Modelling Vlasov equations on complex geometries using the Semi-Lagrangian scheme

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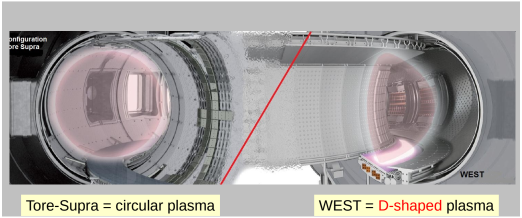
Table of contents

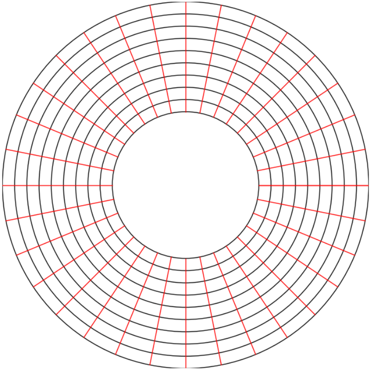
Motivation

The Gyrokinetic Semi-Lagrangian (**GYSELA**) code:



- **Gyrokinetic model**: 5D kinetic equation on the charged particules distribution
- 5 Dimensions: 2 in velocity space, 3 in configuration space
- **Simplified geometry**: concentric toroidal magnetic flux surfaces with circular cross-sections
- Based on the **Semi-Lagrangian** scheme

Standard poloidal plane mesh



Current representation of the poloidal plane :

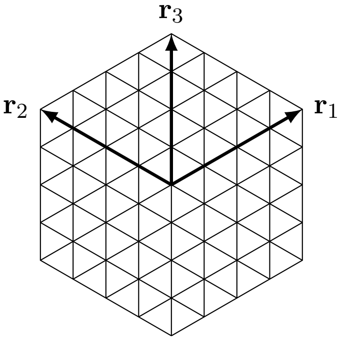
- Annular geometry
- **Polar mesh** (r, θ)

Some limitations of this choice :

- Geometric (and numeric) **singular point** at origin of mesh
- Unrepresented area and very costly to minimize that area
- Impossible to represent **complex geometries**

The hexagonal mesh[1]

Idea: Use a new mapping: **hexagon** $\longrightarrow$ **circle** (*thanks to B.D. Scott and T.T. Ribeiro*).



Some advantages:

- No singular points
- (Hopefully) no need of multiple patches for the core of the tokamak
- Twelve-fold symmetry $\Rightarrow$ more efficient programming
- Easy transformation from cartesian to hexagonal coordinates
- Easy mapping to a disk $\Rightarrow$ field aligned physical mesh

- Regularity of the mesh $\Rightarrow$ easy to find characteristic's feet (BSL)

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[1] R. Sadourny, A. Arakawa, and Y. Mintz. "Integration of the nondivergent barotropic vorticity equation with an icosahedral-hexagonal grid for the sphere". *Monthly Weather Review* 6 (1968).

The Backward Semi-Lagrangian Method

We consider the advection equation

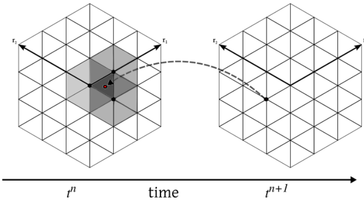
$$\frac{\partial f}{\partial t} + \mathbf{a}(x,t) \cdot \nabla_{\mathbf{x}} f = 0 \qquad (1)$$

The scheme:

- Fixed grid on phase-space
- Method of characteristics : ODE $\longrightarrow$ origin of characteristics
- Density f is conserved along the characteristics

$$\textit{i.e.} \qquad f^{n+1}(\mathbf{x}_i) = f^n(X(t_n; \mathbf{x}_i, t_{n+1})) \qquad (2)$$

- Interpolate on the origin using known values of previous step at mesh points (initial distribution f^0 known).



The guiding center model: general algorithm

We consider a reduced model of the gyrokinetic model – a simplified 2D Vlasov equation coupled with Poisson:

$$\begin{cases} \dfrac{\partial f}{\partial t} + E_\perp \cdot \nabla_X f = 0 \\ -\Delta\phi = \nabla \cdot E = f \end{cases} \tag{3}$$

The global scheme:

- Known: initial distribution function f^0 and electric field E^0
- For every time step :
  - Solve (Leap frog, RK4, ...) ODE for origin of characteristics X
  - Solve poisson equation $\Rightarrow E^{n+1}$
  - Interpolate distribution in $X^n \Rightarrow f^{n+1}$

For interpolation step: Box-splines interpolation.

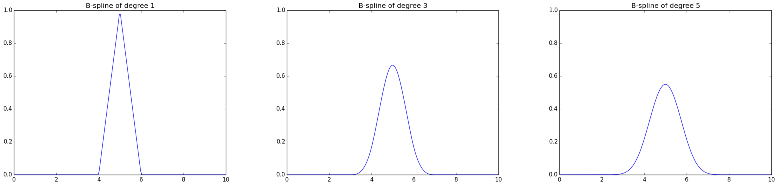
B(asis)-Splines basis*

B-Splines of degree d are defined by the **recursion** formula:

$$B_j^{d+1}(x) = \frac{x - x_j}{x_{j+d} - x_j} B_j^d(x) + \frac{x_{j+1} - x}{x_{j+d+1} - x_{j+1}} B_{j+1}^d(x) \tag{4}$$

Some important properties about B-splines:

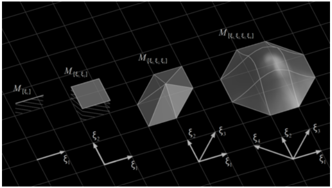
- Piecewise polynomials of degree d $\Rightarrow$ **smoothness**
- Compact support $\Rightarrow$ **sparse matrix system**
- Partition of unity $\sum_j Bj(x) = 1, \forall x$ $\Rightarrow$ **conservation laws**



Box-splines and quasi-interpolation

Box-Spline's properties:

- Generalization of B-Splines;
- depends on the vectors that define the mesh (*i.e.* **triangular meshes**);
- has compact support;
- is positive and symmetric.



Quasi-interpolation:

- Of order L if perfect reconstruction of a polynomial of degree $L-1$
- No exact interpolation at mesh points $f_h(x_i) = f(x_i) + O(\|\Delta x_i\|^L)$

$$f_h(x) = \sum_j c_j B_{\underline{\Xi}}(x - x_j) \tag{5}$$

$\Rightarrow$ Additional freedom to choose the coefficients c_j

Poisson solver : FEM based solver

The Poisson equation in cartesian coordinates:

$$-\Delta\phi = f(t,x) \quad \text{in} \quad \Omega$$

Which in weak formulation gives

$$\int_\Omega \nabla\phi \cdot \nabla\psi \, \mathrm{d}x = -\int_\Omega f(t,x)\psi \, \mathrm{d}x \tag{6}$$

with ψ test function, a **box-spline** B_j. We discretize ϕ and f as follows

$$\phi^h(\mathbf{x}) = \sum_i \phi_i B_i(\mathbf{x}), \qquad f^h(\mathbf{x}) = \sum_i f_i B_i(\mathbf{x}), \qquad \psi^h(\mathbf{x}) = B_j(\mathbf{x})$$

We obtain

$$\sum_{i,j} \phi_i \left(\int_\Omega \partial_x B_i \partial_x B_j + \int_\Omega \partial_y B_i \partial_y B_j \right) = -\sum_{i,k} f_i \int_\Omega B_i B_k \tag{7}$$

$\Rightarrow$ **SELALIB**'s general coordinate elliptic solver (developed by A. Back) or Jorek (**Django** version, developed by A. Ratnani) solver

Circular advection test case

Advection model :

$$\partial_t f + y\partial_x f - x\partial_y f = 0 \tag{8}$$

Taking a gaussian pulse as an initial distribution function

$$f^n = \exp\left(-\frac{1}{2}\left(\frac{(x^n - x_c)^2}{\sigma_x^2} + \frac{(y^n - y_c)^2}{\sigma_y^2}\right)\right) \tag{9}$$

Constant CFL ($CFL = 2$) , $\sigma_x = \sigma_y = \frac{1}{2\sqrt{2}}$, hexagonal radius : 8.
Null Dirichlet boundary condition.
Box-splines ($deg = 2$) for circular advection:

Cells	**dt**	**loops**	L_2 **error**	L_∞ **error**	points/μ-seconds
40	0.05	60	3.53E-2	7.74E-2	0.162
80	0.025	120	1.88E-3	4.66E-3	0.162
160	0.0125	240	6.77E-5	1.35E-4	0.162

Guiding center model : Diocotron instability test case

The Guiding-center model[2]:

$$\begin{cases} \dfrac{\partial f}{\partial t} + E_\perp \cdot \nabla_X f = 0 \\ -\Delta \phi = f \end{cases} \tag{10}$$

with initial distribution function (the diocotron instability in polar coordinates):

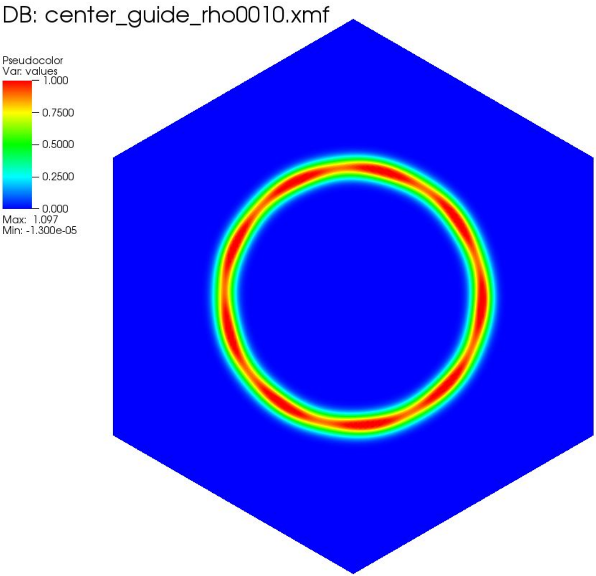
$$f(0, r, \theta) = \begin{cases} 1 + \varepsilon \cos(l \cdot \theta), & r^- \leq r \leq r^+ \\ 0, & \text{otherwise} \end{cases} \tag{11}$$

with

- $\varepsilon = 0.1$
- $l = 9.$
- radius $= 10$
- $r^- = 5$ and $r^+ = 8$
- Null Dirichlet boundary condition.

[2] L. S. Mendoza et al. *Solving the guiding-center model on a regular hexagonal mesh*. Research Report. 2015 (in progress).

Diocotron instability – Time evolution of the distribution



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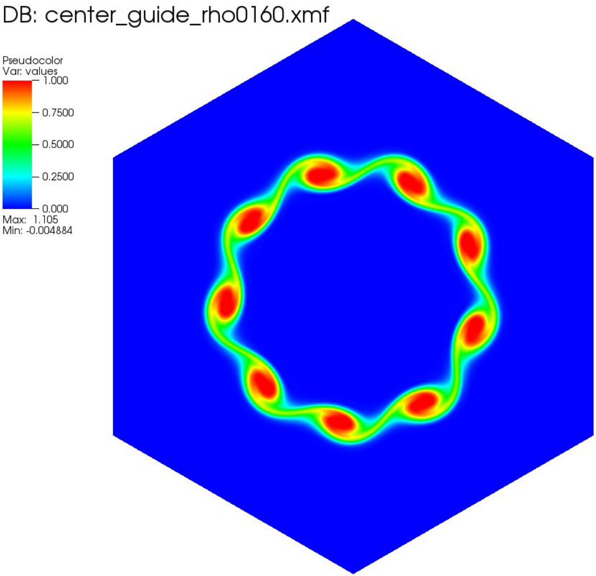

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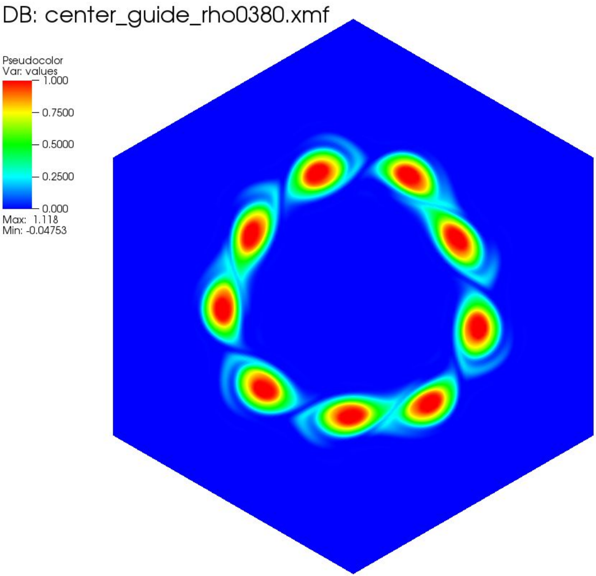

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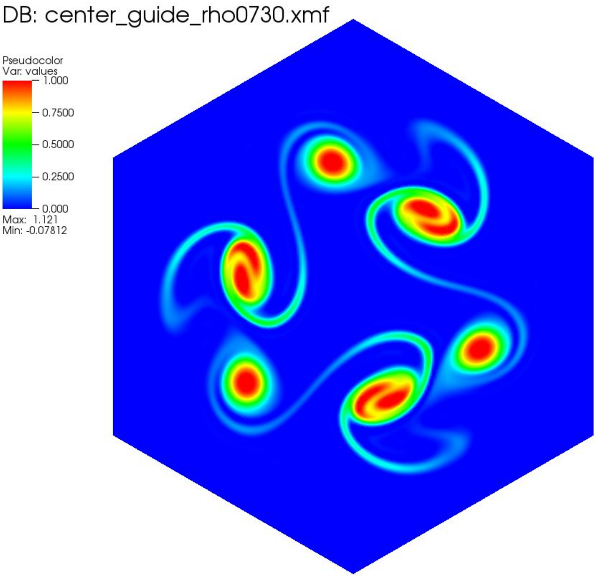

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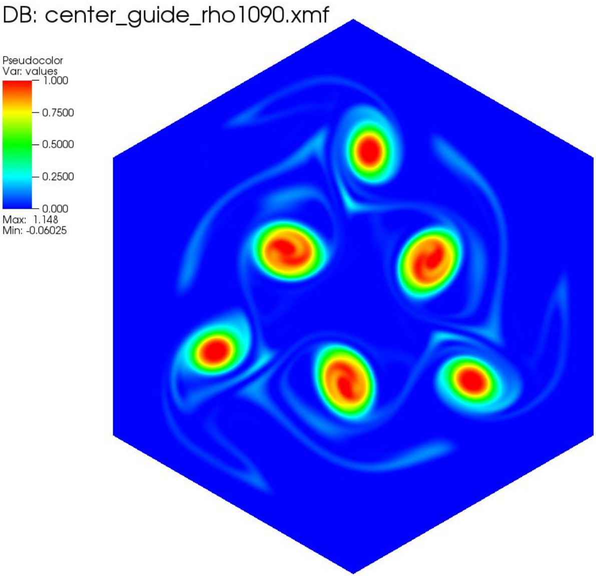
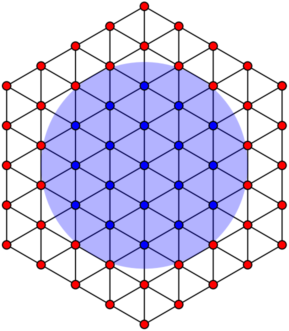


Handling boundary conditions : Main problem

Non interpolating splines $\longrightarrow$ Problems with Dirichlet boundary conditions



We can differentiate three different types of elements:

- Interior/Exterior elements
- Boundary elements

New questions arise:

- How to derive the equation such that BC intervene?
- Which elements should be considered as interior/exterior?

Nitsche's method[a] $\longrightarrow$ Adding additional terms to weak formulation

[a] A. Embar, J. Dolbow, and I. Harari. *International Journal for Numerical Methods in Engineering* 7 (2010).

Conclusions and perspectives

Conclusions:

- New mesh with no singular points for modelling the poloidal plane;
- Interpolation scheme adapted to hexagonal meshes:
  - Box-splines adapted to mesh;
  - Quasi-interpolation scheme: efficient scheme.
- Method stable for the Guiding-center model;
- Competitive results (precision/time) with:
  - Multi-patch approach;
  - Hermite Finite Elements method.

Perspectives:

- More complex models to be tested;
- Introduction of mapping to a disk to be done;
- Boundary conditions to be defined properly;
- Other geometry problems: X-point, Scrape-off layer, ...
- Hexagonal mesh for other methods: PIC, ...