square\ \blacksquare\blacksquare\ \square\blacksquare\square$$
$$\mu_q(0) = \square\blacksquare\square\ \blacksquare\blacksquare\ \square\blacksquare\square$$
$$\mu_q(1) = \blacksquare\square\ \blacksquare\blacksquare\ \square\blacksquare\square$$
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Now view μ_q as a substitution $\{0, \ldots, n-1\}^* \to \Sigma^*$.

Example

$q =$ □■□ ■■ □■□

$\mu_q(0) =$ □■□ ■■ □■□

$\mu_q(1) =$ ■□ ■■ □■□

$\mu_q(2) =$ ■■ □■□

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Theorem (Mouchard, 2000)

A word $\mathbf{w}$ is q-coverable iff
$\exists \mathbf{u}$ such that $\mathbf{w} = \mu_q(0 \cdot \mathbf{u})$

Example

$$q = \square\blacksquare\square\ \blacksquare\blacksquare\ \square\blacksquare\square$$
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Irregular coverable words

Remark

For most q, μ_q preserves interesting properties

For instance,

- Non-repetitivity
- Positive entropy
- Divergence of frequencies

Thus we can create **irregular coverable words**

Irregular coverable words

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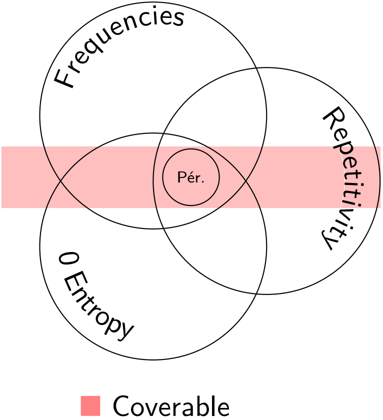

[Marcus, Monteil 2006]

"Trivial" covers

If $q = \square$, there is only one q-coverable word: $\square^{\mathbb{Z}}$.

“Trivial” covers

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Theorem

If μ_q is not injective (on infinite words) then $\forall \mathbf{u},\ \mu_q(\mathbf{u}) = q^{\mathbb{Z}}$.

We have a **dichotomy**:

- either there exist irregular q-coverable words,
- or all q-coverable words are periodic.

Besides, injectivity of μ_q is equivalent to an easy combinatorial condition on q. (More on this later.)

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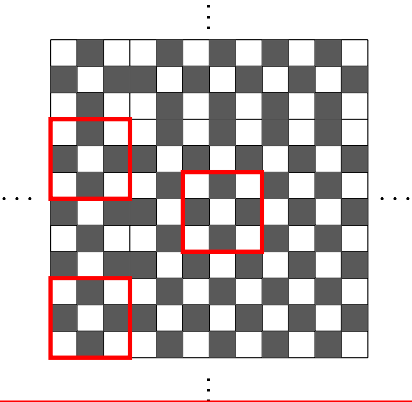
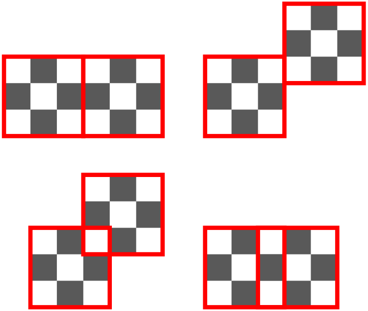
Coverability in 2 dimensions

A **configuration** is a coloring of $\mathbb{Z}^2$. A **block** is a coloring of a finite rectangle.

Definition

Let q be a block.
A configuration $\mathbf{w}$ is q-coverable if it is covered with copies of q.





Notions of regularity

Definitions

- **Block complexity**
  $P_{\mathbf{w}}(m,n) = \#$ blocs (m,n) in $\mathbf{w}$
- **Entropy**
  $\mathrm{Ent}(\mathbf{w}) = \lim \log(P_{\mathbf{w}}(n,n))/n^2$
- **Block frequencies**
  $f_{\mathbf{w}}(b) =$ average number of b-occurrences per cell
- **Repetitivity**
  Each block occurs ∞ often with bounded gaps

Plan

Show that **coverability is independent of these...**
... but we have **no more normal form!**

Ruling out "trivial" covers

The cover $\square$ only allows $\square^{\mathbb{Z}^2}$.

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Theorem (Richomme and G.)

Let q be a block.
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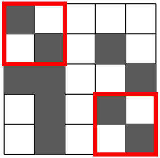
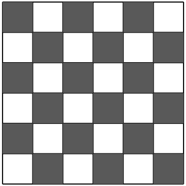


Border

Block in two opposite corners



Primitive root

Unique minimal v such that $u = v^{m \times n}$

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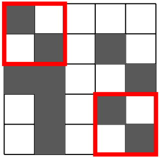

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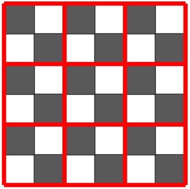

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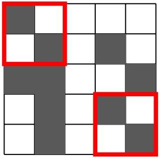
There exists an aperiodic, q-coverable configuration iff the primitive root of q has a nonempty border.

Ideas of the proof

1. If the root has no border,
   all overlaps are multiples of the root
2. Build **tiles** from q and freely tile the plane



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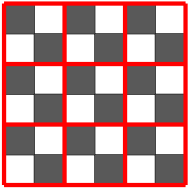

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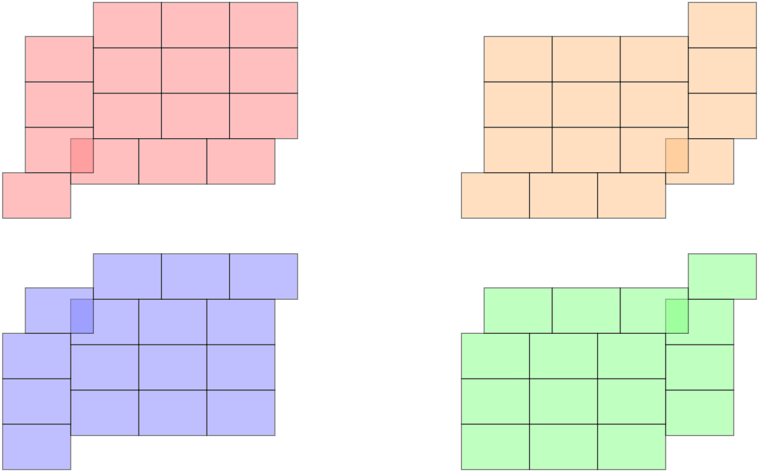
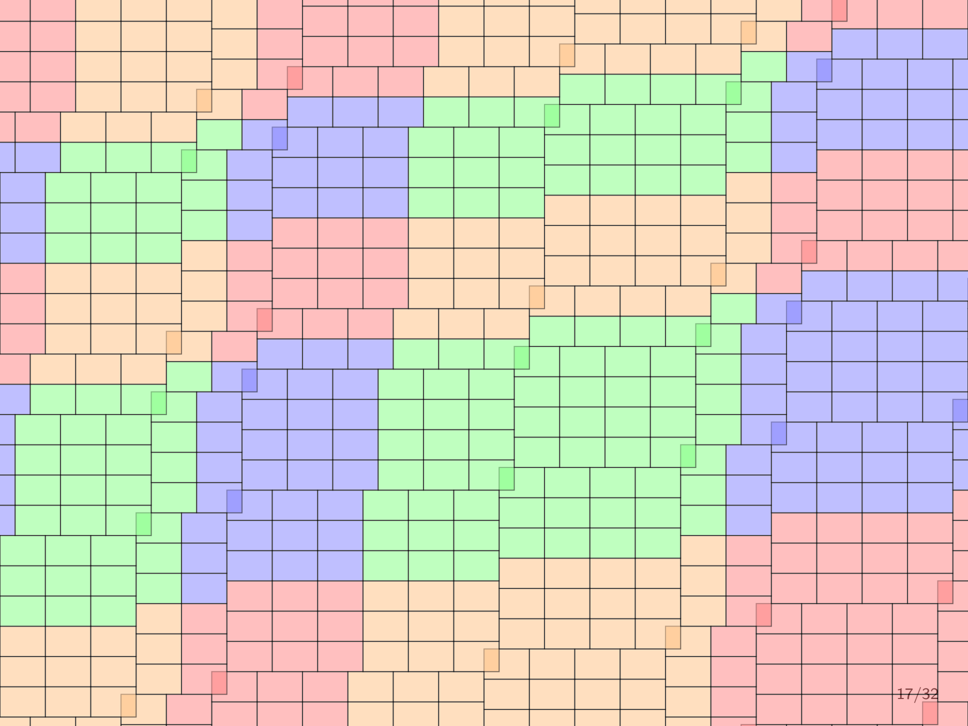
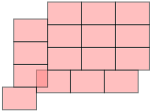
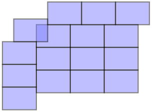
The tiles

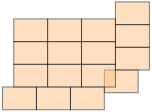


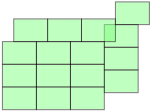


Coverable configurations

$f(a) =$ 

$f(b) =$ 

$f(c) =$ 

$f(d) =$ 

Remark

$f(\mathbf{w})$ is defined for $\mathbf{w} \in \{a, b, c, d\}^{\mathbb{Z}^2}$ only if $\mathbf{w}$ satisfies some local rules

(More about this on the next slide)

Proposition (Richomme and G.)

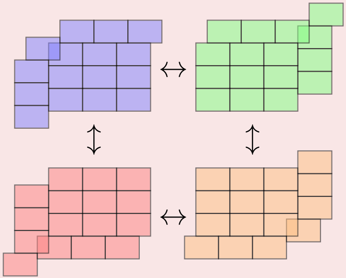
$\forall \mathbf{w},\ f(\mathbf{w})$ is q-coverable if it exists

Moreover, f preserves

- periodicity
- repetitivity
- existence of frequencies

Local rules and entropy

Local rules



Remark

There are configurations

- aperiodic
- non-repetitive
- without frequencies

and matching these rules.

Remark

The rules force zero entropy.

Which covers allow positive entropy?

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Forcing entropy with covers

Fix some block q.

What we want

Conditions on q implying

1. zero entropy for all configurations
2. positive entropy for some configurations

which are q-coverable.

Tool: **interchangeable pairs**

Forcing entropy with covers

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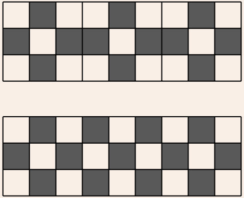
Tool: **interchangeable pairs**

Definition

An **interchangeable pair** is a pair of q-coverable patterns with the same shape. (Not always rectangles.)

Definition

An interchangeable pair is **valid** if its shape can tile the plane.



Interchangeable pairs

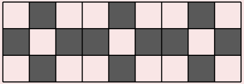
Fix a cover q and let $h = \max\{\text{Ent}(\mathbf{w}), \mathbf{w} \text{ is } q-\text{coverable}\}$.

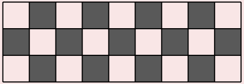
Theorem

If there is a *valid* pair for q, then $h > 0$.

If there is no valid pair for q, then $h = 0$.

Let $\mathbf{u}$ be a configuration with positive entropy. Consider $\mu(\mathbf{u})$.

$\mu(0) =$ 

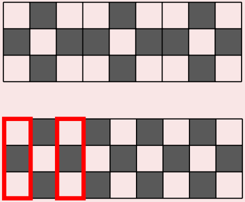
$\mu(1) =$ 

- Let v be an $n \times n$-square in a q-coverable configuration $\mathbf{w}$.
- Let $\bar{v}$ be the smallest q-coverable pattern in $\mathbf{w}$ containing v.
- Then v is determined by the shape of $\bar{v}$ and coordinates.
- We have less than $|\Sigma|^{4n|q|} \times n^2$ possibilities.

Covers allowing positive entropy

Lemma 1

Any cover with full-width or full-height border allows positive entropy.



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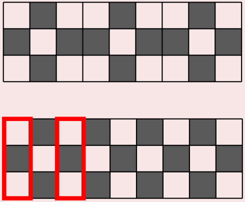
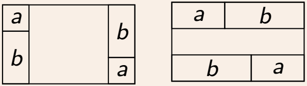


Lemma 2

Any cover with one of these shapes



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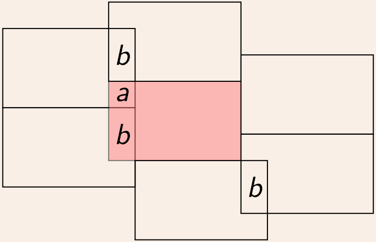

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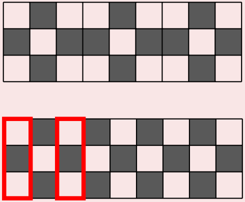

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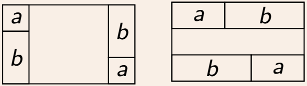

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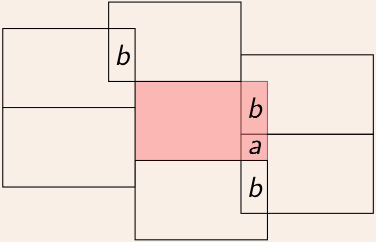
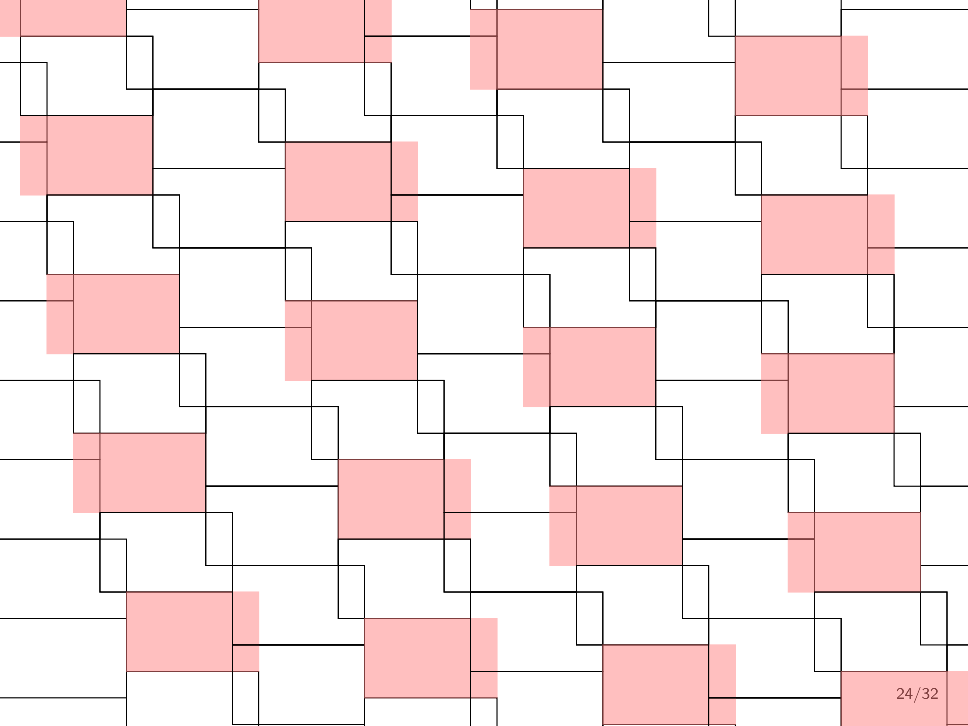
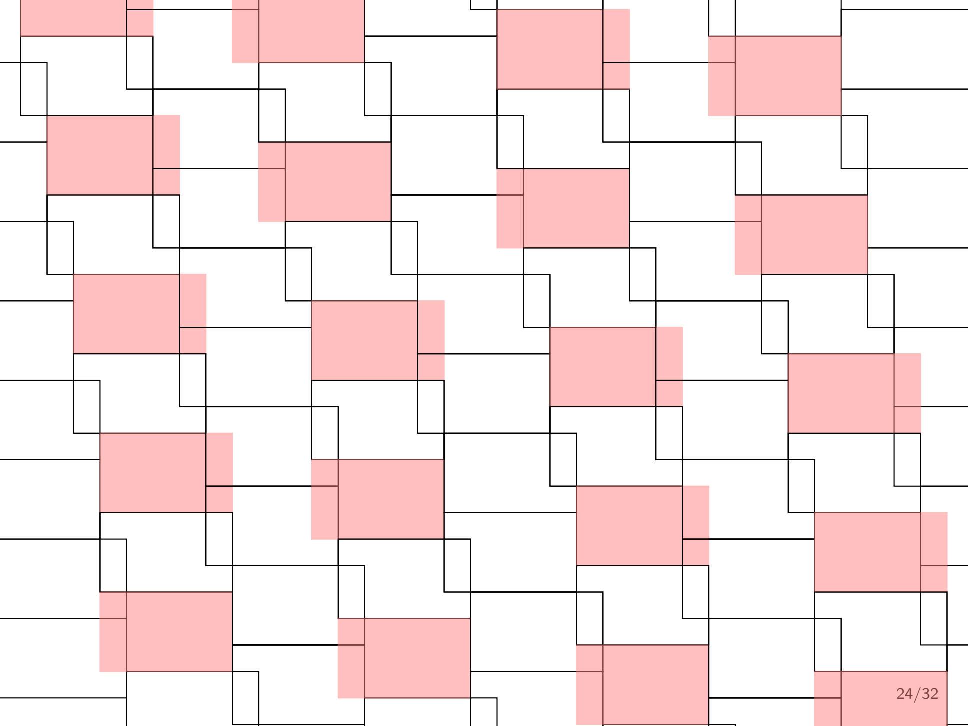






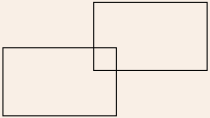
A sufficent condition for zero entropy

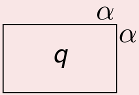
Theorem (Richomme and G.)

If q has a corner without borders, then any q-coverable configuration has zero entropy.

Example

Suppose there are no overlaps like:





What occurrences are covering the α's?

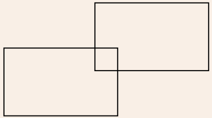
A sufficent condition for zero entropy

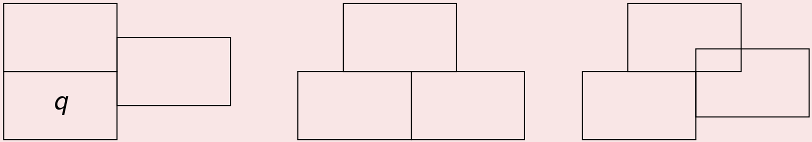
Theorem (Richomme and G.)

If q has a corner without borders, then any q-coverable configuration has zero entropy.

Example

Suppose there are no overlaps like:





There are three cases.

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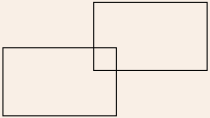

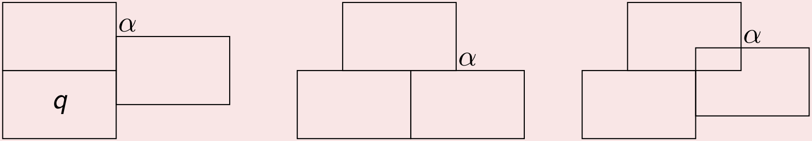

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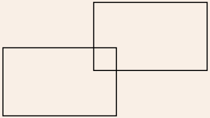

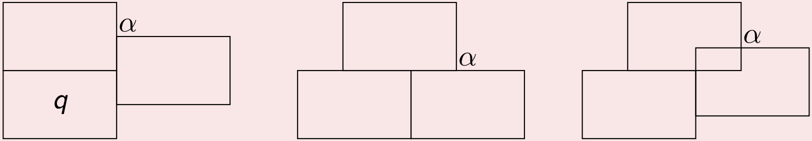

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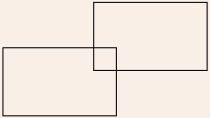

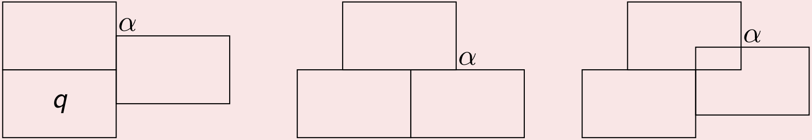

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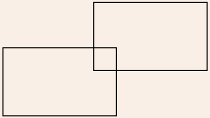
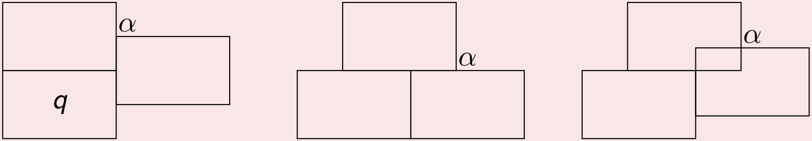




The occurrence covering α is unique in all cases.

$\implies$ the shape of a q-coverable pattern determines the pattern itself

$\implies$ no interchangeable pairs

Another condition

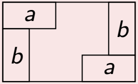
Lemma

Suppose q has **no** pairs of borders (a, b) such that

$$w(a) + w(b) \geq w(q) \qquad \text{or}$$
$$h(a) + h(b) \geq h(q)$$

then any q-coverable configuration has zero entropy.

Not



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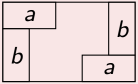
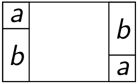
Not 

Proof

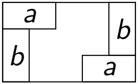
Same ideas as previous proof, but more cases to check.

Recap about entropy

We have



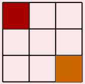
$\Longrightarrow$ positive entropy

Not 

$\Longrightarrow$ zero entropy

Not quite an "if and only if", but we're getting close.

Remark

The duality in 1D does not apply in 2D: the cover  have aperiodic configurations, but all with zero entropy.

Plan

1. Introduction
2. Coverability in $\mathbb{Z}$
3. Coverability in $\mathbb{Z}^2$
4. Forcing entropy with covers
5. Multi-scale coverability

Multi-scale coverability

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A {word, configuration} is

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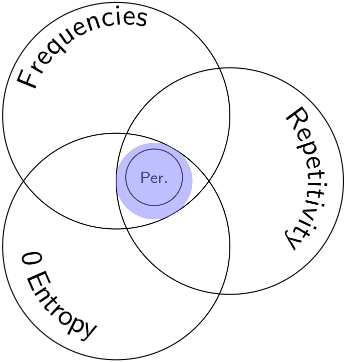

[Marcus, Monteil 2006]

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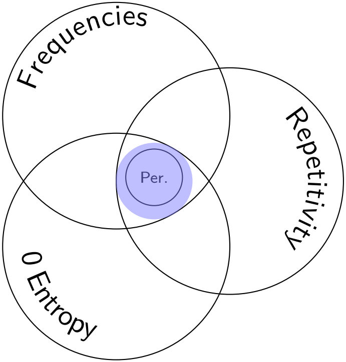
- **Good notion of regularity**



[Marcus, Monteil 2006]

Multi-scale coverability in 2D

Reminder (Marcus and Monteil)

Any **1D** multi-scale word has

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Question

What about multi-scale configurations?

Multi-scale coverability in 2D

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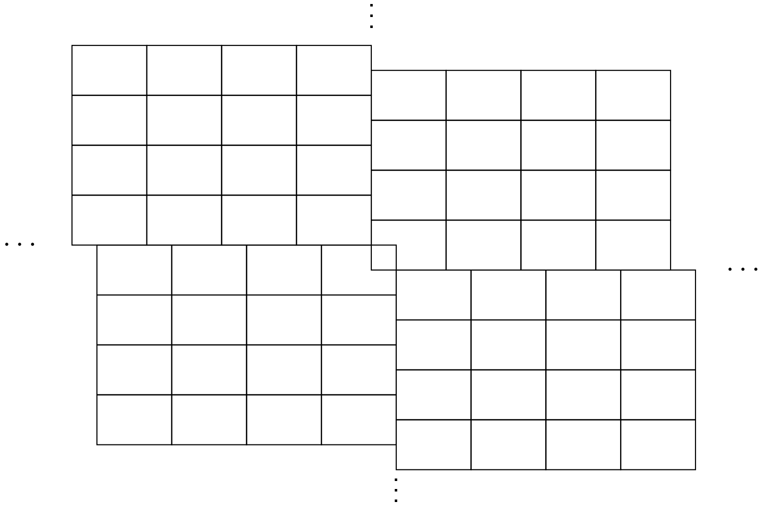
Proof sketch

1. Direct adaptation of 1D proof
2. Lots of calculations

Repetitivity of multi-scale configurations



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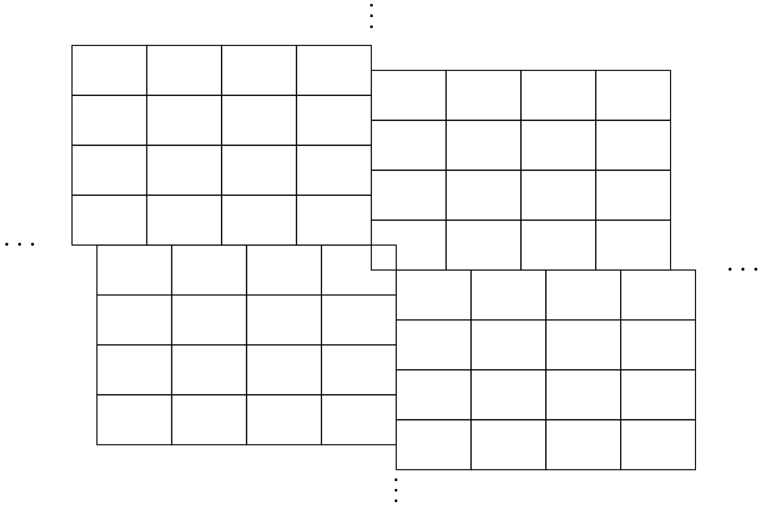

Conclusion

- **Coverability** comes from the study of finite and $\mathbb{Z}$-words
- On $\mathbb{Z}^2$: characterization of *trivial* covers
- Ongoing characterization of *covers forcing zero entropy*
- **Multi-scale coverability** is a good notion of regularity

Many possible extensions:

- as a local rule
- as a notion of regularity

Thank you for your attention!