# Descriptive complexity on non-Polish spaces II

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**Abstract**

This article is a study of descriptive complexity of subsets of represented spaces. Two competing measures of descriptive complexity are available. The first one is topological and measures how complex it is to obtain a set from open sets using boolean operations. The second one measures how complex it is to test membership in the set, and we call it symbolic complexity because it measures the complexity of the symbolic representation of the set. While topological and symbolic complexity are equivalent on countably-based spaces, they differ on more general spaces. Our investigation is aimed at explaining this difference and highly suggests that it is related to the well-known mismatch between topological and sequential aspects of topological spaces.

## 1 Introduction

This article fits in the line of research extending descriptive set theory, mainly developed on Polish spaces, to other classes of topological spaces relevant to theoretical computer science, such as domains [Sel04], quasi-Polish spaces [dB13], and represented spaces [PdB15, dBSS16, CH20]. We pursue our investigation of descriptive set theory on represented spaces, started in [CH20].

Theoretical computer science, logic and descriptive set theory closely interact, providing different ways of describing properties, by programs, formulas or boolean operation from basic properties, all intimately related. For instance, a property of real numbers that is decidable in the limit must belong to the class $\boldsymbol{\Delta}^0_2$, and every $\boldsymbol{\Delta}^0_2$-property is decidable in the limit relative to some oracle.

This correspondence works very well on Polish spaces and more generally countably-based topological spaces. However, little is known for other topological spaces whose points can be represented and processed by a program, and it has been shown in [CH20] that the correspondence fails, even on natural spaces such as the space of polynomials with real coefficients: there is a property which can be decided with 2 mind-changes, but which is not a difference of two open sets, and is in no level below $\boldsymbol{\Delta}^0_2$.

We introduce *symbolic* descriptive complexity, which captures the algorithmic complexity of a set, and compare it to topological descriptive complexity.

Our general goal is to understand when and why these two measures of complexity differ, and what topological properties of the underlying space cause this disagreement. Our results suggest that the mismatch between the two measures of complexity reflects the discordance between the sequential and the topological aspects of the space, so that symbolic complexity may be interpreted as a measure of *sequential* complexity rather than topological complexity, in the same way as many topological notions have a sequential counterpart (sequential continuity, sequential compactness, sequential closure, etc.).

More precisely, we show that among Hausdorff spaces, the spaces that are not Fréchet-Urysohn exhibit a disagreement between symbolic and topological complexity at the lowest level above the open sets, namely the differences of open sets. This result extends a similar result obtained in [CH20] for the subclass of coPolish spaces.

We focus on the space of open sets of a Polish space, and relate the disagreement between symbolic and topological complexity to the compactness properties of the Polish space, by dividing Polish spaces into 4 classes, ranging from the locally compact to the non $\sigma$-compact spaces, and giving a detailed analysis of descriptive complexity of sets in each case.

Along the way, we develop several tools and techniques that are needed to prove our results and are interesting on their own right. In particular we argue that the classical notion of hardness, which makes sense on countably-based spaces, is too restrictive on other spaces and we solve the problem by introducing the weaker notion of hard* set.

We finally observe that the discordance between topological and sequential aspects is already at the core of the theory of admissibly represented topological spaces. These spaces, also characterized as the $T_0$ quotients of countably-based spaces, are all sequential and form a subclass of topological spaces which behave particularly well from a categorical perspective: for instance, contrary to general topological spaces, they form a cartesian closed category. More concretely, in this category, the space constructions such as product space or subspaces do not coincide with the ones in the category of topological spaces, but with their sequentializations. Our separation results between symbolic and topological complexity heavily rely on the disagreement between sequential and topological space constructions.

## 1.1 Summary of the main results

We give a quick overview of the main results, stated informally.

In a represented space $\mathbf{X} = (X, \delta_X)$, we introduce the *symbolic* complexity of a set $A \subseteq X$. If $\Gamma$ is a descriptive complexity class, such as $\boldsymbol{\Sigma}^0_n$ or $\mathbf{D}_n$ (difference of $n$ open sets), then we define the corresponding symbolic complexity class $[\Gamma]$ as follows:

$$A \in [\Gamma](\mathbf{X}) \iff \delta_{\mathbf{X}}^{-1}(A) \in \Gamma(\mathrm{dom}(\delta_{\mathbf{X}})).$$

In a topological space with an admissible representation, one usually has

$$\Gamma(\mathbf{X}) \subseteq [\Gamma](\mathbf{X})$$

and our goal is to understand when and why the other inclusion does not hold, i.e. when and why the topological and symbolic measures of complexity differ. It is know from [dB13] that they coincide when $\mathbf{X}$ is a countably-based space.

We first observe that the classical notion of hardness, which is very useful to identify the complexity of a set, is closely related to symbolic rather than topological complexity. We introduce a weaker version, called hard* set and prove:

**Theorem** (Theorem 4.1)**.** *For a Borel subset $A$ of an analytic space $\mathbf{X}$,*

$$\begin{aligned} A \text{ is } \Gamma\text{-hard} &\iff A \notin [\check{\Gamma}](\mathbf{X}), \\ A \text{ is } \Gamma\text{-hard*} &\iff A \notin \check{\Gamma}(\mathbf{X}). \end{aligned}$$

A topological subspace of a sequential space is not always sequential, so the subspace constructions differ in the categories of topological and sequential spaces. This difference implies a difference between symbolic and topological complexity.

The sequential spaces whose subspaces are sequential are called the Fréchet-Urysohn spaces. The class $\underset{\sim}{\mathbf{D}}_2$ consists of differences of two open sets.

**Theorem** (Theorem 5.1)**.** *If $\mathbf{X}$ is admissibly represented, Hausdorff and not Fréchet-Urysohn, then*

$$[\underset{\sim}{\mathbf{D}}_2](\mathbf{X}) \nsubseteq \underset{\sim}{\mathbf{D}}_2(\mathbf{X}).$$

The assumption that the space is Hausdorff is needed. Indeed, spaces of open sets behave better at low complexity levels.

**Theorem** (Theorem 6.1)**.** *If $\mathbf{X}$ is admissibly represented then*

$$[\underset{\sim}{\mathbf{D}}_n](\mathcal{O}(\mathbf{X})) = \underset{\sim}{\mathbf{D}}_n(\mathcal{O}(\mathbf{X})).$$

However, the proof is not constructive and we show that the corresponding effective classes disagree. The class $\mathrm{D}_2$ consists of differences of two *effective* open sets. Let $\mathcal{N}_1$ be the space of functions $\mathbb{N} \to \mathbb{N}$ having at most 1 non-zero value.

**Theorem** (Theorem 6.2)**.** *One has $[\mathrm{D}_2](\mathcal{O}(\mathcal{N}_1)) \nsubseteq \mathrm{D}_2(\mathcal{O}(\mathcal{N}_1))$.*

Finally, we give a rather detailed study of descriptive complexity on the spaces $\mathcal{O}(\mathbf{X})$ when $\mathbf{X}$ is Polish. More precisely, we connect the relationship between symbolic and topological complexity classes to the compactness properties of $\mathbf{X}$. Some of the proofs heavily rely on the fact that the product topology is not sequential in general, so product space constructions differ in the categories of topological and sequential spaces.

In particular, symbolic and topological complexity differ at higher levels when $\mathbf{X}$ is Polish and not locally compact.

**Theorem** (Theorem 7.2)**.**

- *There exists $A \in [\underset{\sim}{\mathbf{D}}_\omega](\mathcal{O}(\mathcal{N}_1))$ which is $\underset{\sim}{\mathbf{\Delta}}^0_3$-complete*.*

- *There exists* $A \in [\underset{\sim}{\mathbf{\Sigma}}^0_k](\mathcal{O}(\mathbb{N}\times\mathcal{N}_1))$ *which is* $\underset{\sim}{\mathbf{\Sigma}}^0_{k+1}$*-complete*, for each* $k \geq 2$.
- *There exists* $A \in [\underset{\sim}{\mathbf{\Sigma}}^0_2](\mathcal{O}(\mathcal{N}))$ *which is not Borel.*

**Theorem** (Classification - Positive results, Theorem 7.1)**.** *Let* $\mathbf{X}$ *be Polish.*

- *If* $\mathbf{X} \in$ *Class I, then* $\mathcal{O}(\mathbf{X})$ *is countably-based,*
- *If* $\mathbf{X} \in$ *Class II, then for all* $k \geq 3$, $[\underset{\sim}{\mathbf{\Sigma}}^0_k](\mathcal{O}(\mathbf{X})) = \underset{\sim}{\mathbf{\Sigma}}^0_k(\mathcal{O}(\mathbf{X}))$*,*
- *If* $\mathbf{X} \in$ *Class III, then for all* $k \geq 2$, $[\underset{\sim}{\mathbf{\Sigma}}^0_k](\mathcal{O}(\mathbf{X})) \subseteq \underset{\sim}{\mathbf{\Sigma}}^0_{k+2}(\mathcal{O}(\mathbf{X}))$.

The paper is organized as follows. In Section 2, after giving the needed background on represented spaces. In Section 3 we introduce symbolic complexity and provide simple tools for its study. In Section 4 we introduce and study the notion of hard* set, used to capture the topological complexity of sets. In Section 5 we prove that Hausdorff spaces that are not Fréchet-Urysohn exhibit a disagreement between symbolic and topological complexity at the lowest level. In Section 6, we study spaces of open sets. In particular, in Section 7 we focus on open subsets of Polish spaces and locate symbolic complexity classes depending on the compactness properties of the Polish space.

# 2 Background

The Baire space is $\mathcal{N} = \mathbb{N}^\mathbb{N}$ with the product topology generated by the cylinders $[\sigma]$, with $\sigma \in \mathbb{N}^*$. A **represented space** is a pair $(X, \delta_X)$ where $X$ is a set and $\delta_X :\subseteq \mathcal{N} \to X$ is onto. A **realizer** of a function $f : (X, \delta_X) \to (Y, \delta_Y)$ is any function $F : \mathrm{dom}(\delta_X) \to \mathrm{dom}(\delta_Y)$ such that $f \circ \delta_X = \delta_Y \circ F$. $f$ is **computable** if it has a computable realizer.

A represented space $(X, \delta_X)$ is **admissible** if the continuously realizable functions $f :\subseteq \mathcal{N} \to X$ are precisely the continuous functions (for the final topology of $\delta_X$).

An effective countably-based space is a countably-based topological space $X$ with a numbered basis of the topology $(B_i)_{i\in\mathbb{N}}$ such that intersection of basic open sets is computable: $B_i \cap B_j = \bigcup_{k\in W_{f(i,j)}} B_k$ for some computable $f : \mathbb{N}^2 \to \mathbb{N}$, where $(W_e)_{e\in\mathbb{N}}$ is an effective enumeration of the c.e. subsets of $\mathbb{N}$. The standard representation, which is admissible, is defined by representing $x \in X$ by any listing of the set $\{i \in \mathbb{N} : x \in B_i\}$. A particularly useful property of these spaces is that the standard representation is **effectively open**: $\delta([\sigma]) = \bigcup_{i\in W_{g(\sigma)}} B_i$, for some computable $g$. The class $\Sigma^0_1(X)$ of effective open sets consists of c.e. unions of basic open sets. More details can be found in [Wei00, Pau15].

A topological space has an admissible representation if and only if it is $T_0$ and is a quotient of a countably-based space, written $\mathrm{QCB}_0$-space [Sch02]. Hence most of the results can be read in two ways, whether one prefers starting with a represented set or a topological space. The results are stated for admissibly represented spaces. However, they could be stated for qcb-spaces with the $T_0$ property. If $X$ is such a topological space, then the symbolic complexity classes can be defined because $X$ has an admissible representation and the symbolic classes that are closed under continuous preimages do not depend on the choice of an admissible representation.

## 2.1 Hierarchies on topological spaces

**Definition 2.1.** A **complexity class**, or simply **class**, is a family $\Gamma = \{\Gamma(X)\}$ indexed by topological spaces $X$, where $\Gamma(X)$ is a collection of subsets of the topological space $X$.

One of the simplest examples is the class of open sets $\boldsymbol{\Sigma}^0_1 = \{\boldsymbol{\Sigma}^0_1(X)\}$.

We say that a complexity class $\Gamma$ is closed under continuous preimages if for all topological spaces $X, Y$ and continuous $f : X \to Y$, $A \in \Gamma(Y)$ implies $f^{-1}(A) \in \Gamma(X)$. Complexity classes are often closed under continuous preimages, and we will always explicitely write this assumption when needed.

### 2.1.1 Borel hierarchy

The Borel hierarchy, usually defined on Polish spaces, can be extended immediately to any topological space $X$, with a slight modification to handle correctly the non-Hausdorff spaces, in which open sets are not always unions of closed sets [Sel04]. Let $X$ be a topological space.

- $\boldsymbol{\Sigma}^0_1(X)$ is the class of open sets,
- For $1 < \alpha < \omega_1$, $A \in \boldsymbol{\Sigma}^0_\alpha(X)$ if $A = \bigcup_{i\in\mathbb{N}} A_i \setminus B_i$ where $A_i, B_i \in \boldsymbol{\Sigma}^0_{\alpha_i}$ with $\alpha_i < \alpha$.

We define $\boldsymbol{\Pi}^0_\alpha(X)$ as the class of complements of sets in $\boldsymbol{\Sigma}^0_\alpha(X)$, as well as $\boldsymbol{\Delta}^0_\alpha(X) = \boldsymbol{\Sigma}^0_\alpha(X) \cap \boldsymbol{\Pi}^0_\alpha(X)$.

### 2.1.2 Difference hierarchy

Let $X$ be a topological space. The difference hierarchy $(\mathbf{D}_\alpha(\boldsymbol{\Sigma}^0_\beta(X)))_{1\le\alpha<\omega_1}$ based on $\boldsymbol{\Sigma}^0_\beta(X)$ is defined by transfinite induction as follows [Sel04]:

- $\mathbf{D}_1(\boldsymbol{\Sigma}^0_\beta(X)) = \boldsymbol{\Sigma}^0_\beta(X)$,
- $A \in \mathbf{D}_{\alpha+1}(\boldsymbol{\Sigma}^0_\beta(X))$ if $A = U \setminus B$ where $U \in \boldsymbol{\Sigma}^0_\beta(X)$ and $B \in \mathbf{D}_\alpha(\boldsymbol{\Sigma}^0_\beta(X))$,
- For a limit ordinal $\lambda$, $A \in \mathbf{D}_\lambda(\boldsymbol{\Sigma}^0_\beta(X))$ if
$$A = \bigcup_{\substack{\alpha<\lambda,\\ \alpha \text{ even}}} B_{\alpha+1} \setminus B_\alpha,$$
where $(B_\alpha)_{\alpha<\lambda}$ is a growing sequence of sets in $\boldsymbol{\Sigma}^0_\beta(X)$.

Let $\check{\mathbf{D}}_\alpha(\mathbf{\Sigma}^0_\beta(X))$ be the class of complements of sets in the class $\mathbf{D}_\alpha(\mathbf{\Sigma}^0_\beta(X))$. In order to avoid heavy notations, we write $\mathbf{D}_\alpha(X)$ to denote $\mathbf{D}_\alpha(\mathbf{\Sigma}^0_1(X))$.

In any topological space $X$, the difference hierarchy based on $\mathbf{\Sigma}^0_\beta(X)$ is contained in $\mathbf{\Delta}^0_{\beta+1}(X)$. On Polish spaces and even quasi-Polish spaces, the Hausdorff-Kuratowski Theorem states that the hierarchy entirely exhausts the class $\mathbf{\Delta}^0_{\beta+1}(X)$ (Theorem 70 in [dB13]).

## 2.2 Space of open sets

Let $\mathbf{X} = (X, \delta_\mathbf{X})$ be a represented space. It is also a topological space in the following way. First, $\mathrm{dom}(\delta_\mathbf{X}) \subseteq \mathcal{N}$ is a topological subspace of $\mathcal{N}$, whose open sets are obtained as the the intersections of the open subsets of $\mathcal{N}$ with $\mathrm{dom}(\delta_\mathbf{X})$. Next, the topology on $X$ is the final topology of $\delta_\mathbf{X}$, whose open sets are the sets $U \subseteq X$ such that $\delta_\mathbf{X}^{-1}(U)$ is open in $\mathrm{dom}(\delta_\mathbf{X})$.

The set $\mathcal{O}(\mathbf{X})$ has a canonical representation, where the names of $U \in \mathcal{O}(\mathbf{X})$ are the names of the open subsets $V$ of $\mathcal{N}$ such that $\delta_\mathbf{X}^{-1}(U) = V \cap \mathrm{dom}(\delta_\mathbf{X})$. It makes $\mathcal{O}(\mathbf{X})$ an admissibly represented space (without assuming that $\mathbf{X}$ is admissibly represented).

The represented space $\mathcal{O}(\mathbf{X})$ is in turn a topological space, whose topology is the Scott topology when $\mathbf{X}$ is admissible [Sch15].

Note that the finite levels of the Borel and difference hierarchies can in turn be equipped with representations in an obvious way. For instance, a set in $\mathbf{D}_2(\mathbf{X})$ is represented by pairing two names of open subsets of $\mathbf{X}$. A set in $\mathbf{\Sigma}^0_{n+1}(\mathbf{X})$ is inductively represented by two sequences of names of sets in $\mathbf{\Sigma}^0_n(\mathbf{X})$. How to represent sets in a given descriptive complexity class has been investigated in [Bra05, Sel13].

## 2.3 Product spaces

Each admissiby represented space $\mathbf{S}$ comes with a topology $\tau_\mathbf{S}$. If $\mathbf{S}, \mathbf{T}$ are admissibly represented spaces, then $\mathbf{S}^\mathbb{N}$ and $\mathbf{S} \times \mathbf{T}$ have natural admissible representations. However, the corresponding topologies are not in general the product topologies of $\tau_\mathbf{S}$ and $\tau_\mathbf{T}$, but their sequentializations. In this section, we identify a case when the topologies of the product spaces coincide with the product topologies. This case is when $\mathbf{S}$ and $\mathbf{T}$ are the spaces of open sets of quasi-Polish spaces.

For a topological space $X$, its space of open sets $\mathcal{O}(X)$ has at least two natural topologies: the Scott topology, defined from the ordering structure, and the compact-open topology. A topological space $X$ is called **consonant** if the Scott topology and the compact-open topology coincide on $\mathcal{O}(X)$. It is proved in [dB13] that every quasi-Polish space is consonant. Note that every quasi-Polish space has an admissible representation.

**Proposition 2.1.** *If $\mathbf{X}$ and $\mathbf{Y}$ are quasi-Polish, then the topologies on the admissiby represented spaces $\mathcal{O}(\mathbf{X})^\mathbb{N}$ and $\mathcal{O}(\mathbf{X}) \times \mathcal{O}(\mathbf{Y})$ are the product topologies.*

*Proof.* As represented spaces, one has $\mathcal{O}(\mathbf{X})^{\mathbb{N}} \cong \mathcal{O}(\mathbb{N}\times\mathbf{X})$ and $\mathcal{O}(\mathbf{X})\times\mathcal{O}(\mathbf{Y}) \cong \mathcal{O}(\mathbf{X}\sqcup\mathbf{Y})$. The topologies on the admissibly represented spaces $\mathcal{O}(\mathbb{N}\times\mathbf{X})$ and $\mathcal{O}(\mathbf{X}\sqcup\mathbf{Y})$ are the Scott topologies.

On the other hand, for any topological spaces $X, Y$, it is easy to see that the compact-open topology on $\mathcal{O}(\mathbb{N}\times X)$ and $\mathcal{O}(X\sqcup Y)$ is the product topology on $\mathcal{O}(X)^{\mathbb{N}}$ and $\mathcal{O}(X)\times\mathcal{O}(Y)$ respectively, where $\mathcal{O}(X)$ and $\mathcal{O}(Y)$ are endowed with the compact-open topology.

When $\mathbf{X}$ and $\mathbf{Y}$ are quasi-Polish, so are $\mathbb{N}\times\mathbf{X}$ and $\mathbf{X}\sqcup\mathbf{Y}$, so $\mathbf{X}$, $\mathbf{Y}$, $\mathbb{N}\times\mathbf{X}$ and $\mathbf{X}\sqcup\mathbf{Y}$ are consonant, i.e. the Scott topology and the compact-open topology coincide on their spaces of open sets. As a result the topology on the represented spaces $\mathcal{O}(\mathbf{X})^{\mathbb{N}}$ and $\mathcal{O}(\mathbf{X})\times\mathcal{O}(\mathbf{Y})$ is the product of the topologies on $\mathcal{O}(\mathbf{X})$ and $\mathcal{O}(\mathbf{Y})$. □

# 3 Symbolic descriptive complexity

The main goal of this article is to explore the relationship between the descriptive complexity of subsets of represented spaces, and the descriptive complexity of their names sets, which we call the *symbolic* complexity of the set.

Let $\mathbf{X} = (X, \delta_{\mathbf{X}})$ be a represented space. As said before, $\mathbf{X}$ is also a topological space and if $\Gamma$ is a complexity class, then we write $\Gamma(\mathbf{X})$ for the corresponding topological descriptive complexity class on the topological space $\mathbf{X}$.

We also define another notion of complexity of sets, called symbolic complexity, obtained directly from the representation.

**Definition 3.1.** Let $\Gamma$ a complexity class. For any represented space $\mathbf{X}$, we define the corresponding **symbolic** complexity class $[\Gamma](\mathbf{X})$ as follows: for $A \subseteq \mathbf{X}$,

$$A \in [\Gamma](\mathbf{X}) \iff \delta_{\mathbf{X}}^{-1}(A) \in \Gamma(\mathrm{dom}(\delta_{\mathbf{X}})).$$

Observe that the conditions $B \in \Gamma(\mathrm{dom}(\delta_{\mathbf{X}}))$ and $B = C \cap \mathrm{dom}(\delta_{\mathbf{X}})$ for some $C \in \Gamma(\mathcal{N})$ are a priori different. The latter implies the former if $\Gamma$ is closed under continuous preimages. When $\Gamma = \underset{\sim}{\mathbf{\Delta}}^0_2$, these two conditions are in general different.

**Proposition 3.1.** *If $\Gamma$ is a complexity class that is closed under continuous preimages, then $[\Gamma](\mathbf{X})$ does not depend on the choice of an admissible representation of $\mathbf{X}$.*

*Proof.* Let $\delta_1, \delta_2$ be such that $\delta_1$ is continuously reducible to $\delta_2$: there exists a continuous function $F : \mathrm{dom}(\delta_1) \to \mathrm{dom}(\delta_2)$ such that $\delta_1 = \delta_2 \circ F$. If $\delta_2^{-1}(A) \in \Gamma(\mathrm{dom}(\delta_2))$ then $\delta_1^{-1}(A) = F^{-1}(\delta_2^{-1}(A)) \in \Gamma(\mathrm{dom}(\delta_1)$.

As a result, if $\delta_1$ and $\delta_2$ are both admissible, then they are continuously reducible to each other so they induce the same class $[\Gamma](\mathbf{X})$. □

Symbolic complexity is usually more fine-grained than topological complexity.

**Proposition 3.2.** *If $\Gamma$ is a complexity class that is closed under continuous preimages, then*

$$\Gamma(\mathbf{X}) \subseteq [\Gamma](\mathbf{X}).$$

*Proof.* By definition of the final topology of $\delta_{\mathbf{X}}$, the function $\delta_{\mathbf{X}} : \mathrm{dom}(\delta_{\mathbf{X}}) \to \mathbf{X}$ is continuous, so if $A \in \Gamma(\mathbf{X})$ then $\delta_{\mathbf{X}}^{-1}(A) \in \Gamma(\mathrm{dom}(\delta_{\mathbf{X}}))$. □

The main topic of the article is to investigate when the other inclusion also holds, and more generally to better understand the symbolic complexity classes. Observe that usually, the definition of a descriptive complexity class is extensional, in the sense that it describes how its elements are built.

On Polish spaces with an admissible representation, the Borel hierarchy built from the open sets coincides with the hierarchy lifted from $\mathcal{N}$ by the representation: for instance a set is a countable union of closed sets if and only if its pre-image is a countable union of closed sets (Theorem 6.7 in [Bra05]).

On quasi-Polish spaces, the same holds if the definition of the Borel hierarchy is slightly amended: for instance $\utilde{\mathbf{\Sigma}}^0_2$-sets are not countable unions of closed sets, but countable unions of differences of open sets.

**Theorem 3.1** ([dB13])**.** *If $\mathbf{X}$ is a countably-based topological space, then*

$$[\utilde{\mathbf{\Sigma}}^0_\alpha](\mathbf{X}) = \utilde{\mathbf{\Sigma}}^0_\alpha(\mathbf{X}).$$

**Definition 3.2** (Analytic set)**.** For a topological space $X$, a set $A \subseteq X$ is **analytic**, written $A \in \utilde{\mathbf{\Sigma}}^1_1(X)$, if $A = \emptyset$ or there exists a continuous function $f : \mathcal{N} \to X$ such that $A = f(\mathcal{N})$.

*Remark* 3.1. The class $\utilde{\mathbf{\Sigma}}^1_1$ is *not* closed under continuous preimages. Indeed, let $D \subseteq \mathcal{N}$ with $D \notin \utilde{\mathbf{\Sigma}}^1_1(\mathcal{N})$. In the space $D$, one has $D \notin \utilde{\mathbf{\Sigma}}^1_1(D)$, but $D$ is the preimage of $\mathcal{N} \in \utilde{\mathbf{\Sigma}}^1_1(\mathcal{N})$ under the continuous function $\mathrm{id} : D \to \mathcal{N}$.

However, $\utilde{\mathbf{\Sigma}}^1_1$ is obviously closed under continuous *images*. So for any represented space $\mathbf{X}$, one has $[\utilde{\mathbf{\Sigma}}^1_1](\mathbf{X}) \subseteq \utilde{\mathbf{\Sigma}}^1_1(\mathbf{X})$, which contrasts with lower complexity classes, usually closed under continuous preimages and satisfying Proposition 3.2.

**Lemma 3.1.** *One has*

$$[\utilde{\mathbf{\Sigma}}^1_1](\mathcal{P}(\omega)) = \utilde{\mathbf{\Sigma}}^1_1(\mathcal{P}(\omega)).$$

*Proof.* As observed above, one obviously has $[\utilde{\mathbf{\Sigma}}^1_1](\mathcal{P}(\omega)) \subseteq \utilde{\mathbf{\Sigma}}^1_1(\mathcal{P}(\omega))$.

Conversely, assume that $A$ is the image of a continuous function $f : \mathcal{N} \to \mathcal{P}(\omega)$. The set $\{(p, q) \in \mathcal{N} \times \mathcal{N} : \delta(p) = f(q)\}$ belongs to $\Pi^0_2(\mathcal{N} \times \mathcal{N})$, in particular it is analytic, and $\delta^{-1}(A)$ is the first projection of that set, so it is also analytic. □

## 3.1 Tools

We give a simple way of locating a symbolic complexity class. A **network** in a topological space $X$ is a family $\mathcal{N}$ of subsets of $X$ such that every open set is a union of elements of $\mathcal{N}$ [Eng89].

**Proposition 3.3.** *Let* $\mathbf{X}$ *be admissibly represented. Assume that* $\mathbf{X}$ *has a countable network of sets in* $\utilde{\mathbf{\Sigma}}^0_{i+1}(\mathbf{X})$*. One has*

$$[\utilde{\mathbf{\Sigma}}^0_n](\mathbf{X}) \subseteq \utilde{\mathbf{\Sigma}}^0_{n+i}(\mathbf{X}).$$

Every admissibly represented space has a countable network, given by the images of cylinders under the admissible representation.

*Proof.* Let $\mathbf{Y}$ be the topological space with underlying set $X$ and whose topology is generated by the countable network of $\mathbf{X}$. $\mathbf{Y}$ is countably-based, let $\delta_{\mathbf{Y}}$ be its standard representation.

By definition of network, every open subset of $\mathbf{X}$ is an open subset of $\mathbf{Y}$. Conversely, every open subset of $\mathbf{Y}$ belongs to $\utilde{\mathbf{\Sigma}}^0_{i+1}(\mathbf{X})$. It implies, by induction on $n$, that $\utilde{\mathbf{\Sigma}}^0_n(\mathbf{Y}) \subseteq \utilde{\mathbf{\Sigma}}^0_{n+i}(\mathbf{X})$.

One has $[\utilde{\mathbf{\Sigma}}^0_n](\mathbf{X}) \subseteq [\utilde{\mathbf{\Sigma}}^0_n](\mathbf{Y}) = \utilde{\mathbf{\Sigma}}^0_n(\mathbf{Y})$ because we can apply Theorem 3.1 to the countably-based space $\mathbf{Y}$. As a result, $[\utilde{\mathbf{\Sigma}}^0_n](\mathbf{X}) \subseteq \utilde{\mathbf{\Sigma}}^0_{n+i}(\mathbf{X})$. □

**Corollary 3.1.** *Let* $\mathbf{X} = 2^{\mathbb{N}^{\mathbb{N}}}$ *or* $\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}$*. One has*

$$[\utilde{\mathbf{\Sigma}}^0_n](\mathbf{X}) \subseteq \utilde{\mathbf{\Sigma}}^0_{n+1}(\mathbf{X}).$$

*Proof.* The images of cylinders under the reprentation are closed subsets of $\mathbf{X}$. □

A common technique to prove a separation result in a space $\mathbf{Y}$ is to prove it in a simpler space $\mathbf{X}$ and then transfer the result to $\mathbf{Y}$ by including $\mathbf{X}$ into $\mathbf{Y}$.

**Proposition 3.4.** *Let* $\Gamma, \Gamma'$ *be complexity classes that are closed under continuous (resp. computable) preimages.*

*Let* $\mathbf{X}$ *be a continuous (resp. computable) retract of* $\mathbf{Y}$*. If* $[\Gamma](\mathbf{X}) \not\subseteq \Gamma'(\mathbf{X})$*, then* $[\Gamma](\mathbf{Y}) \not\subseteq \Gamma'(\mathbf{Y})$*.*

*Proof.* Let $r : \mathbf{Y} \to \mathbf{X}$ and $s : \mathbf{X} \to \mathbf{Y}$ be continuous (resp. computable) functions such that $r \circ s = \mathrm{id}_{\mathbf{X}}$. Let $A \in [\Gamma](\mathbf{X})$ with $A \notin \Gamma'(\mathbf{X})$. Take $B = r^{-1}(A)$. As $r$ is continuous hence continuously realizable (resp. computable), one has $B \in [\Gamma](\mathbf{Y})$. As $s$ is continuous (resp. computable) and $A = s^{-1}(B)$, $B \notin \Gamma'(\mathbf{Y})$. □

If $\mathbf{Y} = (Y, \delta_{\mathbf{Y}})$ is a represented space and $X \subseteq Y$, then $\mathbf{X} := (X, \delta_{\mathbf{X}})$ is a represented space by taking $\delta_{\mathbf{X}}$ as the restriction of $\delta_{\mathbf{Y}}$ to $\delta_{\mathbf{Y}}^{-1}(X)$. Observe that as a topological space, $\mathbf{X}$ is not always a topological subspace of $\mathbf{Y}$, but the sequentialization of the topological subspace [].

**Proposition 3.5.** *Let* $\Gamma$ *be closed under finite intersections and continuous (resp. computable) preimages, and* $\Gamma'$ *be closed under continuous (resp. computable) preimages. Let* $X \in \Gamma(\mathbf{Y})$*. If* $[\Gamma](\mathbf{X}) \not\subseteq \Gamma'(\mathbf{X})$*, then* $[\Gamma](\mathbf{Y}) \not\subseteq \Gamma'(\mathbf{Y})$*.*

*Proof.* The representation $\delta_{\mathbf{X}}$ of $\mathbf{X}$ is the restriction of $\delta_{\mathbf{Y}}$ to $\delta_{\mathbf{Y}}^{-1}(X)$.

Let $A \in [\Gamma](\mathbf{X})$ with $A \notin \Gamma'(\mathbf{X})$. One has $A \in [\Gamma](\mathbf{Y})$. Indeed, $\delta_{\mathbf{Y}}^{-1}(A) = \delta_{\mathbf{X}}^{-1}(A) = S \cap \mathrm{dom}(\delta_{\mathbf{X}})$ for some $S \in \Gamma(\mathcal{N})$, and $\mathrm{dom}(\delta_{\mathbf{X}}) = \delta_{\mathbf{Y}}^{-1}(X) = T \cap \mathrm{dom}(\delta_{\mathbf{Y}})$ for some $T \in \Gamma(\mathcal{N})$. By assumption, $U := S \cap T \in \Gamma(\mathcal{N})$ so $\delta_{\mathbf{Y}}^{-1}(A) = U \cap \mathrm{dom}(\delta_{\mathbf{Y}})$ and $A \in [\Gamma](\mathbf{Y})$.

If $A \in \Gamma'(\mathbf{Y})$, then by continuity (resp. computability) of the identity from $\mathbf{X}$ to $\mathbf{Y}$, $A \in \Gamma'(\mathbf{X})$ which is a contradiction. $\square$

# 4 Hardness

An important tool to pinpoint the descriptive complexity of a set is provided by the notions of hardness and completeness. If $\Gamma$ is a descriptive complexity class, then in any topological space $X$, one can define a set $A \subseteq X$ to be **$\Gamma$-hard** if for each $C \in \Gamma(\mathcal{N})$, there is a continuous reduction from $C$ to $A$, i.e. a continuous function $f : \mathcal{N} \to X$ such that $C = f^{-1}(A)$. Note that the reduction always starts from $\mathcal{N}$. It contrasts with the generalizations of Wadge reducibility between subsets of a topological or represented spaces investigated in [Peq15, RSS15].

As is well known in descriptive set theory on Polish (and even quasi-Polish) spaces, the hardness of a set is closely related to its complexity: Wadge's Lemma implies that for any class $\Gamma \neq \check{\Gamma}$ of Borel sets and any Borel subset $A$ of a Polish space $X$,

$$A \text{ is } \Gamma\text{-hard} \iff A \notin \check{\Gamma}(X).$$

However, outside countably-based spaces it turns out that the hardness of a set is related to its symbolic rather than topological complexity, which usually differ as we will see shortly.

Therefore, we need another notion of hardness which reflects the topological complexity of a set.

**Definition 4.1.** Let $(X, \tau)$ be a topological space and $\Gamma$ a descriptive complexity class. We say that $A \subseteq X$ is **$\Gamma$-hard*** if for every countably-based topology $\tau' \subseteq \tau$, $A$ is $\Gamma$-hard in $(X, \tau')$. A set is **$\Gamma$-complete*** if it belongs to $\Gamma(X)$ and is $\Gamma$-hard*.

Note that when $(X, \tau)$ is countably-based, these notions coincide with the standard notions of hardness and completeness.

Now we can state the main result of this section, making clear that hardness is related to symbolic complexity, while hardness* is related to topological complexity. Say that a topological space is analytic if it is a continuous image of $\mathcal{N}$.

**Theorem 4.1.** *Let $\Gamma = \mathbf{D}_\alpha(\mathbf{\Sigma}^0_\beta)$, $\alpha, \beta < \omega_1$. For an analytic admissibly represented space $X$ and $A \subseteq X$ Borel,*

$$A \textit{ is } \Gamma\textit{-hard} \iff A \notin [\check{\Gamma}](X),$$
$$A \textit{ is } \Gamma\textit{-hard*} \iff A \notin \check{\Gamma}(X).$$

For $\beta = 1$, the assumptions that the space is analytic and that $A$ is Borel can be dropped. The proof assumes $\underset{\sim}{\boldsymbol{\Sigma}}^1_1$-determinacy.

*Proof.* The first equivalence is obtained by apply Wadge's theorem to $\delta_{\mathbf{X}}^{-1}(A)$ and observe that by admissibility, any continuous function $f : \mathcal{N} \to \mathbf{X}$ factors into $f = \delta_{\mathbf{X}} \circ F$ for some continuous $F : \mathcal{N} \to \mathcal{N}$, so that $A$ is $\underset{\sim}{\boldsymbol{\Pi}}^0_\alpha$-hard if and only if $\delta_{\mathbf{X}}^{-1}(A)$ is $\underset{\sim}{\boldsymbol{\Pi}}^0_\alpha$-hard.

The proof of the second equivalence relies on the following straightforward generalization of the Louveau-Saint Raymond Theorem (see Theorem 28.19 in [Kec95]).

**Lemma 4.1.** *Let $X$ be an analytic subset of $\mathcal{P}(\omega)$. For any $A \in \underset{\sim}{\boldsymbol{\Delta}}^1_1(\mathcal{P}(\omega))$ and $\alpha < \omega_1$,*

$$A \notin \underset{\sim}{\boldsymbol{\Sigma}}^0_\alpha(\mathcal{P}(\omega)) \iff A \text{ is } \underset{\sim}{\boldsymbol{\Pi}}^0_\alpha\text{-hard.}$$

*Proof.* Let $\delta$ be the standard representation of $X := \mathcal{P}(\omega)$ and $A \notin \underset{\sim}{\boldsymbol{\Sigma}}^0_\alpha(X)$. The sets $\delta^{-1}(A)$ and $\delta^{-1}(X \setminus A)$ are disjoint analytic subsets of $\mathcal{N}$ by Lemma 3.1, and cannot be separated by a set in $\underset{\sim}{\boldsymbol{\Sigma}}^0_\alpha(\mathcal{N})$ by Theorem 3.1. We can apply the Louveau-Saint Raymond theorem, which given $C \in \underset{\sim}{\boldsymbol{\Pi}}^0_\alpha(\mathcal{N})$ yieds a continuous function $f : \mathcal{N} \to \delta^{-1}(X)$ such that $C = f^{-1}(A)$. The function $\delta \circ f : \mathcal{N} \to X$ is a continuous reduction from $C$ to $A$. □

We now prove Theorem 4.1. Let $f : \mathcal{N} \to X$ be continuous onto. If $A$ is Borel, then there is a countably-based topology $\tau' \subseteq \tau$ such that $A$ is already Borel in $(X, \tau')$. The function $f$ is *a fortiori* continuous when $X$ is endowed with the topology $\tau'$. The topology $\tau'$ is not necessarily $T_0$. Let $Y$ be the Kolmogorov quotient of $(X, \tau')$. It is a $T_0$ countably-based space, so it embeds in $\mathcal{P}(\omega)$. It is a continuous image of $\mathcal{N}$, so it is analytic. As $A$ is Borel in $(X, \tau')$, every point of $A$ is separated from every point of $X \setminus A$. As a result, the images of $A$ and $X \setminus A$ form a partition of $Y$ into two analytic sets $A_Y$ and $Y \setminus A_Y$.

As $A \notin \underset{\sim}{\boldsymbol{\Sigma}}^0_\alpha(X)$, $A_Y \notin \underset{\sim}{\boldsymbol{\Sigma}}^0_\alpha(Y)$ so $A_Y$ is $\underset{\sim}{\boldsymbol{\Pi}}^0_\alpha$-hard in $Y$ by Lemma 4.1. Let $h : \mathcal{N} \to Y$ be a continuous reduction from some $C \in \underset{\sim}{\boldsymbol{\Pi}}^0_\alpha(\mathcal{N})$ to $A_Y$. Let $i : Y \to (X, \tau')$ be any function mapping an equivalence class to an arbitrary representative. This function is continuous, and $i \circ h : \mathcal{N} \to (X, \tau')$ is a continuous reduction from $C$ to $A$, so $A$ is $\underset{\sim}{\boldsymbol{\Pi}}^0_\alpha$-hard in $(X, \tau')$. □

## 4.1 Hausdorff-Kuratowski Theorem

On Polish and even quasi-Polish spaces, there is no $\underset{\sim}{\boldsymbol{\Delta}}^0_n$-complete set because of the Hausdorff-Kuratowski theorem. Other spaces may admit $\underset{\sim}{\boldsymbol{\Delta}}^0_n$-complete* sets, and this possibility is again tightly related to the validity of the Hausdorff-Kuratowski Theorem for $\underset{\sim}{\boldsymbol{\Delta}}^0_n$-sets.

**Theorem 4.2.** *Let $X$ be an analytic topological space.*

*For each $n \geq 2$, the Hausdorff-Kuratowski Theorem holds for $\underset{\sim}{\boldsymbol{\Delta}}^0_n$ if and only if there is no $\underset{\sim}{\boldsymbol{\Delta}}^0_n$-complete* set.*

*For $n = 2$, the analyticity assumption can be droppped.*

*Proof.* If the HK Theorem holds, then there is no $\utilde{\mathbf{\Delta}}^0_n$-complete* set. Indeed, such a set $A$ would be in $\utilde{\mathbf{D}}_\alpha(\utilde{\mathbf{\Sigma}}^0_{n-1})$ for some $\alpha < \omega_1$ and some countably-based topology, and $\utilde{\mathbf{\Delta}}^0_n$-hard for that topology, which would imply that $\utilde{\mathbf{\Delta}}^0_n(\mathcal{N}) \subseteq \utilde{\mathbf{D}}_\alpha(\utilde{\mathbf{\Sigma}}^0_{n-1})(\mathcal{N})$, which is known to be false (the difference hierarchies do not collapse on $\mathcal{N}$).

Conversely, if the HK does not hold, then there exists $A \in \utilde{\mathbf{\Delta}}^0_n(X)$ such that $A \notin \utilde{\mathbf{D}}_\alpha(\utilde{\mathbf{\Sigma}}^0_{n-1})$ for any $\alpha < \omega_1$. If $X$ is analytic or $n = 2$, then $A$ is $\check{\utilde{\mathbf{D}}}_\alpha(\utilde{\mathbf{\Sigma}}^0_{n-1})$-hard* for each $\alpha < \omega_1$ by Theorem 4.1. As a result, $A$ is $\utilde{\mathbf{\Delta}}^0_n$-hard*, hence $\utilde{\mathbf{\Delta}}^0_n$-complete*. □

We now give a criterion for the validity of the Hausdorff-Kuratowski Theorem at a given level.

**Theorem 4.3.** *Let $(X, \tau)$ be a topological space. If there exists a Polish topology $\tau'$ such that $\tau \subseteq \tau' \subseteq \utilde{\mathbf{\Sigma}}^0_n(\tau)$, then the Hausdorff-Kuratowski theorem holds for $(X, \tau)$ from level $n + 1$ onwards: for $k \geq n$,*

$$\utilde{\mathbf{\Delta}}^0_{k+1}(X, \tau) = \bigcup_{\alpha<\omega_1} \utilde{\mathbf{D}}_\alpha(\utilde{\mathbf{\Sigma}}^0_k(X, \tau)).$$

The proof follows the line of the argument in [Kec95], reducing the case of $\utilde{\mathbf{\Delta}}^0_n$ to $\utilde{\mathbf{\Delta}}^0_2$ by enriching the topology. However, some care is needed because we have to deal with two topologies.

*Proof.* We first prove the following result.

*Claim* 4.1. For any $k \leq n$ and any countable family $\mathcal{F} \subseteq \utilde{\mathbf{\Sigma}}^0_k(X, \tau)$, there exists a Polish topology $\tau'' \subseteq \utilde{\mathbf{\Sigma}}^0_n(X, \tau)$ containing $\mathcal{F}$.

*Proof of the Claim.* We prove it by induction on $k$. For $k = 1$, the result is immediate by taking $\tau'' = \tau'$, as $\mathcal{F}$ is already contained in $\tau'$. Assume the result for $k < n$ and let $\mathcal{F} \subseteq \utilde{\mathbf{\Sigma}}^0_{k+1}(X, \tau)$. There exists a countable family $\mathcal{G} \subseteq \utilde{\mathbf{\Sigma}}^0_k(X, \tau)$ such that each element of $\mathcal{F}$ is a countable union of differences of elements of $\mathcal{G}$. By induction, there is a Polish topology $\tau'' \subseteq \utilde{\mathbf{\Sigma}}^0_n(X, \tau)$ containing $\mathcal{G}$. Let $\tau'''$ be generated by $\tau''$ and the complements of the elements of $\mathcal{G}$. As the latter sets are closed in $\tau''$ which is Polish, $\tau'''$ is Polish. Moreover, those sets belong to $\utilde{\mathbf{\Pi}}^0_k(X, \tau) \subseteq \utilde{\mathbf{\Sigma}}^0_n(X, \tau)$, so $\tau''' \subseteq \utilde{\mathbf{\Sigma}}^0_n(X, \tau)$. Finally, each element of $\mathcal{F}$ is open in $\tau'''$, and the claim is proved. □

We now prove the theorem. Let $A \in \utilde{\mathbf{\Delta}}^0_{n+1}(X, \tau)$. There exists a countable family $\mathcal{F} \subseteq \utilde{\mathbf{\Sigma}}^0_n(X, \tau)$ such that $A$ and its complement are countable unions of differences of elements of $\mathcal{F}$. Applying the claim, there exists a Polish topology $\tau'' \subseteq \utilde{\mathbf{\Sigma}}^0_n(X, \tau)$ containing $\mathcal{F}$. Therefore, $A \in \utilde{\mathbf{\Delta}}^0_2(X, \tau'')$ so applying the Hausdorff-Kuratowski theorem for Polish spaces, one has $A \in \utilde{\mathbf{D}}_\alpha(X, \tau'')$ for some $\alpha < \omega_1$. We conclude by observing that $\tau'' \subseteq \utilde{\mathbf{\Sigma}}^0_n(X, \tau)$.

□

We give two simple applications of this result.

On $\mathbb{R}[X]$, hence on $\mathbb{R}[X]^{\mathbb{N}}$, there is a set in $[\mathrm{D}_\omega]$ which is $\boldsymbol{\Delta}^0_2$-complete* (Theorem 5.8 in [CH20]). Theorem 4.3 implies that there is no $\boldsymbol{\Delta}^0_k$-complete* set for $k \geq 3$.

**Corollary 4.1.** *On $\mathbb{R}[X]^{\mathbb{N}}$, for all $k \geq 3$, the Hausdorff-Kuratowski Theorem holds for level $\boldsymbol{\Delta}^0_k$, therefore there is no $\boldsymbol{\Delta}^0_k$-complete* set.*

*Proof.* For each $n, d \in \mathbb{N}$, the set $C_{n,d} := \{(P_i)_{i\in\mathbb{N}} : \deg(P_i) \leq d\}$ is closed. Enriching the topology on $\mathbb{R}[X]^{\mathbb{N}}$ with these sets results in a Polish topology contained in $\boldsymbol{\Sigma}^0_2(\mathbb{R}[X]^{\mathbb{N}})$ (the space becomes homeomorphic to $\mathbb{R}^n$). □

We will see later that on $\mathcal{O}(\mathcal{N}_1)$, hence on $\mathcal{O}(\mathbb{N} \times \mathcal{N}_1)$, there is a set in $[\mathrm{D}_\omega]$ which is $\boldsymbol{\Delta}^0_3$-complete* (Theorem 7.2). Theorem 4.3 implies that there is no $\boldsymbol{\Delta}^0_k$-complete* set for $k \geq 4$.

**Corollary 4.2.** *On $\mathcal{O}(\mathbb{N}\times\mathcal{N}_1)$, for all $k \geq 4$ the Hausdorff-Kuratowski Theorem holds at level $\boldsymbol{\Delta}^0_k$, therefore there is no $\boldsymbol{\Delta}^0_k$-complete* set.*

*Proof.* We add the following sets to the topology: for each $(n, f) \in \mathbb{N} \times \mathcal{N}_1$, the closed set $\{U : (n, f) \notin U\}$; for each $(n, p) \in \mathbb{N}^2$, the $\boldsymbol{\Pi}^0_2$-set $\{U : \{n\}\times[0^p] \subseteq U\}$. The resulting topological space is homeomorphic to the Cantor space, so it is Polish, and its topology is contained in $\boldsymbol{\Sigma}^0_3(\mathcal{O}(\mathbb{N} \times \mathcal{N}_1))$. □

# 5 Fréchet-Urysohn property

In [CH20] we have given a characterization of the coPolish spaces on which the symbolic complexity differs from the topological complexity at the level $\mathbf{D}_2$: they are exactly the spaces that are not Fréchet-Urysohn.

We can extend part of the argument from coPolish spaces to Hausdorff admissibly represented spaces. We will see later (Theorem 6.1) that the assumption that the space is Hausdorff cannot be dropped.

**Theorem 5.1.** *Let $\mathbf{X}$ be admissibly represented and Hausdorff. If $\mathbf{X}$ is not Fréchet-Urysohn, then*

$$[\mathbf{D}_2](\mathbf{X}) \not\subseteq \mathbf{D}_2(\mathbf{X}).$$

We use the Arens' space $\mathbf{S_2}$, which is the inductive limit of $\{0\}\cup\{\frac{1}{n} + \frac{1}{k}X^n : n \leq N, k \in \mathbb{N}\}$. As in [CH20], one has $[\mathbf{D}_2](\mathbf{X}) \not\subseteq \mathbf{D}_2(\mathbf{X})$ and a witness is the set $A = \{0\} \cup \{\frac{1}{n} + \frac{1}{k}X^n : n, k \in \mathbb{N}\}$. Therefore, Theorem 5.1 is an immediate corollary of the next result together with Proposition 3.5.

**Proposition 5.1.** *Let $\mathbf{X}$ be admissibly represented and Hausdorff. $\mathbf{X}$ is not Fréchet-Urysohn if and only if $\mathbf{X}$ contains a closed copy of $\mathbf{S_2}$.*

*Remark* 5.1 (Historical remark about Proposition 5.1). Franklin [Fra67] proved that when $X$ is a Hausdorff sequential space, $X$ is Fréchet-Urysohn if and only if it does not contain a set which, endowed with the sequentialization of the subspace topology, is homeomorphic to $\mathbf{S_2}$ (Proposition 7.3 in [Fra67]). It implies that if $\mathbf{X}$ is a Hausdorff admissibly represented space, then $\mathbf{X}$ is not Fréchet-Urysohn if and only if $\mathbf{X}$ does not contain $\mathbf{S_2}$ as a represented subspace.

In [Tan94] and [Lin97] it is proved that when $X$ is a Hausdorff sequential space having a point-countable $k$-network, $X$ is not Fréchet-Urysohn if and only if it does not contain a *closed* set homeomorphic to $\mathbf{S_2}$ (Theorem 2.12 in [Lin97]). Observe that the subspace topology on a closed subset of a sequential space is always sequential, so there is no need to take the sequentialization of the subspace topology as in Franklin's result. This result implies ours, because admissibly represented spaces are sequential and the images of cylinders under the representation give a countable $k$-network. However we provide a proof in our setting for self-containedness.

The result was also recently proved in [dBPS19] for the subclass of coPolish spaces (Proposition 66 in [dBPS19], where $\mathbf{S_2}$ is called $\mathbf{S_{min}}$).

*Proof of Proposition 5.1.* The proof takes inspiration from the proof of Proposition 3.3.2 in [Sch02].

If $\mathbf{X}$ is not Fréchet-Urysohn, then there exist points $x, x_n, x_{n,k} \in X$ such that $\lim_n x_n = x$ and $\lim_k x_{n,k} = x_n$, but $x$ is not the limit of a sequence of points in $\{x_{n,k} : n, k \in \mathbb{N}\}$. Those points taken together are a candidate for being a copy of $\mathbf{S_2}$ in $\mathbf{X}$. However, they might not form a closed set. We show how to extract subsequences which form a closed set. First, we can assume w.l.o.g. that all these points are pairwise distinct, using that $\mathbf{X}$ is Hausdorff.

Let $(C_i)_{i\in\mathbb{N}}$ be a countable pseudobase of $\mathbf{X}$ (for instance, $C_i$ is the image under $\delta_{\mathbf{X}}$ of the cylinder number $i$ in $\mathcal{N}$). As in the proof of Proposition 3.3.2 in [Sch02], let

$$J = \{(i, j) \in \mathbb{N} \times \mathbb{N} : \exists U, V \text{ disjoint open sets}, C_i \subseteq U, C_j \subseteq V\},$$

and for $(i, j) \in J$, choose disjoint open sets $U_{i,j}$ and $V_{i,j}$ such that $C_i \subseteq U_{i,j}$ and $C_j \subseteq V_{i,j}$.

One can extract subsequences so that every $U_{i,j}$ containing $x$ contains $x_n$ and $x_{n,k}$ for almost every $n$ and all $k$. Indeed, if $x \in U_{i,j}$ then for almost every $n$, $x_n \in U_{i,j}$ and for each such $n$, $x_{n,k} \in U_i$ for almost every $k$. We now rename the sequences so that we work with the extracted subsequences.

Let $C = \{x\} \cup \{x_n : n \in \mathbb{N}\} \cup \{x_{n,k} : n, k \in \mathbb{N}\}$. Let us show that $C$ is closed.

The sets $K_x = \{x\} \cup \{x_n : n \in \mathbb{N}\}$ and $K_{x_n} = \{x_n\} \cup \{x_{n,k} : n, k \in \mathbb{N}\}$ are closed, because they are compact and $\mathbf{X}$ is Hausdorff. One has $C = K_x \cup \bigcup_n K_{x_n}$, so it is sufficient to show that $\overline{\bigcup_n K_{x_n}} = \{x\} \cup \bigcup_n K_{x_n}$.

Of course, the right-hand side is contained in the left-hand side. Conversely, let $y \in \overline{\bigcup_n K_{x_n}}$ with $y \neq x$. As $\mathbf{X}$ is Hausdorff and $y \neq x$, there exist disjoint open sets $U, V$ with $x \in U$ and $y \in V$. There exist pseudobasic sets $C_i, C_j$ such that $x \in C_i \subseteq U$ and $y \in C_j \subseteq V$, so $(i, j) \in J$. One has $x \in U_{i,j}$, so for almost every $n$, $K_{x_n} \subseteq U_{i,j}$ which is disjoint from the neighborhood $V_{i,j}$ of $y$. As a result, there exists $n_0$ such that $y \notin \overline{\bigcup_{n\geq n_0} K_{x_n}}$, so $y \in \overline{\bigcup_{n<n_0} K_{x_n}} = \bigcup_{n<n_0} K_{x_n}$.

The obvious injection $e : \mathbf{S_2} \to C$ is continuous. We show that its inverse is continuously realizable.

For each compact subset of $\mathbf{S_2}$, the restriction of $e$ has a continuous, hence continuously realizable inverse. We show that given a name of $y \in C$, we can find in finite time a compact set containing $y$, which we then use to produce a name of $e^{-1}(y)$.

Let

$$\begin{aligned} U &= \mathcal{N} \setminus \overline{\delta_{\mathbf{X}}^{-1}(C \setminus K_x)}, \\ V &= \mathcal{N} \setminus \delta_{\mathbf{X}}^{-1}(x). \end{aligned}$$

One has $\delta_{\mathbf{X}}^{-1}(x) \subseteq U$ because $x \notin [C \setminus K_x]_{\mathrm{seq}}$, so $\mathrm{dom}(\delta_{\mathbf{X}}) \subseteq U \cup V$. Moreover, $\delta_{\mathbf{X}}(U) \cap C \subseteq K_x$.

Let $p$ be a name of $y \in C$. Test in parallel whether $p \in U$ and $p \in V$. If $p \in U$ answers first, then we know that $y \in K_x$.

If $p \in V$ answers first, then one can find $n$ such that $y \in K_{x_n}$. Indeed, find $(i,j) \in J$ such that $x \in U_{i,j}$ and $y \in V_{i,j}$, and take $n_0$ such that $\bigcup_{n \geq n_0} K_{x_n} \subseteq U_{i,j}$, which implies that $y \in \bigcup_{n<n_0} K_n$. Reject all $n < n_0$ such that $y \notin K_{x_n}$, and the only remaining one is such that $y \in K_{x_n}$.

As $\mathbf{X}$ is sequential and $C \subseteq \mathbf{X}$ is closed, the topology on $C$ is the subspace topology, so $e$ is a topological embedding of $\mathbf{S_2}$ into $\mathbf{X}$. $\square$

# 6 Spaces of open sets

We now focus on a particular class of spaces, namely the spaces of open subsets of a represented space. We first show that for such spaces, symbolic complexity and topological complexity coincide for the finite levels of the difference hierarchy. It implies in particular that the assumption that the space is Hausdorff in Theorem 5.1 cannot be dropped.

## 6.1 Finite levels of the difference hierarchy

The results of this section are reminiscent of results by Grassin [Gra74] and Selivanov [Sel84] about numbered sets.

**Theorem 6.1.** *For any admissibly represented space $\mathbf{X}$, one has for all $n \in \mathbb{N}$,*

$$[\utilde{\mathbf{D}}_n](\mathcal{O}(\mathbf{X})) = \utilde{\mathbf{D}}_n(\mathcal{O}(\mathbf{X})).$$

The section is devoted to the proof of this result. We first need to isolate one of the two conditions for a set to be Scott open.

**Definition 6.1.** Say that $A \subseteq \mathcal{O}(\mathbf{X})$ is ***approximable*** if for every directed set $\Delta \subseteq \mathcal{O}(\mathbf{X})$ such that $\bigcup_{U \in \Delta} U \in A$, there exists $U \in \Delta \cap A$.

The poset $\mathcal{O}(\mathbf{X})$ has additional properties which imply a simpler characterization of this notion.

**Proposition 6.1.** *Let $A \subseteq \mathcal{O}(\mathbf{X})$. The following conditions are equivalent:*

- *$A$ is approximable,*

- *For every growing sequence $(U_i)_{i\in\mathbb{N}}$ such that $\bigcup_{i\in\mathbb{N}} U_i \in A$, there exists $i$ such that $U_i \in A$,*

- *For every growing sequence $(U_i)_{i\in\mathbb{N}}$ such that $\bigcup_{i\in\mathbb{N}} U_i \in A$, one has $U_i \in A$ for almost all $i$.*

*Proof.* The space $\mathbf{X}$ is hereditarily Lindelöf, so from $\Delta$ one can extract a countable subset with the same union, from which we can define a growing sequence using the fact that $\Delta$ is directed.

If $U_i \notin A$ for infinitely many $i$, then one can extract an infinite subsequence $U_{\varphi(i)} \notin A$ which contradicts the approximability condition. $\square$

We follow the same strategy as in [Gra74]. We show that if $A \in [\utilde{\mathbf{D}}_n](\mathcal{O}(\mathbf{X}))$, then $A$ and $A^c$ are approximable and $A$ has no $n+1$-chain, i.e. no sequence $U_0 \subseteq U_1 \subseteq \ldots \subseteq U_n$ with $U_i \in A$ exactly when $i$ is even, and we show that these properties characterize the sets in $\utilde{\mathbf{D}}_n(\mathcal{O}(\mathbf{X}))$.

**Lemma 6.1.** *If $A \subseteq \mathcal{O}(\mathbf{X})$ is not approximable then $A$ is $\utilde{\mathbf{\Pi}}^0_2$-hard.*

*If $A \subseteq \mathcal{O}(\mathbf{X})$ has an $n+1$-chain then $A$ is $\check{\utilde{\mathbf{D}}}_n$-hard.*

*Proof.* In the space $\overline{\mathbb{N}}_< = \mathbb{N} \cup \{\infty\}$ with the admissible representation $\delta(p) = \sup\{p(n) : n \in \mathbb{N}\}$, one easily shows that the set $\{\infty\}$ is $\Pi^0_2$-complete. In the space $[0,n] <$ with the admissible representation $\delta'(p) = \min(n, \delta(p))$, one can show that $E_n := \{i \leq n :$ is even$\}$ is $\check{\utilde{\mathbf{D}}}_n$-complete.

If $A$ is not approximable then there exists a growing sequence $U_i \in \mathcal{O}(\mathbf{X})$ such that $U_i \notin A$ but $U_\infty := \bigcup_i U_i \in A$. Let $f : \overline{\mathbb{N}}_< \to \mathcal{O}(\mathbf{X})$ be defined by $f(x) = U_x$. The function $f$ is Scott continuous and it reduces $\{\infty\}$ in $\overline{\mathbb{N}}_<$ to $A$, so $A$ is $\utilde{\mathbf{\Pi}}^0_2$-hard.

If $A$ has an $n+1$-chain $U_0 \subseteq U_1 \subseteq \ldots \subseteq U_n$ then the function $g : [0,n]_< \to \mathcal{O}(\mathbf{X})$ defined by $g(i) = U_i$ is Scott continuous and reduces $E_n$ to $A$, so $A$ is $\check{\utilde{\mathbf{D}}}_n$-hard. $\square$

In particular, if $A \in [\utilde{\mathbf{\Delta}}^0_2](\mathcal{O}(\mathbf{X}))$ then $A$ and $A^c$ are approximable, and if $A \in [\utilde{\mathbf{D}}_n](\mathcal{O}(\mathbf{X}))$ then $A$ has no $n+1$-chain.

The first part of the argument is completed. We now show the second part, which needs more development, in particular a reformulation of chains. To a set $A \subseteq \mathcal{O}(\mathbf{X})$, we associate a decreasing sequence of upwards closed sets as follows:

$$\mathcal{U}_0(A) = \mathcal{O}(\mathbf{X})$$
$$\mathcal{U}_{n+1}(A) = \begin{cases} \uparrow(A \cap \mathcal{U}_n(A)) & \text{if } n \text{ is even,} \\ \uparrow(A^c \cap \mathcal{U}_n(A)) & \text{if } n \text{ is odd.} \end{cases}$$

Observe that $A$ has no $n+1$-chain if and only if $\mathcal{U}_{n+1}(A) = \emptyset$.

Observe that $\mathcal{U}_{n+1}(A) = \mathcal{U}_n(A^c \cap \uparrow A)$. One easily has

$$\mathcal{U}_n(A) \setminus \mathcal{U}_{n+1}(A) \subseteq A \text{ if } n \text{ odd},$$
$$\mathcal{U}_n(A) \setminus \mathcal{U}_{n+1}(A) \subseteq A^c \text{ if } n \text{ even}.$$

We first show that if $A$ and $A^c$ are approximable, then these sets are Scott open.

**Lemma 6.2.** *If $A \subseteq \mathcal{O}(\mathbf{X})$ is approximable and $B$ is Scott open, then*

- *$\uparrow A$ is approximable, hence Scott open,*
- *$A \cap B$ is approximable, so $\uparrow(A \cap B)$ is Scott open.*

*Proof.* Let $(U_i)_{i\in\mathbb{N}}$ be a growing sequence whose union $U$ belongs to $\uparrow A$. Let $V \in A$ be contained in $U$. Let $V_i = V \cap U_i$. As $A$ is approximable, there exists $i$ such that $V_i \in A$. Therefore, $U_i \in \uparrow A$.

Let $(U_i)_{i\in\mathbb{N}}$ be a growing sequence whose union $U$ belongs to $A \cap B$. As $B$ is Scott open, then exists $i_0$ such that $U_i \in B$ for all $i \geq i_0$. Using the fact that $A$ is approximable, applied to the sequence $(U_i)_{i\geq i_0}$, gives some $i \geq i_0$ such that $U_i \in A$, and we know that $U_i \in B$. □

**Corollary 6.1.** *Let $A \subseteq \mathcal{O}(\mathbf{X})$). If both $A$ and $A^c$ are approximable, then the sets $\mathcal{U}_\alpha(A)$ are Scott open.*

*Proof.* Easy induction using Lemma 6.2 . □

We can now prove the characterization of the class $\underset{\sim}{\mathbf{D}}_n(\mathcal{O}(\mathbf{X}))$.

**Proposition 6.2.** *Let $n \in \mathbb{N}$. For $\mathbf{A} \subseteq \mathcal{O}(\mathbf{X})$, the following conditions are equivalent:*

1. *$A \in \underset{\sim}{\mathbf{D}}_n(\mathcal{O}(\mathbf{X}))$,*
2. *$A \in [\underset{\sim}{\mathbf{D}}_n](\mathcal{O}(\mathbf{X}))$,*
3. *$A$ and $A^c$ are approximable and $\mathcal{U}_{n+1}(A) = \emptyset$.*

*Proof.* We have already proved 1. $\implies$ 2. $\implies$ 3.

Assume that $\mathcal{U}_{n+1}(A) = \emptyset$. If $n$ is even then $A = (\mathcal{U}_1 \setminus \mathcal{U}_2) \cup \ldots \cup (\mathcal{U}_{n-1} \setminus \mathcal{U}_n)$. If $n$ is odd then $A = (\mathcal{U}_1 \setminus \mathcal{U}_2) \cup \ldots \cup \mathcal{U}_n$. If $A$ and $A^c$ are moreover approximable, then the sets $\mathcal{U}_i$ are Scott open, so $A \in \underset{\sim}{\mathbf{D}}_n(\mathcal{O}(\mathbf{X}))$. □

This result is similar to a characterization of $\underset{\sim}{\mathbf{D}}_\alpha$ proved in [CH20] using sets $H_{\alpha+1}(A)$: $A \in \underset{\sim}{\mathbf{D}}_\alpha(X) \iff H_{\alpha+1}(A) = \emptyset$, in any topological space $X$. One may ask whether the result presented here could be proved using the sets $H_n(A)$ rather than $\mathcal{U}_n(A)$. It is probably not possible, because the result does not extend to levels higher than $\omega$ (Theorem 7.3 below), whereas the result in [CH20] holds for all $\alpha < \omega_1$.

## 6.2 Higher levels

At least, we can extend the coincidence result to the very first level of the infinite difference hierarchy.

**Proposition 6.3.** *One has*

$$\begin{aligned}[\underset{\sim}{\mathbf{D}}_\omega](\mathcal{O}(\mathbf{X})) \cap [\check{\underset{\sim}{\mathbf{D}}}_\omega](\mathcal{O}(\mathbf{X})) &= \underset{\sim}{\mathbf{D}}_\omega(\mathcal{O}(\mathbf{X})) \cap \check{\underset{\sim}{\mathbf{D}}}_\omega(\mathcal{O}(\mathbf{X})) \\ &= \bigcup_{n\in\mathbb{N}} \underset{\sim}{\mathbf{D}}_n(\mathcal{O}(\mathbf{X})).\end{aligned}$$

*Proof.* Let $A \in [\underset{\sim}{\mathbf{D}}_\omega](\mathcal{O}(\mathbf{X})) \cap [\check{\underset{\sim}{\mathbf{D}}}_\omega](\mathcal{O}(\mathbf{X}))$. Let $p_0$ be a name of $\emptyset$. After reading some finite prefix $\sigma$, one obtains an upper bound $n$ on the number of mind-changes. The restriction of the representation to $[\sigma]$ is equivalent to the representation, so $A \in [\underset{\sim}{\mathbf{D}}_n](\mathcal{O}(\mathbf{X})) = \underset{\sim}{\mathbf{D}}_n(\mathcal{O}(\mathbf{X}))$. □

Another consequence of the preceding development is a characterization of the class $[\underset{\sim}{\mathbf{\Delta}}^0_2]$ in certain cases.

**Proposition 6.4.** *Let $\mathbf{X}$ be countably-based. The class $[\underset{\sim}{\mathbf{\Delta}}^0_2](\mathcal{O}(\mathbf{X}))$ is the class of approximable and co-approximable sets.*

*Proof.* We know from Lemma 6.1 that if $A \in [\underset{\sim}{\mathbf{\Delta}}^0_2](\mathcal{O}(\mathbf{X}))$ then both $A$ and $A^c$ are approximable.

Conversely, assume that $A \subseteq \mathcal{O}(\mathbf{X})$ and its complement are approximable. Observe that if $U_i$ is a growing sequence of open sets with union $U$, then $\mathbf{1}_A(U_i)$ converges to $\mathbf{1}_A(U)$ as $i \to \infty$, as both $A$ and $A^c$ are approximable. Let $(B_i)_{i\in\mathbb{N}}$ be a countable basis of $\mathbf{X}$, closed under finite intersections and unions. Let $E = \{i \in \mathbb{N} : B_i \in A\}$. From a name of an open set $U \in \mathcal{O}(\mathbf{X})$, one can continuously derive a sequence $(i_n)_{n\in\mathbb{N}}$ such that $B_{i_n} \subseteq B_{i_{n+1}}$ and $\bigcup_n B_{i_n} = U$. Therefore, whether $U \in A$ can be tested with finitely mind changes, by testing whether $i_n \in E$. □

## 6.3 Effectiveness

The proof of Theorem 6.1 is not effective. We show that there is no effective argument by proving that $[\mathrm{D}_2](\mathcal{O}(\mathbf{X})) \nsubseteq \mathrm{D}_2(\mathcal{O}(\mathbf{X}))$ for some particular $\mathbf{X}$.

**Theorem 6.2.** *One has*

$$[\mathrm{D}_2](\mathcal{O}(\mathcal{N}_1)) \nsubseteq \mathrm{D}_2(\mathcal{O}(\mathcal{N}_1)).$$

The argument is based on the proof of a result in [CH20], stating that when $\mathbf{X}$ is not countably-based, for sets $A \in \underset{\sim}{\mathbf{D}}_2(\mathbf{X})$, there is no continuous way of converting a $\underset{\sim}{\mathbf{D}}_2$-description of $\delta^{-1}(A)$ into a $\underset{\sim}{\mathbf{D}}_2$-description of $A$ in $\mathbf{X}$.

From the argument, we can extract the following:

**Proposition 6.5.** *There exists a computable fixed-point free multifunction $h : \underset{\sim}{\mathbf{D}}_2(\mathcal{O}(\mathcal{N}_1)) \rightrightarrows [\underset{\sim}{\mathbf{D}}_2](\mathcal{O}(\mathcal{N}_1))$.*

*Proof.* Given names of two open sets $\mathcal{U}, \mathcal{V} \in \mathcal{O}(\mathcal{O}(\mathcal{N}_1))$, we describe some set $A \in [\undertilde{\mathbf{D}}_2](\mathcal{O}(\mathcal{N}_1))$ such that $A \neq \mathcal{U} \setminus \mathcal{V}$. A name of $A$ consists in names of two open sets $E_0, E_1 \in \mathcal{O}(\mathcal{N})$ such that $\delta^{-1}(A) = E_1 \setminus E_0$, where $\delta$ is the representation of $\mathcal{O}(\mathcal{N}_1)$.

Let $f_0 \in \mathcal{N}_1$ be the null function. We start with $A = \mathcal{U}_0 = \{U \in \mathcal{O}(\mathcal{N}_1) : f_0 \in U\}$, $E_1 = \delta^{-1}(A)$ and $E_0 = \emptyset$. If we eventually see that $\mathcal{N}_1 \in \mathcal{U}$, then we stop our enumeration of $E_1$, so that $E_1$ is a finite union $C$ of cylinders, and let $E_0 = C$ and $A = \emptyset$. If we eventually see that $\mathcal{N}_1 \in \mathcal{V}$, then we can find some $V \in \mathcal{O}(\mathcal{N}_1)$ such that $V \in \mathcal{V} \setminus \delta(C)$. The set $\delta(C)$ is upwards closed, so no open subset of $V$ belongs to $\delta(C)$. We let then $A = \{U \in \mathcal{O}(\mathcal{N}_1) : U \subseteq V\}$, $E_1 = \mathcal{N}$ and $E_0 = \delta^{-1}(A^c)$, which is possible as $E_1$ and $E_0$ both contain $C$ (which has already being enumerated in them).

We can do that because $\delta(C)$ has empty interior: there exists $n$ such that every element of $\delta(C)$ contains $[0^n]$, however in every non-empty open subset of $\mathcal{O}(\mathcal{N}_1)$, there exist $V$ that does not contain $[0^n]$. Moreover, we can choose $V$ so that it is a finite union of cylinders from $\mathcal{N}_1$, so that we can effectively enumerate $E_0$. □

*Proof of Theorem 6.2.* We first use $h$ from Proposition 6.5 to perform a diagonalization and build a set in $[\mathrm{D}_2](\mathbb{N} \times \mathcal{O}(\mathcal{N}_1))$ which is not in $\mathrm{D}_2(\mathbb{N} \times \mathcal{O}(\mathcal{N}_1))$. As $\mathbb{N}$ is contained in $\mathcal{O}(\mathbb{N})$ as a $\mathrm{D}_2$-set, and $\mathcal{O}(\mathbb{N}) \times \mathcal{O}(\mathcal{N}_1) \cong \mathcal{O}(\mathbb{N} \sqcup \mathcal{N}_1) \cong \mathcal{O}(\mathcal{N}_1)$, we can include $\mathbb{N} \times \mathcal{O}(\mathcal{N}_1)$ as a $D_2$-subset of $\mathcal{O}(\mathcal{N}_1)$ and transfer the result from $\mathbb{N} \times \mathcal{O}(\mathcal{N}_1)$ to $\mathcal{O}(\mathcal{N}_1)$ by applying Proposition 3.5.

Observe that $\mathrm{D}_2(\mathbb{N} \times \mathcal{O}(\mathcal{N}_1)) \cong \mathcal{C}(\mathbb{N}, \mathrm{D}_2(\mathcal{O}(\mathcal{N}_1)))$, so we can identify a set $A \in \mathrm{D}_2(\mathbb{N} \times \mathcal{O}(\mathcal{N}_1))$ with a computable function $n \mapsto A(n) \in \mathrm{D}_2(\mathcal{O}(\mathcal{N}_1))$. Let $(A_i)_{i \in \mathbb{N}}$ be an effective enumeration of $\mathrm{D}_2(\mathbb{N} \times \mathcal{O}(\mathcal{N}_1))$. We can build some $B \in [\mathrm{D}_2](\mathbb{N} \times \mathcal{O}(\mathcal{N}_1))$ such that for all $i$, $B(i) \in h(A_i(i))$. Indeed, let $H :\subseteq \mathcal{N} \to \mathcal{N}$ be a computable realizer of $h$. Let $p_i$ be a name of $A_i(i)$, which is computable uniformly in $i$. We define $B(i)$ as the element whose name is $H(p_i)$. Finally, one has $B \notin \mathrm{D}_2(\mathbb{N} \times \mathcal{O}(\mathcal{N}_1))$, otherwise $B = A_i$ for some $i$, so $A_i(i) = B(i) \in h(A_i(i))$, contradicting the fact that $h$ has no fixed-point. □

**Corollary 6.2.** *If $\mathcal{N}_1$ embeds as a $\mathrm{D}_2$-subset of $\mathbf{X}$, then*

$$[\mathrm{D}_2](\mathcal{O}(\mathbf{X})) \nsubseteq \mathrm{D}_2(\mathcal{O}(\mathbf{X})).$$

*Proof.* $\mathcal{O}(\mathcal{N}_1)$ is a computable retract of $\mathcal{O}(\mathbf{X})$, so the separation result (Theorem 6.2) about $\mathcal{O}(\mathcal{N}_1)$ extends to $\mathcal{O}(\mathbf{X})$ by Proposition 3.4. □

# 7 Open subsets of Polish spaces

We now focus on spaces of open subsets of Polish spaces, for which we can establish a rather precise picture of the relationship between symbolic and topological complexity, depending on the compactness properties of the space.

## 7.1 The 4 classes

The first observation is that when $X$ is locally compact, for instance $X = \mathbb{R}$, $\mathcal{O}(X)$ is countably-based so it behaves very well in terms of descriptive complexity: symbolic and topological complexity coincide. We split the whole class of Polish spaces into 4 disjoint classes, ranging from the locally compact spaces to the non $\sigma$-compact spaces.

Let $X_{\mathrm{nk}} = \{x \in X : x \text{ has no compact neighborhood}\}$, which is a closed subset of $X$.

**Definition 7.1.** Let $X$ be a Polish space.

1. $X \in$ Class I if $X_{\mathrm{nk}} = \emptyset$, i.e. $X$ is locally compact,
2. $X \in$ Class II if $X_{\mathrm{nk}} \neq \emptyset$ is finite,
3. $X \in$ Class III if $X_{\mathrm{nk}} \neq \emptyset$ is infinite and $X$ is $\sigma$-compact,
4. $X \in$ Class IV if $X$ is not $\sigma$-compact.

Observe that the union of Classes I, II, III is the class of $\sigma$-compact spaces.

*Example* 7.1. Let us give one example for each class:

1. $\mathbb{R}$ belongs to Class I,
2. $\mathcal{N}_1 = \{f \in \mathcal{N} : f \text{ takes at most one positive value}\}$ belongs to Class II, with one element having no compact neighborhood, namely the zero function $f_0$,
3. $\mathbb{N} \times \mathcal{N}_1$ belongs to Class III, where the elements with no compact neighborhood are the pairs $(n, f_0)$,
4. $\mathcal{N}$ belongs to Class IV.

Moreover, the three latter spaces are minimal in their respective classes, i.e. embed into every space of their classes.

**Proposition 7.1.** *Let $X$ be Polish.*

- $X \notin$ *Class I* $\iff$ $X$ *contains a closed copy of* $\mathcal{N}_1$,
- $X \notin$ *Classes I or II* $\iff$ $X$ *contains a* $\underset{\sim}{\mathbf{D}}_2$ *copy of* $\mathbb{N} \times \mathcal{N}_1$,
- $X \notin$ *Classes I, II or III* $\iff$ $X$ *contains a closed copy of* $\mathcal{N}$.

*Proof.* The backwards implications are easy, because if $C$ is a closed subset, or even a $\underset{\sim}{\mathbf{D}}_2$-subset of $X$ and $x \in C$ has no compact neighborhood in the subspace $C$, then $x$ has no compact neighborhood in $X$.

Assume that $X$ is not locally compact and let $x_0 \in X_{\mathrm{nk}}$. We define a double-sequence $x_{i,n}$ by induction on $i$. Let $B_0$ be a basic neighborhood of $x_0$. As $\overline{B_0}$ is not compact, it contains a sequence $x_{0,n}$ with no converging subsequence. In particular, there exists a neighborhood $B_1$ of $x_0$ such that $\overline{B_1}$ does not contain any $x_{0,n}$. Again, $\overline{B_1}$ is not compact so it contains a sequence $x_{i,n}$ with no converging subsequence. We continue, making sure that the radius of $B_i$ converges to 0. One easily checks that the set $\{x_0\} \cup \{x_{i,n} : i, n \in \mathbb{N}\}$ is closed and homeomorphic to $\mathcal{N}_1$, by sending $x_0$ to the zero function, and $x_{i,n}$ to the function $f$ such that $f(i) = n$.

Assume that $X_{\mathrm{nk}}$ is infinite. It contains a copy $D$ of $\mathbb{N}$ with $D \in \underset{\sim}{\mathbf{D}}_2(X)$. Each point $x \in D$ is contained in a neighrbohood $B_x$ such that $\overline{B_x} \cap \overline{B_y} = \emptyset$ for $x \neq y$. Around each point $x$ of $D$ and inside $B_x$ we can build a closed copy of $\mathcal{N}_1$ as in the previous case. Their union is a copy of $\mathbb{N} \times \mathcal{N}_1$ and belongs to $\underset{\sim}{\mathbf{D}}_2(X)$.

The third statement is a particular case of Hurewicz theorem (Theorem 7.10 in [Kec95]). $\square$

## 7.2 Classification

We now relate the behavior of symbolic complexity on $\mathcal{O}(X)$ to the class of $X$. We first locate the symbolic complexity classes.

**Theorem 7.1** (Classification – Positive results)**.** *Let $X$ be Polish.*

1. *If $X \in$ Class I, then $[\underset{\sim}{\mathbf{\Sigma}}^0_k](\mathcal{O}(X)) = \underset{\sim}{\mathbf{\Sigma}}^0_k(\mathcal{O}(X))$ for all $k$,*
2. *If $X \in$ Class II, then $[\underset{\sim}{\mathbf{\Sigma}}^0_k](\mathcal{O}(X)) = \underset{\sim}{\mathbf{\Sigma}}^0_k(\mathcal{O}(X))$ for $k \geq 3$,*
3. *If $X \in$ Class III, then $[\underset{\sim}{\mathbf{\Sigma}}^0_k](\mathcal{O}(X)) \subseteq \underset{\sim}{\mathbf{\Sigma}}^0_{k+2}(\mathcal{O}(X))$ for $k \geq 2$.*

We then identify gaps between symbolic and topological complexity.

**Theorem 7.2** (Classification – Negative results)**.** *Let $X$ be Polish.*

1. *If $X \notin$ Class I, then $[\underset{\sim}{\mathbf{D}}_\omega](\mathcal{O}(X))$ contains a $\underset{\sim}{\mathbf{\Delta}}^0_3$-complete* set,*
2. *If $X \notin$ Class II, then $[\underset{\sim}{\mathbf{\Sigma}}^0_k](\mathcal{O}(X))$ contains a $\underset{\sim}{\mathbf{\Sigma}}^0_{k+1}$-complete*-set for $k \geq 2$,*
3. *If $X \notin$ Class III, then $[\underset{\sim}{\mathbf{\Sigma}}^0_2](\mathcal{O}(X))$ contains a non-Borel set.*

We observe that two phenomena are possible. For some spaces $\mathbf{X}$, the classes $[\underset{\sim}{\mathbf{\Sigma}}^0_k](\mathcal{O}(X))$ and $\underset{\sim}{\mathbf{\Sigma}}^0_k(\mathcal{O}(X))$ differ for low values of $k$ and then coincide after some rank. For other spaces, the classes never coincide.

It is open whether $\underset{\sim}{\mathbf{\Sigma}}^0_k(O(X)) \subseteq \underset{\sim}{\mathbf{\Sigma}}^0_{k+1}(\mathcal{O}(X))$ when $X$ belongs to Class III. A similar study should be done when $X$ is not Polish.

*Proof of Theorem 7.1.* Let $\mathbf{X}$ be $\sigma$-compact, i.e. belong to Class I, II or III. $\mathbf{X}$ has a countable network of compact sets: if $\mathbf{X} = \bigcup_n X_n$ with $X_n$ compact, then the intersections of the rational closed balls with the $X_n$'s give such a network.

*Claim* 7.1*.* For each open set $U$, the set $\uparrow U = \{O \in \mathcal{O}(\mathbf{X}) : U \subseteq O\}$ belongs to $\underset{\sim}{\mathbf{\Pi}}^0_2(\mathcal{O}(\mathbf{X}))$.

*Proof of Claim 7.1.* Let $(K_i)_{i\in\mathbb{N}}$ be a countable network of compact sets. For each $i$, the set $\mathcal{U}_{K_i} = \{O \in \mathcal{O}(\mathbf{X}) : K_i \subseteq O\}$ is open. Define $E = \{i \in \mathbb{N} : K_i \subseteq U\}$. One has $\uparrow U = \bigcap_{i\in E} \mathcal{U}_{K_i}$ which belongs to $\underset{\sim}{\mathbf{\Pi}}^0_2(\mathcal{O}(\mathbf{X}))$. $\square$

Let $(U_i)_{i\in\mathbb{N}}$ be an enumeration of the finite unions of basic open subsets of $\mathbf{X}$. The sets $\uparrow U_i$ form a countable network of $\mathcal{O}(\mathbf{X})$ and they belong to $\underset{\sim}{\mathbf{\Sigma}}^0_3(\mathcal{O}(\mathbf{X}))$. Proposition 3.3 implies that $[\underset{\sim}{\mathbf{\Sigma}}^0_k](\mathcal{O}(\mathbf{X})) \subseteq \underset{\sim}{\mathbf{\Sigma}}^0_{k+2}(\mathcal{O}(\mathbf{X}))$ for all $k$.

Let $\mathbf{X}_k = \mathbf{X} \setminus \mathbf{X}_{\mathrm{nk}}$. $\mathbf{X}_k$ is locally compact. For each open set $U \subseteq \mathbf{X}$, let $\mathcal{O}_U(\mathbf{X}) = \{O \in \mathcal{O}(\mathbf{X}) : U \subseteq O \text{ and } O \cap X_{\mathrm{nk}} = U \cap X_{\mathrm{nk}}\}$. First, $\mathcal{O}(\mathbf{X}) = \bigcup_U \mathcal{O}_U(\mathbf{X})$, for the obvious reason that each open set $U$ belongs to $\mathcal{O}_U(\mathbf{X})$. We will later see that when $\mathbf{X}$ is in Class II, this union can be reduced to a *countable* union. For the moment, $\mathbf{X}$ is just $\sigma$-compact.

*Claim* 7.2. For each $U$, one has $\mathcal{O}_U(\mathbf{X}) \in \underset{\sim}{\mathbf{\Pi}}^0_2(\mathcal{O}(\mathbf{X}))$.

*Proof of Claim 7.2.* Let $(K_i)_{i\in\mathbb{N}}$ be a countable network of compact sets. Define $E = \{i \in \mathbb{N} : K_i \subseteq U\}$ and $F = \{i \in \mathbb{N} : K_i \cap \mathbf{X}_{\mathrm{nk}} \nsubseteq U\}$. One has $O \in \mathcal{O}_U(\mathbf{X})$ iff $\forall i \in E$, $K_i \subseteq O$ and $\forall i \in F$, $K_i \cap \mathbf{X}_{\mathrm{nk}} \nsubseteq O$. Therefore, $\mathcal{O}_U(\mathbf{X})$ is a countable intersection of open and closed subsets of $\mathcal{O}(\mathbf{X})$. $\square$

*Claim* 7.3. For each $U$, $\mathcal{O}_U(\mathbf{X})$ is countably-based.

*Proof of Claim 7.3.* Indeed, $\mathcal{O}_U(\mathbf{X})$ is a continuous retract of $\mathcal{O}(\mathbf{X}_k)$, via the retraction $r(O) = O \cup U$ and the section $s(O) = O \setminus \mathbf{X}_{\mathrm{nk}}$. As $\mathbf{X}_k$ is locally compact, $\mathcal{O}(\mathbf{X}_k)$ is countably-based, so $\mathcal{O}_U(\mathbf{X})$ is countably-based as well. $\square$

Let now $A \in [\underset{\sim}{\mathbf{\Sigma}}^0_k](\mathcal{O}(\mathbf{X}))$. For each open set $U$, $A \cap \mathcal{O}_U(\mathbf{X}) \in [\underset{\sim}{\mathbf{\Sigma}}^0_k](\mathcal{O}_U(\mathbf{X})) = \underset{\sim}{\mathbf{\Sigma}}^0_k(\mathcal{O}_U(\mathbf{X}))$ as $\mathcal{O}_U(\mathbf{X})$ is countably-based. As $\mathcal{O}_U(\mathbf{X}) \in \underset{\sim}{\mathbf{\Pi}}^0_2(\mathcal{O}(\mathbf{X}))$, one has $A \cap \mathcal{O}_U(\mathbf{X}) \in \underset{\sim}{\mathbf{\Sigma}}^0_k(\mathcal{O}(\mathbf{X}))$ for $k \geq 3$.

Now assume that $\mathbf{X}$ is in Class II. $\mathbf{X}_{\mathrm{nk}}$ is finite. Let $(U_i)_{i\in\mathbb{N}}$ be an enumeration of the finite unions of basic balls of $\mathbf{X}$. One has $\mathcal{O}(\mathbf{X}) = \bigcup_i \mathcal{O}_{U_i}(\mathbf{X})$, which is a countable union. Indeed, if $O \in \mathcal{O}(\mathbf{X})$ then $O \cap \mathbf{X}_{\mathrm{nk}}$ is finite so when expressing $O$ as a union of basic balls, this finite part is already covered by some finite union $U_i$. Therefore, for $k \geq 3$, if $A \in [\underset{\sim}{\mathbf{\Sigma}}^0_k](\mathcal{O}(\mathbf{X}))$ then $A = \bigcup_i A \cap \mathcal{O}_{U_i}(\mathbf{X})$ which is a countable union of sets in $\underset{\sim}{\mathbf{\Sigma}}^0_k(\mathcal{O}(\mathbf{X}))$, so $A \in \underset{\sim}{\mathbf{\Sigma}}^0_k(\mathcal{O}(\mathbf{X}))$. $\square$

*Proof of Theorem 7.2.* In the next sections, we will prove the separation results for the spaces $\mathcal{N}_1$ (Theorem 7.3), $\mathbb{N} \times \mathcal{N}_1$ (Proposition 7.2) and $\mathcal{N}$ (Theorem 7.4). As they embed as $\underset{\sim}{\mathbf{D}}_2$-subsets of spaces of each corresponding class by Proposition 7.1, these spaces inherit the separation results. Indeed, if $D \in \underset{\sim}{\mathbf{D}}_2(\mathbf{X})$ then $\mathcal{O}(D)$ is a continuous retract of $\mathcal{O}(\mathbf{X})$: the retraction is $r(O) = O \cap D$ and the section is $s(O) = O \cup V$ if $D = U \setminus V$ with $U, V$ open. $\square$

### 7.3 Open subsets of $\mathcal{N}_1$

The next result implies in particular that Theorem 6.1 does not extend to higher levels of the difference hierarchy.

**Theorem 7.3.** *The class $[\mathrm{D}_\omega](\mathcal{O}(\mathcal{N}_1))$ contains a $\underset{\sim}{\mathbf{\Delta}}^0_3$-complete* set.*

We need two preliminary results. The first one will help defining the set.

**Lemma 7.1.** *$\mathcal{N}$ is a $[\Sigma^0_2]$-retract of $\mathcal{O}(\mathbb{N})$: there exists $r : \mathcal{O}(\mathbb{N}) \to \mathcal{N}$ which is $[\Sigma^0_2]$-measurable, $s : \mathcal{N} \to \mathcal{O}(\mathbb{N})$ which is computable, such that $r \circ s = \mathrm{id}_{\mathcal{N}}$.*

*Proof.* Let $\langle .,. \rangle : \mathbb{N}^2 \to \mathbb{N}$ be a computable bijection. Let $r(E) = f_E$ be defined by

$$f_E(i) = \begin{cases} \min\{j \in \mathbb{N} : \langle i,j \rangle \in E\} & \text{if that set is non-empty,} \\ 0 & \text{otherwise.} \end{cases}$$

Let $s(f) = \{\langle i, f(i) \rangle : i \in \mathbb{N}, f(i) \geq 1\}$. One easily checks that $r$ and $s$ satisfy the required conditions. □

The next result helps showing that the set is hard*. For $h \in \mathcal{N}$, we define the compact set $K_h = \{f \in \mathcal{N}_1 : f \leq h\}$.

**Lemma 7.2.** *Let $\tau' \subseteq \tau$ be a countably-based topology containing the point-open sets $\mathcal{U}_{\{x\}} = \{U \in \mathcal{O}(\mathcal{N}_1) : x \in U\}$, for all $x \in \mathcal{N}_1$. There exists $h$ such that on $\mathcal{U}_{K_h} = \{U \in \mathcal{O}(\mathcal{N}_1) : K_h \subseteq U\}$, the bijection $\mathcal{O}(\mathcal{N}_1) \to \mathcal{O}(\mathcal{N}_1 \setminus \{0^\omega\})$ is a homeomorphism.*

Observe that $\mathcal{N}_1 \setminus \{0^\omega\} \cong \mathbb{N}$ so $\mathcal{O}(\mathcal{N}_1 \setminus \{0^\omega\} \cong \mathcal{O}(\mathbb{N})$.

*Proof.* As $\mathcal{N}_1$ is Polish, it is consonant, i.e. the Scott topology on $\mathcal{O}(\mathcal{N}_1)$ is the compact-open topology. As a result, there exists a sequence of compact sets $(K_i)_{i \in \mathbb{N}}$ such that the corresponding sets $\mathcal{U}_{K_i} = \{U \in \mathcal{O}(\mathcal{N}_1) : K_i \subseteq U\}$ generate $\tau'$. For each $i$, there exists a function $f_i \in \mathcal{N}$ such that for all $f \in K_i$, $f \leq f_i$. Let $h$ eventually dominate each $f_i$: $h(i) = \max\{f_0(i), \dots, f_i(i)\}$. For each $i$, the set $F_i := K_i \setminus K_h$ is finite hence compact, so $\mathcal{U}_{F_i}$ is open in $\mathcal{O}(\mathcal{N}_1 \setminus \{0^\omega\})$ and $\mathcal{U}_{K_i} \cap \mathcal{U}_{K_h} = \mathcal{U}_{F_i} \cap \mathcal{U}_{K_h}$, so on $\mathcal{U}_{K_h}$, the $\tau'$-open set $\mathcal{U}_{K_i}$ coincides with $\mathcal{U}_{F_i}$. □

*Proof of Theorem 7.3.* Let $A \subseteq \mathcal{O}(\mathcal{N}_1)$ be defined as follows. To $E \in \mathcal{O}(\mathbb{N})$ one can associate a function $f_E = r(E) : \mathbb{N} \to \mathbb{N}$ by Lemma 7.1. For $U \in \mathcal{O}(\mathcal{N}_1)$, let $U_i = \{f(i) : f \in U\}$. Let

$$A = \{U \in \mathcal{O}(\mathcal{N}_1) : 0^\omega \in U \text{ and } |\{i : [0, f_{U_0}(i)] \nsubseteq U_i\}| \text{ is even}\}.$$

We first show that $A \in [\Delta^0_2](\mathcal{O}(\mathcal{N}_1))$. We are given the jump of a name of $U$. We first decide whether $0^\omega \in U$. In that case, we get some $n_0$ such that $[0^{n_0}] \subseteq U$, which implies that $U_n = \mathbb{N}$ for all $n \geq n_0$. For each $n < n_0$, we can compute $f_{U_0}(i)$ and decide whether $[0, f_{U_0}(i)]$ is contained in $U_i$, and then compute the parity of the corresponding set.

We now observe that one even has $A \in [\mathrm{D}_\omega](\mathcal{O}(\mathcal{N}_1))$, which means that there is an algorithm taking a name of $U \in \mathcal{O}(\mathcal{N}_1)$, immediately outputs "false" and may eventually produce some number $n$ and change its mind at most $n$ times. Indeed, given a name of $U$, either $0^\omega \notin U$, which implies $U \notin A$, or $0^\omega \in U$, in which case we obtain some $n_0$, from which we can decide in the limit where $U \in A$ with at most $2n_0$ mind-changes, because of the following claim.

*Claim* 7.4. For each $i$, the predicate $[0, f_{U_0}(i)] \subseteq U_i$ is decidable with at most 2 mind-changes.

*Proof of the claim.* We start with a guess $f_{U_0}(i) = 0$, in which case we declare the predicate as true. If the value of $f_{U_0}(i)$ eventually changes, then we change our mind and declare the predicate as false. From now, we can converge from above to the actual value of $f_{U_0}(i)$ and enumerate $U_i$ at the same time, so the truth of the predicate $[0, f_{U_0}(i)] \subseteq U_i$ can only change from false to true. Therefore, we will change our mind at most once. $\square$

Therefore, we can compute the number $|\{i < n_0 : [0, f_{U_0}(i)] \nsubseteq U_i\}|$ with at most $2n_0$ mind changes, and decide whether $U \in A$ with this number of mind changes.

As $\mathcal{N}_1$ is in Class II, one has $[\boldsymbol{\Delta}^0_3](\mathcal{O}(\mathcal{N}_1)) = \boldsymbol{\Delta}^0_3(\mathcal{O}(\mathcal{N}_1))$, so $A \in \boldsymbol{\Delta}^0_3(\mathcal{O}(\mathcal{N}_1))$. We now show that for any countably-based topology $\tau' \subseteq \tau$, $A$ is $\boldsymbol{\Delta}^0_3$-hard. We reduce the problem of computing the parity of a finite $\boldsymbol{\Pi}^0_1$-subset of $\mathbb{N} \setminus \{0\}$ to $A$. We can enrich $\tau'$ so that it contains the point-open sets and use Lemma 7.2 giving a function $h$, and work in $\mathcal{U}_{K_h}$, so that the topology $\tau'$ coincides with the topology on $\mathcal{O}(\mathcal{N} \setminus \{0^\omega\}) \cong \mathcal{O}(\mathbb{N})$. We can assume that $h(0) = 0$ for simplicity.

Given $P \in \boldsymbol{\Pi}^0_1(\mathbb{N})$ finite, we define $U \in \mathcal{O}(\mathcal{N}_1)$ containing $K_h$, by defining $U_i$ for each $i \in \mathbb{N}$ as follows:

$$U_0 = \{\langle i, j\rangle : i \notin P \text{ or } j \geq h(i) + 1\},$$

and for $i \geq 1$,

$$U_i = \begin{cases} \mathbb{N} \setminus \{h(i) + 1\} & \text{if } i \in P, \\ \mathbb{N} & \text{if } i \notin P. \end{cases}$$

One can check that

$$f_{U_0}(i) = \begin{cases} h(i) + 1 & \text{if } i \in P, \\ 0 & \text{if } i \notin P. \end{cases}$$

so $[0, f_{U_0}(i)] \subseteq U_i$ if and only if $i \notin P$. As a result, $U \in A$ if and only if $|P|$ is even.

Observe that $U$ indeed contains $K_h$ so the reduction, which is continuous for the topology on $\mathcal{O}(\mathcal{N}_1 \setminus \{0^\omega\})$, is continuous for $\tau'$. Therefore, we have shown that $A$ is $\boldsymbol{\Delta}^0_3$-hard*. $\square$

## 7.4 Open subsets of $\mathbb{N} \times \mathcal{N}_1$

Using Theorem 7.3, we can build a set in $[\Sigma^0_2]$ which is $\boldsymbol{\Sigma}^0_3$-hard*, and then iterate the construction in order to climb the finite levels of the Borel hierarchy, giving the following result.

**Proposition 7.2.** *Let $\mathbf{X} = \mathbb{N} \times \mathcal{N}_1$. For each $n \geq 2$, the class $[\Sigma^0_n](\mathcal{O}(\mathbf{X}))$ contains a $\boldsymbol{\Sigma}^0_{n+1}$-complete* set.*

The iteration is given by the next result.

**Lemma 7.3.** *Let $\mathbf{X}$ be quasi-Polish and $n \in \mathbb{N}$. If $A \subseteq \mathcal{O}(\mathbf{X})$ is $\boldsymbol{\Sigma}^0_n$-hard*, then $B_A := \{(U_i)_{i\in\mathbb{N}} : \exists i, U_i \notin A\}$ is $\boldsymbol{\Sigma}^0_{n+1}$-hard* in $\mathcal{O}(\mathbf{X})^{\mathbb{N}}$.*

*Proof.* Let $\tau$ be a countably-based topology on $\mathcal{O}(\mathbf{X})^{\mathbb{N}}$. Observe that $\mathbf{X}$ is quasi-Polish, so the topology on $\mathcal{O}(\mathbf{X})^{\mathbb{N}}$ is the product topology by Proposition 2.1. As a result, there is a countably-based topology $\tau'$ on $\mathcal{O}(\mathbf{X})$ such that $\tau$ is contained in the product topology of $\tau'$. We can define a $\tau$-continuous reduction from a set in $\boldsymbol{\Sigma}^0_{n+1}(\mathcal{N})$ to $B_A$, by using the fact that $A$ is $\boldsymbol{\Sigma}^0_n$-hard in the topology $\tau'$. Indeed, let $C \in \boldsymbol{\Sigma}^0_{n+1}(\mathcal{N})$. There exists a sequence $C_i \in \boldsymbol{\Pi}^0_n(\mathcal{N})$ such that $C = \bigcup_i C_i$. Each $C_i$ is reducible to $A^c$, so there exists continuous functions $f_i : \mathcal{N} \to (\mathcal{O}(\mathbf{X}), \tau')$ such that $C_i = f_i^{-1}(A^c)$. Let $f : \mathcal{N} \to (\mathcal{O}(\mathbf{X})^{\mathbb{N}}, \tau)$ be defined by $f(p) = (f_i(p))_{i\in\mathbb{N}}$. One has $C = f^{-1}(B_A)$ and $f$ is continuous, because it is continuous for the product of $\tau'$, which contains $\tau$. ☐

*Proof of Proposition 7.2.* The key observation is that we can iterate Lemma 7.3 because $\mathbf{X} \cong \mathbb{N} \times \mathbf{X}$ so $\mathcal{O}(\mathbf{X}) \cong \mathcal{O}(\mathbf{X})^{\mathbb{N}}$.

We prove the result by induction on $n \geq 2$.

For $n = 2$, we apply Lemma 7.3 to the set $A$ provided by Theorem 7.3, which belongs to $[\Delta^0_2](\mathcal{O}(\mathcal{N}_1))$ and is $\boldsymbol{\Delta}^0_3$-complete*. In particular, it is $\boldsymbol{\Sigma}^0_2$-hard*. The set $B_A$ belongs to $[\Sigma^0_2](\mathcal{O}(\mathbf{X}))$ and $\boldsymbol{\Sigma}^0_3(\mathcal{O}(\mathbf{X}))$ and is $\boldsymbol{\Sigma}^0_3$-hard* by Lemma 7.3.

Let $n \geq 2$ and assume that $A \in [\Sigma^0_n](\mathcal{O}(\mathbf{X}))$ is $\boldsymbol{\Sigma}^0_{n+1}$-complete*. The set $B_A$ belongs to $[\Sigma^0_{n+1}](\mathcal{O}(\mathbf{X}))$ and $\boldsymbol{\Sigma}^0_{n+2}(\mathcal{O}(\mathbf{X}))$, and is $\boldsymbol{\Sigma}^0_{n+2}$-hard* by Lemma 7.3. Therefore the induction step is proved. ☐

Observe that Lemma 7.3 relies on the fact that for the space $\mathbf{Y} := \mathcal{O}(\mathbf{X})$, the topology on the space $\mathbf{Y}^{\mathbb{N}}$ of sequences is the product topology. It would be interesting to know whether the result fails for some $\mathbf{Y}$ that does not satisfy this condition, or for some $\mathbf{Y} = \mathcal{O}(\mathbf{X})$ where $\mathbf{X}$ is not quasi-Polish.

### 7.5 Open subsets of $\mathcal{N}$

Here we show that for the space $\mathbf{X} = \mathcal{O}(\mathcal{N})$, the class $[\Sigma^0_2](\mathbf{X})$ contains a set that is not Borel.

**Theorem 7.4.** *There is a set in $[\Sigma^0_2](\mathcal{O}(\mathcal{N}))$ which is not Borel.*

To build this set, we first work on an intermediate space.

We have seen that two represented spaces $\mathbf{X}$ and $\mathbf{Y}$ naturally induce a third represented space $\mathbf{X} \times \mathbf{Y}$. The topology induced by that representation is not in general the product topology, but its sequentialization.

A simple example is given by $\mathbf{X} = \mathcal{N}$ and $\mathbf{Y} = \mathcal{O}(\mathcal{N})$. The evaluation map $\mathcal{N} \times \mathcal{O}(\mathcal{N}) \to \mathbb{S}$ is continuous (and computable), however it is not continuous w.r.t. the product topology, because $\mathcal{N}$ is not locally compact (see [EH02] for more details on this topic). In other words the set $\{(f, O) \in \mathcal{N} \times \mathcal{O}(\mathcal{N}) : f \in O\}$ is not open for the product topology (but it is sequentially open, or open for the topology induced by the representation). It is even worse.

**Proposition 7.3.** *$E = \{(f, O) \in \mathcal{N} \times \mathcal{O}(\mathcal{N}) : f \in O\}$ is not Borel for the product topology.*

*Proof.* We prove that for every Borel set $A$, there exists a dense $G_\delta$-set $G \subseteq \mathcal{N}$ such that for every $f \in G$, $(f, \mathcal{N} \setminus \{f\}) \in A \iff (f, \mathcal{N}) \in A$. It implies the result as it is obviously false for the set $E$. To prove it, we show that the class of sets satisfying this condition contains the open sets in the product topology and is closed under taking complements and countable unions, which implies that this class contains the Borel sets.

First, consider a basic open set $A = [u] \times \mathcal{U}_K$ where $u$ is a finite sequence of natural numbers, $K$ is a compact subset of $\mathcal{N}$ and $\mathcal{U}_K = \{O \in \mathcal{O}(\mathcal{N}) : K \subseteq O\}$. Define $G = [u]^c \cup [u] \setminus K$, which is a dense open set. For $f \in [u]^c$, no $(f, O)$ belongs to $A$. For $f \in [u] \setminus K$, both $(f, \mathcal{N} \setminus \{f\})$ and $(f, \mathcal{N})$ belong to $A$.

If $A$ satisfies the condition with a dense $G_\delta$-set $G$, then $A^c$ satisfies the condition with the same $G$. If $A_i$ satisfy the condition with dense $G_\delta$-sets $G_i$ then $\bigcup_i A_i$ satisfies the condition with $G = \bigcap_i G_i$. □

*Proof of Theorem 7.4.* We show that $\mathcal{N} \times_{\text{prod}} \mathcal{O}(\mathcal{N})$, which is the topological space endowed with the product topology, is a $[\Sigma^0_2]$-retract of $\mathcal{O}(\mathcal{N})$. We build:

- A continuous function $s : \mathcal{N} \times_{\text{prod}} \mathcal{O}(\mathcal{N}) \to \mathcal{O}(\mathcal{N})$,
- A $[\Sigma^0_2]$-measurable function $r : \mathcal{O}(\mathcal{N}) \to \mathcal{N} \times \mathcal{O}(\mathcal{N})$,
- Such that $r \circ s = \text{id}$.

First, these ingredients enable us to derive the result. Indeed, let $E$ be the set from Proposition 7.3 and $F := r^{-1}(E) \subseteq \mathcal{O}(\mathcal{N})$. As $E$ is open in $\mathcal{N} \times \mathcal{O}(\mathcal{N})$, $F$ is $\Sigma^0_2$. However $F$ is not Borel, otherwise $E = s^{-1}(F)$ would be Borel in $\mathcal{N} \times_{\text{prod}} \mathcal{O}(\mathcal{N})$.

Let us now build $s$ and $r$. We identify $\mathcal{O}(\mathcal{N})$ with $\mathcal{O}(\mathcal{N}) \times \mathcal{O}(\mathcal{N})$ and use the fact that the topology on $\mathcal{O}(\mathcal{N}) \times \mathcal{O}(\mathcal{N})$ coincides with the product topology by Proposition 2.1.

By Lemma 7.1, $\mathcal{N}$ is a $[\Sigma^0_2]$-retract of $\mathcal{O}(\mathbb{N})$, which is a computable retract of $\mathcal{O}(\mathcal{N})$, so $\mathcal{N}$ is a $[\Sigma^0_2]$-retract of $\mathcal{O}(\mathcal{N})$. It is witnessed by two functions $r_0 : \mathcal{O}(\mathcal{N}) \to \mathcal{N}$ and $s_0 : \mathcal{N} \to \mathcal{O}(\mathcal{N})$ such that $r_0 \circ s_0 = \text{id}_{\mathcal{N}}$.

Let us simply pair $s_0$ and $r_0$ with the identity on $\mathcal{O}(\mathcal{N})$: let $s(f, O') = (s_0(f), O')$ and $r(O, O') = (r_0(O), O')$. □

In particular, that set is not a countable union of differences of open sets, as it should be on Polish or quasi-Polish spaces. More generally, it is not a countable boolean combination of open sets.

In order to overcome the mismatch between the hierarchy inherited from $\mathcal{N}$ via the representation and the class of Borel sets, one may attempt to change the definition of Borel sets. In [NV97] the Borel sets are redefined as the smallest class containing the open sets and the saturated compact sets, and closed under countable unions and complements. We observe here that this class is too large in the space $\mathcal{O}(\mathcal{N})$. First, if $U \subseteq \mathcal{N}$ is open then the set $\{V \in \mathcal{O}(\mathcal{N}) : U \subseteq V\}$

is compact and saturated in $\mathcal{O}(\mathcal{N})$. From this it is easy to see that the set built above is Borel in this weaker sense. However this notion of Borel sets is too loose, because compact saturated sets do not usually have a Borel pre-image. For instance, the singleton $\{\mathcal{N}\}$ is compact saturated but its pre-image under the representation is a $\boldsymbol{\Pi}^1_1$-complete set, hence is not Borel.

### 7.6 Open questions

We leave the following questions open:

- If $\mathbf{X}$ is admissibly represented and Fréchet-Urysohn, does $[\mathbf{D}_2](\mathbf{X}) = \mathbf{D}_2(\mathbf{X})$ hold?
- When $\mathbf{X}$ is in Class III, in particular for $\mathbf{X} = \mathbb{N} \times \mathcal{N}_1$, we know the inclusion $[\boldsymbol{\Sigma}^0_k](\mathcal{O}(\mathbf{X})) \subseteq \boldsymbol{\Sigma}^0_{k+2}(\mathcal{O}(\mathbf{X}))$. Can it be improved to $[\boldsymbol{\Sigma}^0_k](\mathcal{O}(\mathbf{X})) \subseteq \boldsymbol{\Sigma}^0_{k+1}(\mathcal{O}(\mathbf{X}))$?
- When $\mathbf{X}$ is in Class IV, in particular for $\mathbf{X} = \mathcal{N}$, does $[\boldsymbol{\Delta}^0_2](\mathcal{O}(\mathbf{X})) \subseteq \mathcal{B}(\mathcal{O}(\mathbf{X}))$ hold? In that case, is $[\boldsymbol{\Delta}^0_2](\mathcal{O}(\mathbf{X}))$ contained in some level of the Borel hierarchy? The same questions can be asked for $[\mathbf{D}_\omega](\mathcal{O}(\mathbf{X}))$.
- What can be said about effective classes $[\mathrm{D}_\omega]$, $[\Delta^0_2]$, $[\Sigma^0_k]$?
- Can we extend the classification theorem when $\mathbf{X}$ is not Polish?
- Is there a Hausdorff admissibly represented space $\mathbf{X}$ such that $[\boldsymbol{\Sigma}^0_2](\mathbf{X}) \nsubseteq \boldsymbol{\Sigma}^0_2(\mathbf{X})$?
- What do the sets in $[\boldsymbol{\Sigma}^0_2](\mathcal{O}(\mathcal{N}))$ look like?
- Is it possible to modify the definition of Borel sets on $\mathcal{O}(\mathcal{N})$ to match exactly the sets that have a Borel pre-image under the representation?

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