Applications of Effective Probability Theory to Martin-Löf Randomness

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Effective probability theory ($\in$ computable analysis)

Computable versions of object from probability/measure theory:

- probability measure,
- measurable set,
- almost sure convergence, etc.

Martin-Löf randomness

An individual notion of randomness:

- 0000000000000000000 . . . is not ML-random,
- 1011011011011011011 . . . is not ML-random,
- 0101101001000110101 . . . is ML-random.

Applications of Effective Probability Theory to Martin-Löf Randomness

Two main contributions:

- We provide a new framework to study randomness and strengthen existing results,
- We solve an open question about randomness and Brownian motion.

Martin-Löf randomness

We identify reals in $[0,1]$ with their binary representations.

How to define an individual notion of randomness?

- We expect a random real x to belong to all the sets of measure 1.
  For instance, a random real should be **normal**.
- But no point is in all sets of measure 1: x is not in $[0,1] \setminus \{x\}$.

Solution (Martin-Löf, 1966)

- A point x is random if it lies in all the **effective** sets of measure 1.

Computability on [0, 1]

A real number x is represented by an infinite stream $q_0, q_1, q_2, \dots$ of rational numbers such that $|q_i - x| < 2^{-i}$.

Definition

A function $f : [0,1] \to \mathbb{R}$ is **computable** if there is a program which, on input stream representing $x \in [0,1]$, outputs a stream representing $f(x)$.

Definition

A set $A \subseteq [0,1]$ is **semi-decidable** if there is a program which, on input stream representing $x \in [0,1]$, eventually halts if and only if $x \in A$.

Computability on $[0, 1]$

Theorem (A classical result)

- *Every computable function is continuous.*
- *Every semi-decidable set is open.*

Martin-Löf randomness

Definition (Martin-Löf, 1966)

A **Martin-Löf test** is a sequence $V_n \subseteq X$ such that

- $\lambda(V_n) < 2^{-n}$,
- V_n are semi-decidable, uniformly in n.

A point x is **Martin-Löf random** if $x \notin \bigcap_n V_n$ for all tests (V_n).

Theorem (Martin-Löf, 1966)

There exists a universal test $(U_n)_{n\in\mathbb{N}}$: a point x is ML-random if and only if $x \notin \bigcap_n U_n$.

Martin-Löf randomness

To convert a classical probability theorems like

$$P(x) \text{ holds for almost every } x$$

into

$$P(x) \text{ holds for every ML-random } x,$$

one has to find a ML-test $(V_n)_{n\in\mathbb{N}}$ such that $\{x : \neg P(x)\} \subseteq \bigcap_n V_n$.

Example: Strong Law of Large Numbers (SLLN)

Theorem (Classical)

Let f_i be i.i.d. bounded measurable functions with mean m. For almost every x,

$$\lim_{n\to\infty} \frac{f_0(x) + \ldots + f_{n-1}(x)}{n} = m.$$

Martin-Löf randomness

Theorem (SLLN for random points)

Let f_i be i.i.d. bounded computable functions with mean m. For every random x,

$$\lim_{n\to\infty} \frac{f_0(x)+\ldots+f_{n-1}(x)}{n} = m.$$

Sketch of the proof.

Let $\delta > 0$ be a rational number and
$D_n(\delta) := \left\{x : \exists k \geq 2^n\delta^{-4}, \left|\frac{f_0(x)+\ldots+f_{k-1}(x)}{k} - m\right| > \delta\right\}$.

- if f_i are computable then $D_n(\delta)$ are semi-decidable,
- $\lambda(D_n(\delta)) < 2^{-n}$,
- so $D_n(\delta)$ is a ML-test.

□

Martin-Löf randomness

Let's compare:

Theorem (Classical SLLN)

Let f_i be i.i.d. bounded measurable functions with mean m. For almost every x,

$$\lim_{n\to\infty} \frac{f_0(x) + \ldots + f_{n-1}(x)}{n} = m.$$

Theorem (SLLN for random points)

Let f_i be i.i.d. bounded computable functions with mean m. For every random x,

$$\lim_{n\to\infty} \frac{f_0(x) + \ldots + f_{n-1}(x)}{n} = m.$$

Every computable function is continuous. What about discontinuous functions?

Martin-Löf randomness

In [H. and Rojas, CiE09], we study several effective versions of notions from measure theory:

- Measurable set,
- Measurable function,
- Integrable function

and we introduce the new framework of layerwise computability.

Principle

Computability	=	Effective topology
Layerwise computability	=	Effective measure theory

Layerwise computability

- We are still working on $[0,1]$ with the Lebesgue measure λ.
- Let $(U_n)_{n\in\mathbb{N}}$ be a universal Martin-Löf test, fixed once for all.
- We call the sets $K_n := [0,1] \setminus U_n$ the **Martin-Löf layers**. One has:

$$\begin{aligned} K_n &\subseteq K_{n+1} \\ \mathrm{ML} &= \bigcup_n K_n. \end{aligned}$$

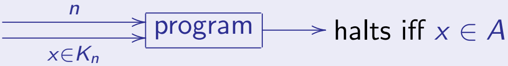
Layerwise computability

Definition

A function $f : [0, 1] \to \mathbb{R}$ is **layerwise computable** if for all n, f is computable on K_n, uniformly in n.

Definition

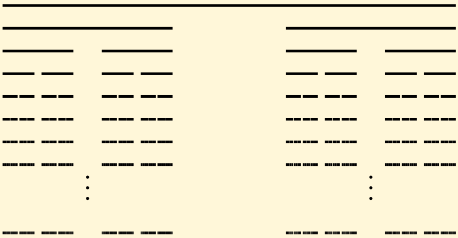
A set A is **layerwise semi-decidable** if for all n, A is semi-decidable on K_n, uniformly in n.



Layerwise computability

Example

The fat Cantor set $A \subset [0,1]$.



$\lambda(A) = \frac{1}{2}$

χ_A is layerwise computable.

Layerwise computability

Let f_i be i.i.d. bounded layerwise computable functions with mean m.

Reminder.

Let $\delta > 0$ be a rational number and
$D_n(\delta) := \left\{x : \exists k \geq 2^n\delta^{-4}, \left|\frac{f_0(x)+\ldots+f_{k-1}(x)}{k} - m\right| > \delta\right\}$.

- if f_i are computable then $D_n(\delta)$ are semi-decidable,
- $\lambda(D_n(\delta)) < 2^{-n}$,
- so $D_n(\delta)$ is a ML-test.

□

Layerwise computability

Let f_i be i.i.d. bounded layerwise computable functions with mean m.

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Let $\delta > 0$ be a rational number and
$D_n(\delta) := \left\{x : \exists k \geq 2^n\delta^{-4}, \left|\frac{f_0(x)+\ldots+f_{k-1}(x)}{k} - m\right| > \delta\right\}$.

- if f_i are layerwise computable then $D_n(\delta)$ are layerwise semi-decidable,
- $\lambda(D_n(\delta)) < 2^{-n}$,
- so $D_n(\delta)$ is a layerwise ML-test.

□

Layerwise computability

Definition

A **layerwise ML-test** is a sequence of sets A_n such that:

- A_n is layerwise semi-decidable,
- $\lambda(A_n) < 2^{-n}$.

Remark

The class of layerwise ML-tests is much larger than the class of plain ML-tests. However...

Theorem (H. and Rojas)

Let $(A_n)_{n\in\mathbb{N}}$ be a layerwise ML-test. If x is random then $x \notin \bigcap_n A_n$.

Layerwise computability

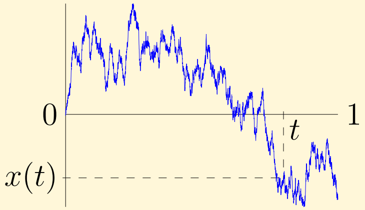
Theorem (SLLN for random points)

Let f_i be i.i.d. bounded layerwise computable functions with mean m. For every random x,

$$\lim_{n\to\infty} \frac{f_0(x) + \ldots + f_{n-1}(x)}{n} = m.$$

About random Brownian motion

- $\mathcal{C}([0,1])$: continuous functions $x : [0,1] \to \mathbb{R}$,
- Wiener measure W on $\mathcal{C}([0,1])$.



Theorem (Classical)

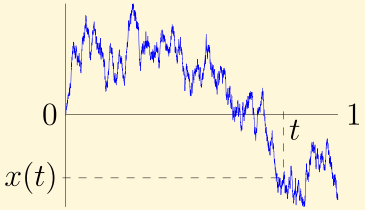
Almost every Brownian path is nowhere differentiable.

Theorem (Algorithmic)

Every Martin-Löf-random Brownian path is nowhere differentiable.

About random Brownian motion

- $x : [0,1] \to \mathbb{R}$ any random path,
- $t \in (0,1]$ any computable real number.



Theorem (Fouché, 2000)

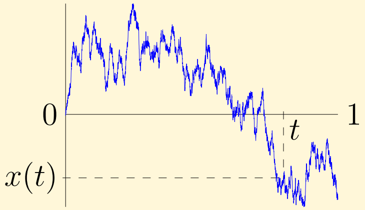
$x(t)$ is not computable.

Question

Can $x(t)$ be lower semi-computable?

About random Brownian motion

- $x : [0,1] \to \mathbb{R}$ any random path,
- $t \in (0,1]$ any computable real number.



Theorem

If $f : (X,\mu) \to (Y,\nu)$ is computable and maps μ to ν, then

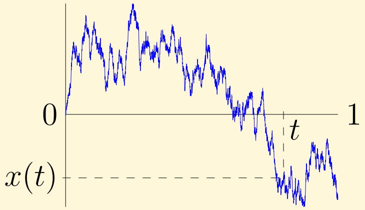
$$f(\mathrm{ML}_{(X,\mu)}) \subseteq \mathrm{ML}_{(Y,\nu)}.$$

Corollary

$x(t)$ is random.

About random Brownian motion

- $x : [0, 1] \to \mathbb{R}$ any random path,
- $t \in (0, 1]$ any computable real number.



Remark

There exist *lower semi-computable* random reals (Chaitin's Ω-numbers).

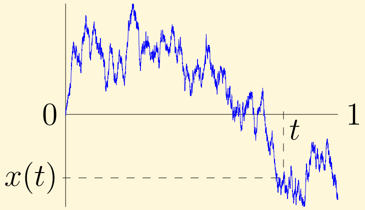
Theorem (H. and Rojas)

Given any computable $t \in (0, 1]$ and any random $y \in \mathbb{R}$, there exists a random path x such that $x(t) = y$.

In particular, every Ω-number y is reached by a random path at a computable time t.

About random Brownian motion

- $x : [0, 1] \to \mathbb{R}$ any random path,
- $t \in (0, 1]$ any computable real number.



Theorem (H. and Rojas)

If $f : (X, \mu) \to (Y, \nu)$ is computable and maps μ to ν, then

$$f(\mathrm{ML}_{(X,\mu)}) = \mathrm{ML}_{(Y,\nu)}.$$

Corollary

For every random $y \in \mathbb{R}$ and every computable $t \in (0, 1]$ there is a random path x such that $x(t) = y$.

Thank you