Theoretical Foundations for Humanoid Multi-Task QP Control

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March 16, 2017

Multi-task control of humanoid robots has been formulated as

$$\text{“}\min_{x \in \mathcal{X}}\text{”} f(x) = (||\ddot{\tau}_1 - \ddot{\tau}_1^d||^2, \dots, ||\ddot{\tau}_p - \ddot{\tau}_p^d||^2). \tag{1}$$

- A task is any “feature” we want to control on the robot (e.g. whole posture, end-effector, CoM)

$$\tau_k : \mathbb{R}^n \to \mathbb{R}^{n_k} \tag{2}$$

- All the tasks cannot be realized perfectly (conflicts)
- Necessity of a compromise between tasks
- Two main approaches fighting against each other in the community : weighted priority vs strict priority

The previous formulation is a particular instance of the general multiobjective optimization problem (aka vector optimization):

$$\text{``}\min_{x\in\mathcal{X}}\text{''}\, f(x) = (f_1(x), \ldots, f_p(x))\,. \tag{3}$$

- There is no "perfect" solution to the problem, the so-called **ideal point** is not a solution in general

$$y^I = (\min_{x\in\mathcal{X}} f_1(x), \ldots, \min_{x\in\mathcal{X}} f_p(x)) \notin \mathcal{Y} = f(\mathcal{X})\,. \tag{4}$$

- There is a whole subset of $\mathcal{X}$, or equivalently of $\mathcal{Y} = f(\mathcal{X})$, of "optimal" solutions.
- There are different names for these solutions: **Pareto-optimal** solution, **efficient solution** (x^*), **nondominated point** $(y^* = f(x^*))$

Definitions

- x^* is an **efficient solution** (resp. $y^* = f(x)$ is a **nondominated point** of $\mathcal{Y}$) if there is no $x \in \mathcal{X}$ (resp. $y \in \mathcal{Y}$) such that $f(x) \leq f(x^*)$ (resp $y \leq y^*$), where $\leq$ denotes the **componentwise order** ($y^1 \leq y^2$ if $\forall k \; y^1_k \leq y^2_k$ and $y^1 \neq y^2$).

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- The set of all nondominated points of $\mathcal{Y}$ is denoted $\mathcal{Y}_N$. It is also known as The **Pareto-optimal front**.

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- The set of all nondominated points of $\mathcal{Y}$ is denoted $\mathcal{Y}_N$. It is also known as The **Pareto-optimal front**.
- x^* is an **weakly efficient solution** (resp. $y^* = f(x)$ is a **weakly nondominated point** of $\mathcal{Y}$) if there is no $x \in \mathcal{X}$ (resp. $y \in \mathcal{Y}$) such that $f(x) < f(x^*)$ (resp $y \leq y^*$), where $<$ denotes the **strict componentwise order** ($y^1 < y^2$ if $\forall k \; y^1_k < y^2_k$.)

Our thesis : **there is no mathematical justification for favoring any efficient solution over another efficient solution, they are all "legitimate", and it is up to the user and depending on the application to favor one over all the others.**

Question: where does the weighted priority vs strict priority debate stand in all of this ?

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They are both so-called **scalarization** schemes of the vector optimization problem.

- The **weighted-sum scalarization** is defined as solving the problem(s)

$$y^{\mathrm{ws}}(w) = \min_{x \in \mathcal{X}} \sum_{k=1}^{p} w_k f_k(x) \,. \tag{5}$$

(note : there is one such problem for each given set of weights w)

- The **lexicographic scalarization** is defined as solving the problem

$$y^{\mathrm{lex}} = \operatorname*{lexmin}_{x \in \mathcal{X}} \left(f_1(x), \ldots, f_p(x) \right) , \tag{6}$$

Theorem

The weighted-sum scalarization optimum with (strictly) **positive** weights, and the lexicographic optimum, are two particular efficient solutions

$$y^{\text{lex}} \in \mathcal{Y}_N, \tag{7}$$

$$\forall w > 0 \quad y^{\text{ws}}(w) \in \mathcal{Y}_N \,. \tag{8}$$

Question : $\exists w > 0 \quad y^{\text{lex}} = y^{\text{ws}}(w) \quad ?$

Definitions

$$\mathcal{S}_0(\mathcal{Y}) = \left\{ y^* \in \mathcal{Y} \mid \sum_{k=1}^{p} w_k y_k^* = \min_{y \in \mathcal{Y}} \sum_{k=1}^{p} w_k y_k, \; 0 \leq w \right\}, \quad (9)$$

$$\mathcal{S}(\mathcal{Y}) = \left\{ y^* \in \mathcal{Y} \mid \sum_{k=1}^{p} w_k y_k^* = \min_{y \in \mathcal{Y}} \sum_{k=1}^{p} w_k y_k, \; 0 < w \right\}. \quad (10)$$

Theorem

$\mathcal{S}_0(\mathcal{Y}) \subset \mathcal{Y}_{wN}$.

Converse:

Theorem

If $\mathcal{Y}$ is $\mathbb{R}^p_{\geqq}$-convex then $\mathcal{S}_0(\mathcal{Y}) = \mathcal{Y}_{wN}$.

(A set is $\mathbb{R}^p_{\geqq}$-convex if its Minkowsky sum with $\mathbb{R}^p_{\geqq}$ is convex. $\mathbb{R}^p_{\geqq}$ is the non-negative quadrant of $\mathbb{R}^p$.)

Theorem

$\mathcal{S}(\mathcal{Y}) \subset \mathcal{Y}_N$.

Converse? not true in general.

Definition

A solution $x^* \in \mathcal{X}$ is called **properly efficient** if it is efficient and $\exists M > 0$ $s.t.$ $\forall x \in \mathcal{X}$, $\forall i \in \{1, \ldots, p\}$: $f_i(x) < f_i(x^*) \Rightarrow \exists j \in \{1, \ldots, p\} \setminus \{i\}$ $s.t.$ $f_j(x^*) < f_j(x)$ and

$$\frac{f_i(x^*) - f_i(x)}{f_j(x) - f_j(x^*)} \leq M\,.$$

In that case the point $f(x^*)$ is said to be **properly nondominated** in $\mathcal{Y}$ and the set of all properly nondominated points of $\mathcal{Y}$ is denoted $\mathcal{Y}_{pN}$.

Theorem (Geoffrion, 1968)

$\mathcal{S}(\mathcal{Y}) \subset \mathcal{Y}_{pN}$.

Converse :

Theorem

If $\mathcal{Y}$ is $\mathbb{R}^p_{\geqq}$-convex then $\mathcal{S}(\mathcal{Y}) = \mathcal{Y}_{pN}$.

Is this the best we can hope for then ?

10 years later...

Theorem (Hartley, 1978)

If $\mathcal{Y}$ is nonempty, $\mathbb{R}^p_{\geqq}$-convex and $\mathbb{R}^p_{\geqq}$-closed then $\mathcal{Y}_N \subset \text{cl}\,(\mathcal{S}(\mathcal{Y}))$.

cl denotes the **topological closure**. A set is $\mathbb{R}^p_{\geqq}$-closed if its Minkowsky sum with $\mathbb{R}^p_{\geqq}$ is closed.

So we end up with, *under the "right" conditions,*

$$\mathcal{S}(\mathcal{Y}) \subset \mathcal{Y}_N \subset \mathrm{cl}\left(\mathcal{S}(\mathcal{Y})\right) . \tag{11}$$

and some Corollaries.

Corollary 1

For any $\epsilon > 0$ and any index k, there exists a set of positive weights $0 < w$ such that $f_k(x^*) - y_k^I < \epsilon$, where x^* denotes a solution of problem (5).

Corollary 2

If a given task τ_k is realizable exactly, i.e. $\exists x \in \mathcal{X}$ s.t. $\ddot{\tau}_k = \ddot{\tau}_k^d$, then it can be reached with weighted-sum scalarization of (1) with positive weights at any given precision, i.e. for any $\epsilon > 0$ there exists $0 < w$ such that

$$||\ddot{\tau}_k(x^*) - \ddot{\tau}_k^d||^2 < \epsilon \,, \tag{12}$$

where x^* is the solution of the w-weighted sum scalarization of (1):

$$\min_{x \in \mathcal{X}} \sum_{l=1}^{p} w_l \, ||\ddot{\tau}_l(x) - \ddot{\tau}_l^d||^2 \,. \tag{13}$$

Corollary 3

The lexicographic (strict priority) optimum can be approached at any given precision by positive weighted sum scalarization, i.e., for any $\epsilon > 0$ there exists a set of positive weights $0 < w$ such that $||f(x^*) - y^L|| < \epsilon$, where x^* is the solution of (13).

Now question : what are the consequences of approaching a desired task **acceleration** at a given precision ϵ, for the **task itself** ?

Definition

The solutions of a system $\dot{\chi} = \varphi(\chi, t)$ are said to be **uniformly ultimately bounded (UUB)** if there exists $b > 0$ and $c > 0$ such that, for every $0 < a < c$, there exists $T(a, b) > 0$ such that

$$||\chi(0)|| < a \;\Rightarrow\; \forall t \geq T(a, b),\; ||\chi(t)|| < b\,. \tag{14}$$

b is called an ultimate bound of the solutions. If a can be arbitrarily large, i.e. if there exists $b > 0$ such that for every $a > 0$ there exists $T(a, b) > 0$ such that

$$||\chi(0)|| < a \;\Rightarrow\; \forall t \geq T(a, b),\; ||\chi(t)|| < b\,, \tag{15}$$

then the solutions are said to be **globally uniformly ultimately bounded** with ultimate bound b.

Definition

The **logarithmic norm** associated with the vector norm $||.||$ in $\mathbb{R}^{2n_k}$ and its subordinate matrix norm $||.||$ in $\mathbb{R}^{2n_k \times 2n_k}$ is defined as

$$\mu(A_k) = \lim_{h \to 0^+} \frac{||I + hA_k|| - 1}{h}. \quad (16)$$

It can be shown that $\mu(A_k) = \lambda_{\max}\left[\frac{1}{2}(A_k + A_k^T)\right]$, the maximum eigenvalue of $\frac{1}{2}(A_k + A_k^T)$.

Proposition

If $\mu(A_k) < 0$ then, for any $\epsilon > 0$, the differential inequality:

$$||\ddot{\tau}_k - \ddot{\tau}_k^d||^2 < \epsilon\,, \tag{17}$$

results in $\eta_k(t)$ globally uniformly ultimately bounded.
Moreover, for any $t \mapsto \varepsilon(t) > 0$ such that $\varepsilon(t) = O\left(e^{2\,\mu(A_k)t}\right)$, the differential inequality

$$||\ddot{\tau}_k - \ddot{\tau}_k^d||^2 < \varepsilon(t)\,, \tag{18}$$

implies, for every initial condition $\eta_k(0)$,

$$\eta_k(t) \xrightarrow[t\to+\infty]{} 0\,. \tag{19}$$

$\eta_k = \begin{pmatrix} e_k \\ \dot{e}_k \end{pmatrix}$, $\dot{\eta}_k = A_k\eta_k$ where $A_k \in \mathbb{R}^{2n_k\times 2n_k}$, $e_k = \tau_k - \tau_k^r$, $\ddot{\tau}_k^d = \ddot{\tau}_k^r - D_k\dot{e}_k - P_k e_k$,
$A_k = \begin{pmatrix} 0 & I_{n_k} \\ -P_k & -D_k \end{pmatrix}$.

1. Multi-task control as a multiobjective optimization problem
2. **Stabilility of multi-task control**
3. Humanoid multi-task control as QP: robustness and continuity properties
4. Numerical Example

Our aim here is to study the behavior of the system of ordinary differential equations (ODEs) defined by

$$\ddot{q} = \operatorname{argmin} \sum_{k=1}^{p} w_k \, ||\ddot{\tau}_k - \ddot{\tau}_k^d||^2 \,. \tag{20}$$

Some notations...

$$\operatorname{vec}\begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nm} \end{pmatrix} = \begin{pmatrix} a_{11} \\ \vdots \\ a_{n1} \\ \vdots \\ \vdots \\ a_{1m} \\ \vdots \\ a_{nm} \end{pmatrix}. \tag{21}$$

Some properties...

For any vector X and matrices A, B and C such that ABC is defined we have

$$\begin{aligned} X &= \operatorname{vec} X\,, && (22)\\ \operatorname{vec}(ABC) &= (C^T \otimes A)\operatorname{vec} B\,, && (23)\\ \operatorname{vec}(AB) &= (I \otimes A)\operatorname{vec} B\,, && (24)\\ \operatorname{vec}(AB) &= (B^T \otimes I)\operatorname{vec} A\,. && (25) \end{aligned}$$

Definition

There exists a so-called commutation matrix K_{nm}, that is the $nm \times nm$ permutation matrix which transforms $\operatorname{vec} A^T$ into $\operatorname{vec} A$ for any $n \times m$ matrix A, i.e. $\forall A \in \mathbb{R}^{n \times m}$
$\operatorname{vec} A^T = K_{mn} \operatorname{vec} A$.

Definition

A matrix function $F : S \subset \mathbb{R}^{n\times m} \to \mathbb{R}^{p\times q}$ is differentiable at $C \in \text{int}(S)$ if there exists a matrix $A(C) \in \mathbb{R}^{mn\times pq}$ such that, for U in a neighborhood of 0 in $\mathbb{R}^{n\times m}$, we have

$$\operatorname{vec} F(C+U) = \operatorname{vec} F(C) + A(C) \operatorname{vec} U + o(||U||)\,. \qquad (26)$$

If $A(C)$ exists it is unique and the $p \times q$ matrix $dF(C;U)$ defined by

$$\operatorname{vec} dF(C;U) = A(C) \operatorname{vec} U\,, \qquad (27)$$

is called the differential of F at C with increment U.

Theorem

If F is differentiable at C then $A(C)$ defined in the previous definition is the Jacobian of $\operatorname{vec} F$ with respect to $\operatorname{vec} X$ (X denoting the variable of F) that we will also call the Jacobian of F at X

$$A(C) = DF(C) = \left. \frac{\partial \operatorname{vec} F}{\partial \operatorname{vec} X} \right|_C . \tag{28}$$

Examples...
The differentials of the mappings $GL_n(\mathbb{R}) \to \mathbb{R}^n$ $X \mapsto X^{-1}$; $\mathbb{R}^{n\times m} \to \mathbb{R}^{m\times n}$, $X \mapsto X^T$; and $\mathbb{R}^{n\times m} \to \mathbb{R}^{n\times n}$, $X \mapsto X^T X$ can be derived respectively as:

$$d(X^{-1}) = -X^{-1} dX\, X^{-1}\,, \tag{29}$$

$$d(X^T) = K_{nm} dX\,, \tag{30}$$

$$d(X^T X) = (K_{mm} + I_{m^2}) \left(I_m \otimes X^T\right) dX\,. \tag{31}$$

This framework allows us to give a precise sense to the expression

$$\text{“}\ \frac{\partial J_k(q)}{\partial q}\ \text{”}, \tag{32}$$

as

$$G_k = DJ_k(q) = \frac{\partial\, \mathrm{vec}\, J_k(q)}{\partial q}\,. \tag{33}$$

Some last notation (for this part)...

$$\gamma_k : \xi \mapsto \eta_k = \gamma_k(\xi) = \begin{pmatrix} g_k(q) - \tau_k^r \\ J_k(q)\dot{q} \end{pmatrix} . \tag{34}$$

$$\mathcal{J}_k(\xi) = \begin{pmatrix} J_k(q) & 0 \\ \frac{\partial [J(q)\dot{q}]}{\partial q} & J_k(q) \end{pmatrix} . \tag{35}$$

$$\Gamma_k = D\mathcal{J}_k(\xi) = \frac{\partial \operatorname{vec} \mathcal{J}_k}{\partial \xi} . \tag{36}$$

$$B(\xi) = \sum_{k=1}^{p} w_k \mathcal{J}_k(\xi)^T \mathcal{J}_k(\xi) , \tag{37}$$

Proposition

Let us suppose $B(\xi) > 0$. The system

$$\dot{\xi} = \operatorname{argmin} \sum_{k=1}^{p} w_k ||\dot{\eta}_k - A_k \eta_k||^2 , \tag{38}$$

has an equilibrium if and only if there exists ξ^0 such that

$$\sum_{k=1}^{p} w_k \mathcal{J}_k(\xi^0)^T A_k \gamma_k(\xi^0) = 0 . \tag{39}$$

In that case, the equilibrium is exponentially stable if and only if the matrix

$$B^{-1} \sum_{k=1}^{p} w_k \left((\gamma_k^T A_k^T \otimes I_{2n_k}) K_{2n_k 2n} \Gamma_k + \mathcal{J}_k^T A_k \mathcal{J}_k \right) , \tag{40}$$

evaluated at ξ^0 is stable.

In a nutshell, testing the stability of the multi-task control system amounts to computing the eingenvalues of the matrix

$$B^{-1}\sum_{k=1}^{p} w_k\left(\left(\gamma_k^T A_k^T \otimes I_{2n_k}\right) K_{2n_k 2n}\Gamma_k + \mathcal{J}_k^T A_k \mathcal{J}_k\right).$$

Corollary

If all the tasks come from a planner, i.e. such that $\forall k \; \gamma_k(\xi^0) = 0$, then ξ^0 is an equilibrium point of (38). It is exponentially stable if and only if

$$\left[\sum_{k=1}^{p} w_k \mathcal{J}_k(\xi^0)^T \mathcal{J}_k(\xi^0)\right]^{-1} \sum_{k=1}^{p} w_k \mathcal{J}(\xi^0)_k^T A_k \mathcal{J}_k(\xi^0) \qquad (41)$$

is stable.

For the case of humanoid robot multi-task control, we can build an explicit representation of the set $\mathcal{X}$ in:

$$\text{“}\min_{x\in\mathcal{X}}\text{”}\, f(x) = (||\ddot{\tau}_1 - \ddot{\tau}_1^d||^2, \ldots, ||\ddot{\tau}_p - \ddot{\tau}_p^d||^2, ||u||^2, ||\hat{\lambda}||^2)\,. \quad (42)$$

Lemma

For the humanoid problem, if $\mathcal{Y} = f(\mathcal{X})$ is nonempty, then it is $\mathbb{R}^{p+2}_{\geqq}$-convex and $\mathbb{R}^{p+2}_{\geqq}$-closed.

By the Theorem of Hartley, we can "safely" consider the weighted sum scalarization, spanning *almost all* Pareto-optimal solutions, that writes as a QP

$$\begin{aligned} &\min_{x} \; x^T Q x + l^T x, \\ &\text{subject to } H_e x = b_e,\; H_i x \leqq b_i \;, \end{aligned} \tag{43}$$

Some propositions...

Proposition

If $\mathcal{X}$ is nonempty then (43) reaches a minimum at a unique point, i.e. the solution exists and is unique.

Definition

The system of equations and inequalities

$$H_e x = b_e, \; H_i x \leqq b_i \, . \tag{44}$$

is said to be regular if H_e has full row rank and there exists x such that $H_e x = b_e$ and $H_i x < b_i$.

Proposition

Let x^0 denote the solution of (43) at an initial point ξ^0. If the system (44) is regular, then there exists $\epsilon_1 > 0$ and $\mathcal{K}_1 > 0$ such that, for any update of the state ξ or modelling error (in particular, in $M(q)$, $N(q, \dot{q})$, and the various Jacobians of the robot) the perturbed system

$$(H_e + \delta H_e)x = b_e + \delta b_e, \; (H_i + \delta H_i)x \leqq b_i + \delta b_i \, , \tag{45}$$

remains solvable and regular for those perturbations $(\delta H_e, \delta H_i, \delta b_e, \delta b_i)$ such that

$$\left|\left|\begin{pmatrix}\delta H_e \\ \delta H_i\end{pmatrix}\right|\right| + \left|\left|\begin{pmatrix}\delta b_e \\ \delta b_i\end{pmatrix}\right|\right| \leq \epsilon_1 \, , \tag{46}$$

and, denoting x any solution of (45) with $\delta x = x - x^0$, we have

$$||\delta x|| \leq \mathcal{K}_1 \left(\left|\left|\begin{pmatrix}\delta H_e \\ \delta H_i\end{pmatrix}\right|\right| + \left|\left|\begin{pmatrix}\delta b_e \\ \delta b_i\end{pmatrix}\right|\right|\right) \max\{1, ||x^0||\}(1 + ||x^0||) \, . \tag{47}$$

Proposition

Let $p = (\delta Q, \delta l, \delta H_e, \delta H_i, \delta b_e, \delta b_i)$ denote a perturbation of the QP (43). We suppose that H_e and $H_e + \delta H_e$ are both full row rank and that the system (44) is regular at the initial state ξ^0. Then there exists $\epsilon_2 > 0$ and $\mathcal{K}_2 > 0$ such that the solution $x^* = x^0 + \delta x$ of the perturbed QP

$$\begin{aligned} &\min_x \; x^T(Q + \delta Q)x + (l + \delta l)^T x, \\ &\text{subject to } (H_e + \delta H_e)x = b_e + \delta b_e, \; (H_i + \delta H_i)x \leqq b_i + \delta b_i \,, \end{aligned} \tag{48}$$

exists and is unique and satisfies, whenever $||p||_\infty < \epsilon_2$

$$||x^* - x^0|| < \mathcal{K}_2 ||p||_\infty \,. \tag{49}$$

Proposition

In the context and with the notations of the previous proposition, the mapping $p \mapsto x^*$ is well defined on a neighborhood of 0 and continuous at 0.

Figure: Example experiment with the HRP-4 humanoid robot.

Numerical Example

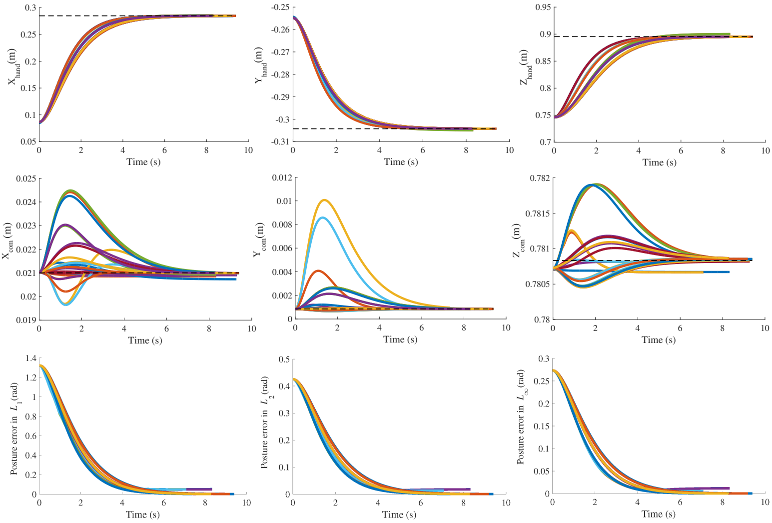

Figure: Tasks and state convergence when varying weights. We plot the trajectories for 27 runs with different sets of weights $w_{hand}, w_{com}, w_q) \in \{10^{-1}, 10^2, 10^3\}^3$.

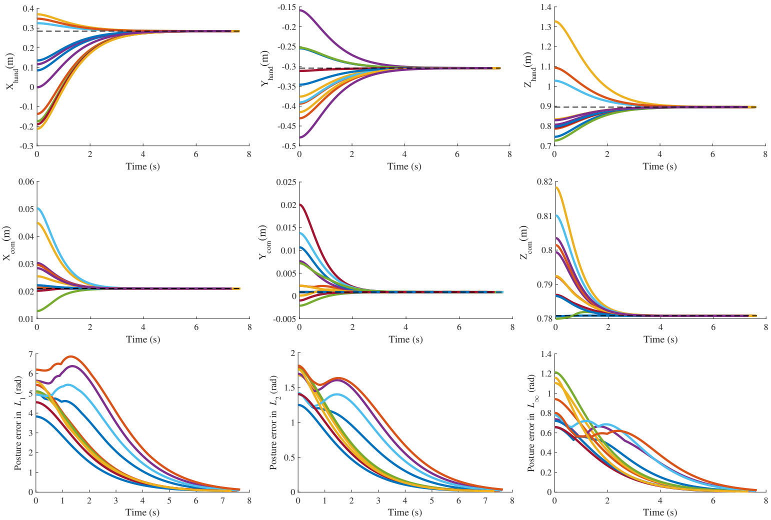

Figure: Tasks and state convergence from random initial states to assess stability of the system (20). We plot the trajectories for 10 runs of the hand reaching experiment of the HRP-4 robot starting from 10 randomly sampled initial configurations in the upper-body of the robot (randomly sampled).

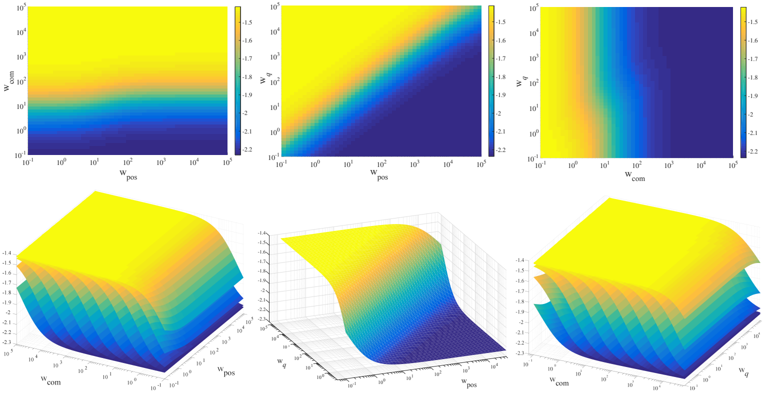

Figure: Matrix stability for assessing the stability of the ODE. We discretized the $(w_{\text{pos}}, w_{\text{com}}, w_q)$ space in $50 \times 50 \times 50$ grid in logartihmic scale ranging from 10^{-1} to 10^5 along each dimension of the weight vector $(w_{\text{pos}}, w_{\text{com}}, w_q) \in \texttt{logspace}(-1, 5, 50)^3$. We plot in color scale and surface plot the maximum real part of the eigenvalues of the matrix from Proposition **??**.

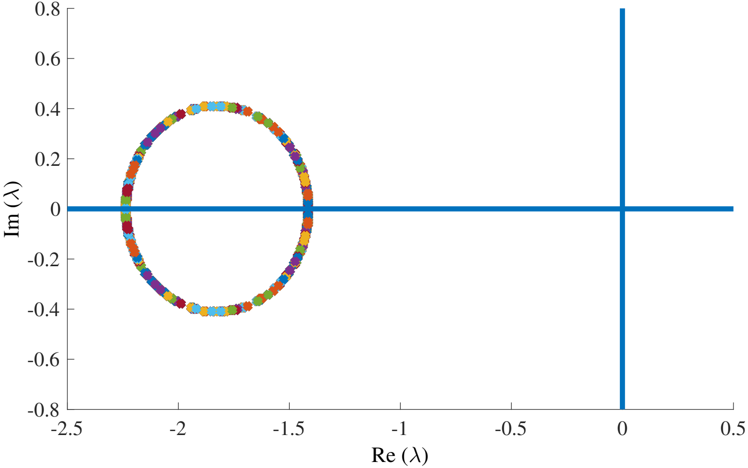

Figure: All 112000 eigenvalues (counting multiplicities) of the 112x112 matrices from Proposition 4 for 1000 set of weights ranging in logarithmic scale from 10^{-1} to 10^5, i.e. $(w_{\text{hand}}, w_{\text{com}}, w_q) \in \texttt{logspace}(-1, 5, 10)^3$. All the eigenvalues are located in the left complex half plane which means they are stable.

Conclusion

1. provide an a posteriori stability criterion, not a controller design methodology
2. pure existence proof on the weights, not constructive ones
3. local asymptotic stability results, not global ones

Thanks for your attention. Questions welcome.