# Computability of invariant measures: two counter-examples

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**Abstract**

We are interested in the computability of the invariant measures in a computable dynamical system. We construct two counter-examples. The first one has a unique SRB measure, which is not computable. The second one has no computable invariant measure at all. The systems are topological, i.e. continuous transformations on compact spaces, so they admit invariant measures.

A topological dynamical system $(X, T)$ is given by:

- a compact metric space $X$,
- a continuous map $T : X \to X$.

The Krylov-Bogolyubov theorem states that every topological system admits an invariant Borel probability measure. This theorem is not constructive. In Section 3 we construct a computable system which admits no computable invariant measure.

*Remark.* The proof of Krylov-Bogolyubov theorem uses the fact that the set of invariant measures is compact for the weak topology over the Borel probability measures. When the system is computable, the set of invariant measures is compact in an effective way, which implies that if the system is moreover uniquely ergodic (i.e. has only one invariant measure) then its unique invariant measure is computable.

## 1 Background

This is a classical result from computable analysis:

**Lemma 1.1.** *Let $X$ be a computable metric space. An open set $U$ is effectively open if and only if there is a computable function $f : X \to [0, 1]$ such that $U = f^{-1}(0, 1]$.*

*Proof.* If $U$ is effectively open then its indicator $\mathbf{1}_U$ is lower semi-computable, so there is a computable sequence of functions $f_n$ such that $\mathbf{1}_U = \sup_n f_n$. Define $f = \sum_n 2^{-n} f_n$. □

## 2 A computable system with a non-computable ergodic SRB measure

Here $X = [0, 1]$. Let $\tau \in (0, 1)$ be a lower semi-computable real number that is not computable. The set $[0, \tau)$ is effectively open, so by Lemma 1.1 there is a computable $f : X \to [0, 1]$ such that $f^{-1}(0, 1] = [0, \tau)$. Instead of using the lemma, we will build $f$ more explicitly in order to make the function $x + f(x)$ increasing.

Let $\tau \in (0, 1)$ be a real number which is lower semi-computable but not computable. Let $\tau_i \nearrow \tau$ (where $\tau_0 = 0$) be a recursive sequence of rationals converging to $\tau$ from below.

Let $\tau \in [0, 1]$ be a lower semi-computable real number which is not computable. There is a computable sequence of rational numbers $\tau_i$ such that $\sup_i \tau_i = \tau$. For each $i$, define $T_i(x) = \max(x, \tau_i)$ and $T(x) = \sum_{i \geq 1} 2^{-i} T_i$. The functions $T_i$ are uniformly computable so $T$ is also computable.

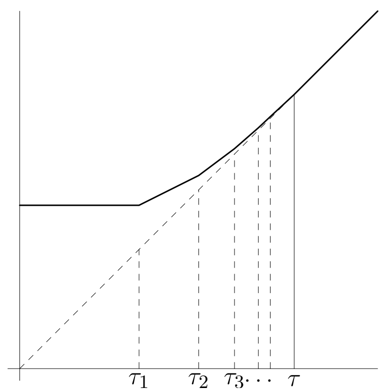


Figure 1: The map $T$.

Now, $T$ is nondecreasing, and $T(x) > x$ if and only if $x < \tau$.

The system $([0, 1], T)$ is hence a computable dynamical system. This system has a SRB ergodic invariant measure which is $\delta_\tau$, the dirac delta placed on $\tau$. Since $\tau$ is not computable then $\delta_\tau$ is not computable. Observe that $\delta_\tau$ is not isolated in the set of invariant measures (otherwise it would be computable).

## 3 A computable system with no computable invariant measure

First observe that on $X = [0, 1]$, every computable transformation admits a computable fixed point $a$, which yields the computable invariant measure $\delta_a$. The situation is different on the unit circle.

**Proposition 3.1.** *On the unit circle, there is a computable map with no computable invariant probability measure.*

On the unit interval $[0,1]$ there is an effective open set $U$ with arbitrary small (yet positive) Lebesgue measure, containing all the computable real numbers (this is very classical. For instance take $U$ from a universal Martin-Löf test). Again, we first construct a computable function as in Lemma 1.1, with some specific properties.

Let $V = U \setminus \{0,1\}$.

Let $a_i < b_i$ ($i \geq 1$) be computable sequences of rational numbers such that $V = \bigcup (a_i, b_i)$. Define $f_i(x) = x$ outside $(a_i, b_i)$, $f_i(x) = 2x - a_i$ on $[a_i, \frac{a_i+b_i}{2}]$ and $f_i(x) = b_i$ on $[\frac{a_i+b_i}{2}, b_i]$ (see Figure 2). $f_i$ is nondecreasing and computable, uniformly in $i$. The map $T_0 : [0,1] \to [0,1]$ defined by $T_0(x) = \sum_{i\geq 1} 2^{-i} f_i$ is computable and nondecreasing, and $T_0(x) > x$ if and only if $x \in V$ (indeed, $f_i(x) > x \iff x \in (a_i, b_i)$).

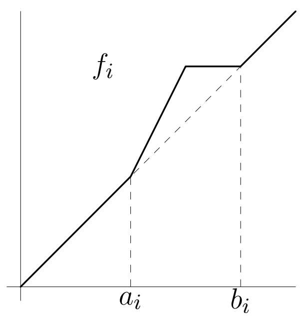


Figure 2: The map $f_i$.

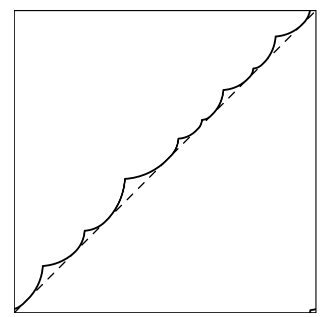

Figure 3: The map $T$.

Let $\epsilon > 0$ be a rational number such that $[0,\epsilon) \cup (1-\epsilon, 1] \subseteq U$. Let $f : [0,1] \to \mathbb{R}$ be defined by $f(x) = \epsilon$ on $[0,\epsilon]$, $f(x) = x$ on $[\epsilon, 1-\epsilon]$, $f(x) = 2x - (1-\epsilon)$ on $[1-\epsilon, 1]$. Let $S = [0,1] \mod 1$ be the unit circle, and define $T : S \to S$ by $T(x) = T_0(x) + f(x) \mod 1$. It is well-defined, continuous and computable.

Let $x \in [0,1]$: the trajectory of $x$ is nondecreasing and converges to the first point above $x$ which is not in $U$, $\inf([x,1] \setminus U)$. More precisely, there are two cases: (i) if $x \notin U$ then it is a fixed point (unstable on the right), (ii) if $x \in U$ then the trajectory of $x$ converges to a lower semi-computable fixed point (non-computable, as it does not belong to $U$).

**Lemma 3.1.** *$U$ is a strictly invariant set: $T^{-1}U = U$.*

*Proof.* If $x \notin U$ then $T(x) = x \notin U$.

If $x \in U$ then $T(x) \in U$. Indeed, if $T(x) \notin U$, $T(x)$ is a fixed point so $T$ is constant on $[x, T(x)]$ ($T$ is nondecreasing). Let $q$ be any rational number in $(x, T(x))$: $T(x) = T(q)$ is then computable, but does not belong to $U$: impossible. $\square$

**Lemma 3.2.** *Let $\mu$ be an invariant probability measure: then $\mu(U) = 0$.*

*Proof.* If $\mu(U) > 0$ then there is an interval $(a,b) = (a_i, b_i)$ from the description of $U$ with positive measure. Now, $\mu(T^{n+1}(a,b)) \geq \mu(T^n(a,b))$ (for any Borel set $A$, $A \subseteq T^{-1}(T(A))$ so $\mu(A) \leq \mu(T(A))$). But $T^n(a)$ and $T^n(b) \nearrow$ to some non computable $\alpha$ (and then are not stationary, as they are computable). Hence, it is possible to find a sequence $n_k$ such that $T^{n_k}(a,b)$ is disjoint from $T^{n_{k+1}}(a,b)$. This is contradictory with the fact that $\mu$ is finite. $\square$

We can conclude: let $\mu$ be a computable invariant probability measure: its support is then included in the complement of $U$. But the support of a computable probability measure always contains computable points: contradiction.

Actually, the set of invariant measures is exactly the set of measures which give null weight to $U$. With this, it is easy to see directly that the set of invariant measures is an effective compact set. Indeed, the function $\mu \to \mu(U)$ is lower semi-computable, so $\{\mu : \mu(U) > 0\}$ is an effective open set. Its complement is then an effective compact set, as the whole space of Borel probability measures is effectively compact.