COMPUTABILITY OF PROBABILITY MEASURES AND MARTIN-LÖF RANDOMNESS OVER METRIC SPACES

MATHIEU HOYRUP AND CRISTÓBAL ROJAS

ABSTRACT. In this paper we investigate algorithmic randomness on more general spaces than the Cantor space, namely computable metric spaces. To do this, we first develop a unified framework allowing computations with probability measures. We show that any computable metric space with a computable probability measure is isomorphic to the Cantor space in a computable and measure-theoretic sense. We show that any computable metric space admits a universal uniform randomness test (without further assumption).

CONTENTS

PARTLY SUPPORTED BY ANR GRANT 05 2452 260 OX

1. Introduction

The theory of algorithmic randomness begins with the definition of individual random infinite sequence introduced in 1966 by Martin-Löf [ML66]. Since then, many efforts have contributed to the development of this theory which is now well established and intensively studied, yet restricted to the Cantor space. In order to carry out an extension of this theory to more general infinite objects as encountered in most mathematical models of physical random phenomena, a necessary step is to understand what means for a probability measure on a general space to be computable (this is very simple expressed on the Cantor Space). Only then algorithmic randomness can be extended.

The problem of computability of (Borel) probability measures over more general spaces has been investigated by several authors: by Edalat for compact spaces using domain-theory ([Eda96]); by Weihrauch for the unit interval ([Wei99]) and by Schröder for sequential topological spaces ([Sch07]) both using representations; and by Gács for computable metric spaces ([Gác05]). Probability measures can be seen from different points of view and those works develop, each in its own framework, the corresponding computability notions. Mainly, Borel probability measures can be regarded as points of a metric space, as valuations on open sets or as integration operators. We express the computability counterparts of these different views in a unified framework, and show them to be equivalent.

Extensions of the algorithmic theory of randomness to general spaces have previously been proposed: on effective topological spaces by Hertling and Weihrauch (see [HW98],[HW03]) and on computable metric spaces by Gács (see [Gác05]), both of them generalizing the notion of randomness tests and investigating the problem of the existence of a universal test. In [HW03], to prove the existence of such a test, ad hoc computability conditions on the measure are required, which *a posteriori* turn out to be incompatible with the notion of computable measure. The second one ([Gác05]), carrying the extension of Levin's theory of randomness, considers *uniform tests* which are tests parametrized by measures. A computability condition on the basis of ideal balls (namely, recognizable Boolean inclusions) is needed to prove the existence of a universal uniform test.

In this article, working in computable metric spaces with any probability measure, we consider both uniform and non-uniform tests and prove the following points:

- uniformity and non-uniformity do not essentially differ,
- the existence of a universal test is assured without any further condition.

Another issue addressed in [Gác05] is the characterization of randomness in terms of Kolmogorov Complexity (a central result in Cantor Space). There, this characterization is proved to hold (for a compact computable metric space X with a computable measure) under the assumption that there exists a computable injective encoding of a full-measure subset of X into binary sequences. In the real line for example, the base-two numeral system (or binary expansion) constitutes such encoding for the Lebesgue measure. This fact was already been (implicitly) used in the definition of random reals (reals with a random binary expansion, w.r.t the uniform measure).

We introduce, for computable metric spaces with a computable measure, a notion of binary representation generalizing the base-two numeral system of the reals, and prove that:

- such a binary representation always exists,
- a point is random if and only if it has a unique binary expansion, which is random.

Moreover, our notion of binary representation allows to identify any computable probability space with the Cantor space (in a computable-measure-theoretic sense). It provides a tool to directly transfer elements of algorithmic randomness theory from the Cantor space to any computable probability space. In particular, the characterization of randomness in terms of Kolmogorov complexity, even in a non-compact space, is a direct consequence of this.

The way we handle computability on continuous spaces is largely inspired by representation theory. However, the main goal of that theory is to study, in general topological spaces, the way computability notions depend on the chosen representation. Since we focus only on Computable Metric Spaces (see [Hem02] for instance) and *Enumerative Lattices* (introduced in setion 2.2) we shall consider only one *canonical* representation for each set, so we do not use representation theory in its general setting.

Our study of measures and randomness, although restricted to computable metric spaces, involves computability notions on various sets which do not have natural metric structures. Fortunately, all these sets become enumerative lattices in a very natural way and the canonical representation provides in each case the right computability notions.

In section 2, we develop a language intended to express computability concepts, statements and proofs in a rigorous but still (we hope) transparent way. The structure of computable metric space is then recalled. In section 3, we introduce the notion of enumerative lattices and present two important examples to be used in the paper. Section 4 is devoted to the detailed study of computability on the set of probability measures. In section 5 we define the notion of binary representation on any computable metric space with a computable measure and show how to construct such a representation. In section 6 we apply all this machinery to algorithmic randomness.

2. Basic definitions

2.1. **Recursive functions.** The starting point of recursion theory was the mathematization of the intuitive notion of function computable by an *effective procedure* or *algorithm*. The different systems and computation models formalizing *mechanical* procedures on natural numbers or symbols have turned out to coincide, and therefore have given rise to a robust mathematical notion which grasps (this is Church-Turing thesis) what means for a (partial) function $\varphi : \mathbb{N} \to \mathbb{N}$ to be *algorithmic*, and which can be made precise using any one of the numerous formalisms proposed. Following the usual denomination, we call such a function a ***(partial) recursive function***. To show that a function $\varphi : \mathbb{N} \to \mathbb{N}$ is recursive, we will exhibit an algorithm $\mathcal{A}$ which on input n halts and outputs $\varphi(n)$ when it is defined, runs forever otherwise.

In the same vein, a robust notion of (partial) recursive function $F : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ can be characterized by different formal definitions:

Via domain theory: (see [AJ94]). This approach takes the notion of recursive function as primitive, which avoids the definition of a new computation model. A partial function $F : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ is recursive if there is a recursive function $F' : \mathbb{N}^* \to \mathbb{N}^*$ which is monotone for the prefix ordering, such that for all $\sigma \in \mathrm{dom}(F)$, $F(\sigma)$ is the infinite sequence obtained at the limit by computing F' on the finite prefixes of σ (precisely, the Baire space can be embedded into the set of finite and infinite sequences of integers ordered by the prefix relation, which is an ω-algebraic domain).

Via oracle Turing machines: (used by Ko and Friedman, see [KF82], [Ko91]). An oracle Turing machine $\mathcal{M}^{[\sigma]}$ is a Turing machine which works with a sequence $\sigma \in \mathbb{N}^{\mathbb{N}}$ provided as oracle and is allowed to read elements σ_n of the oracle sequence. On an input $n \in \mathbb{N}$, it may stop and output a natural number, interpreted as $F(\sigma)_n$.

Via type-two Turing machines: (defined by Weihrauch, see [Wei00]). Expressed differently, it is essentially the same computation model (it works on symbols instead of integers).

Again, to show that a function $F : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ is recursive, we will exhibit an algorithm $\mathcal{A}$ which given $\sigma \in \mathbb{N}^{\mathbb{N}}$ as oracle and n as input, halts and outputs $F(\sigma)_n$. The algorithm together with σ in the oracle is denoted $\mathcal{A}^{[\sigma]}$.

A sequence $\sigma \in \mathbb{N}^{\mathbb{N}}$ is ***recursive*** if the function $n \mapsto \sigma_n$ is recursive. Given a family $(\sigma_i)_{i\in\mathbb{N}}$ of recursive sequences, σ_i is ***recursive uniformly in i*** if the function $\langle i, n\rangle \mapsto \sigma_{i,n}$ is recursive, where $\langle , \rangle$ denotes some computable bijection between tuples and natural numbers.

2.2. **Representations and constructivity.** A representation on a set X is a surjective (partial) function $\rho : \mathbb{N}^{\mathbb{N}} \to X$. Let X and Y be sets with fixed representations ρ_X and ρ_Y.

Definition 2.2.1 (Constructivity notions)**.**

(1) An element $x \in X$ is ***constructive*** if there is a recursive sequence σ such that $\rho_X(\sigma) = x$.
(2) The elements of a sequence $(x_i)_{i\in\mathbb{N}}$ are ***uniformly constructive*** if there is a family $(\sigma_i)_i$ of uniformly recursive sequences such that $\rho_X(\sigma_i) = x_i$ for all i.
(3) A function $f :\subseteq X \to Y$ is ***constructive*** on $D \subseteq X$ if there exists a recursive function $F : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ such that the following diagram commutes on $\rho_X^{-1}(D)$:

$$\begin{array}{ccc} \mathbb{N}^{\mathbb{N}} & \xrightarrow{F} & \mathbb{N}^{\mathbb{N}} \\ \rho_X \downarrow & & \downarrow \rho_Y \\ X & \xrightarrow{f} & Y \end{array} \qquad \text{(that is, } f \circ \rho_X = \rho_Y \circ F \text{ on } \rho_X^{-1}(D)\text{)}$$

We say that y is ***x-constructive*** if there is a function $f :\subseteq X \to Y$ constructive on $\{x\}$ with $f(x) = y$. If x is constructive, x-constructivity and constructivity are equivalent. Note that two sequences of natural numbers can be merged into a single one, so the product $X \times Y$ of two represented sets has a canonical representation. In particular, it makes sense to speak about (x, y)-constructive elements.

2.3. **Objects.** There is a canonical way of defining a representation on a set X when 1) some collection of *elementary objects* of X can be encoded into natural numbers and 2) an element of X can be described by a sequence of these elementary objects. Once encoded into natural numbers, the elementary objects inherit their finite character and may be output by algorithms. Let us make it precise:

Definition 2.3.1. A ***numbered set*** $\mathcal{O}$ is a countable set together with a total surjection $\nu_{\mathcal{O}} : \mathbb{N} \to \mathcal{O}$ called the ***numbering***. We write o_n for $\nu(n)$.

A numbered set $\mathcal{O}$ and a (partial) surjection $\delta : \mathcal{O}^N \to X$ induce canonically a representation $\rho = \delta \circ \nu_{\mathcal{O}}$. At least in this paper, all representations will be obtained in this way. A sequence of finite objects which is mapped by δ to x is called a ***description*** of x.

An algorithm may then be seen as outputting objects:

Given a numbered set $\mathcal{O}$, we say that an algorithm (plain or with oracle) ***enumerates*** a sequence of objects $(o_{n_i})_{i\in\mathbb{N}}$ if on input i it outputs n_i. Given a representation $(\mathcal{O}, \delta)$ on a set X, an algorithm enumerating a description of $x \in X$ is said to ***describe*** x.

An algorithm may also take objects as inputs, with a restriction:

Definition 2.3.2. An algorithm $\mathcal{A}$ is said to be ***extensional*** on an element $x \in X$ if for all σ such that $\rho_X(\sigma) = x$, $\mathcal{A}^{[\sigma]}$ describes the same element $y \in Y$.

We then say that $\mathcal{A}$ x-describes y or that $\mathcal{A}^{[x]}$ describes y.

The constructivity notions of definition 2.2.1 can then be expressed using this language, which will be used throughout this paper.

(1) An element $x \in X$ is constructive if there is an algorithm describing x,
(2) The elements of a sequence $(x_i)_{i\in\mathbb{N}}$ are uniformly constructive if there is an algorithm $\mathcal{A}$ such that $\mathcal{A}(\langle i, .\rangle)$ describes x_i,
(3) A function $f :\subseteq X \to Y$ is constructive on $D \subseteq X$ if there exists an algorithm which x-describes $f(x)$ for all $x \in D$.

A x-constructive element y may be x-described by an algorithm which is extensional only on x, and thus induce a function which is defined only at x.

2.4. Computable Metric Spaces.

Definition 2.4.1. A ***computable metric space*** is a triple $\mathcal{X} = (X, d, \mathcal{S})$, where:
- (X, d) is a separable complete metric space (polish metric space),
- $\mathcal{S} = \{s_i : i \in \mathbb{N}\}$ is a countable dense subset of X,
- The real numbers $d(s_i, s_j)$ are all computable, uniformly in $\langle i, j\rangle$.

The elements of $\mathcal{S}$ are called the ***ideal points***. The numbering $\nu_{\mathcal{S}}$ defined by $\nu_{\mathcal{S}}(i) := s_i$ makes $\mathcal{S}$ a numbered set. Without loss of generality, $\nu_{\mathcal{S}}$ can be supposed to be injective: as $d(s_i, s_j) > 0$ can be semi-decided, ν_S can be effectively transformed into an injective numbering. Then a sequence of ideal points can be uniquely identified with the sequence of their names.

The numbered sets $\mathcal{S}$ and $\mathbb{Q}_{>0}$ induce the numbered set of ***ideal balls*** $\mathcal{B} := \{B(s_i, q_j) : s_i \in \mathcal{S}, q_j \in \mathbb{Q}_{>0}\}$, the numbering being $\nu_{\mathcal{B}}(\langle i, j\rangle) := B(s_i, q_j)$. We write $B_{\langle i,j\rangle}$ for $\nu_{\mathcal{B}}(\langle i, j\rangle)$. The closed ball $\{x \in X : d(s, x) \leq r\}$ is denoted $\overline{B}(s, q)$ and may not coincide with the closure of the open ball $B(s, q)$ (typically, if the space has disconnection).

We now recall some important examples of computable metric spaces:

examples:

1. the Cantor space $(\Sigma^{\mathbb{N}}, d, \mathcal{S})$ where Σ is a finite alphabet, $d(\omega, \omega') := 2^{-\min\{n\in\mathbb{N}:\omega_n\neq\omega'_n\}}$

and $\mathcal{S} := \{w000\ldots : w \in \Sigma^*\}$ where Σ^* is the set of finite words on Σ,
2. $(\mathbb{R}^n, d_{\mathbb{R}^n}, \mathbb{Q}^n)$ with the euclidean metric and the standard numbering of $\mathbb{Q}^n$,
3. if $(X, d_X, \mathcal{S}_X)$ and $(Y, d_Y, \mathcal{S}_Y)$ are two computable metric spaces, $(X \times Y, d, \mathcal{S}_X \times \mathcal{S}_Y)$ has a canonical computable metric space structure, with $d((x,y),(x',y')) = \max\{d_X(x,x'), d_Y(y,y')\}$.

For further examples, like functions spaces $C[0,1]$ and L^p for computable $p \geq 1$ we refer to [Weihrauch]. A sequence $(x_n)_{n\in\mathbb{N}}$ of points is said to be a ***fast Cauchy*** sequence, or simply a ***fast*** sequence if $d(x_n, x_{n+1}) < 2^{-n}$ for all n.

Definition 2.4.2. On a computable metric space $(X, d, \mathcal{S})$, the canonical representation is the Cauchy representation $(\mathcal{S}, \delta_C)$ defined by $\delta_C(\overrightarrow{s}) = x$ for all fast sequence $\overrightarrow{s}$ of ideal points converging to x.

Again, each set X with a computable metric structure $(X, d, \mathcal{S})$ will be implicitly represented using the Cauchy representation. Then canonical constructively notions derive directly from definition 2.2.1. It is usual to call a constructive element of X a ***computable point***, and a constructive function between computable metric space, a ***computable function***. Remark that the computable real numbers are the computable points of the computable metric space $(\mathbb{R}, d, \mathbb{Q})$.

The choice of this representation is justified by the classical result: every computable function between computable metric spaces is continuous (on its domain of computability).

Proposition 2.4.1. *The distance $d : X \times X \to \mathbb{R}$ is a computable function.*

Proposition 2.4.2. *For a point $x \in X$, the following statements are equivalent:*
- *x is a computable point,*
- *all $d(x, s_i)$ are upper semi-computable uniformly in i,*
- *$d_x := d(x, .) : X \to \mathbb{R}$ is a computable function.*

Several metrics and effectivisations of a single set are possible, and induce in general different computability notions: two computable metric structures $(s, \mathcal{S})$ and $(d', \mathcal{S}')$ are said to be *effectively equivalent* if $id : (X, d, \mathcal{S}) \to (X, d', \mathcal{S}')$ is a computable homeomorphism (with computable inverse). In this case, all computability notions are preserved replacing one structure by the other (see [Hem02] for details).

3. Enumerative Lattices

3.1. **Definition.** We introduce a simple structure using basic order theory, on which a natural representation can be defined. The underlying ideas are those from domain theory, but the framework is lighter and (hence) less powerful. Actually, it is sufficient for the main purpose: proposition 3.1.1. This will be applied in the last section on randomness.

Definition 3.1.1. An ***enumerative lattice*** is a triple $(X, \leq, \mathcal{P})$ where $(X, \leq)$ is a complete lattice and $\mathcal{P} \subseteq X$ is a numbered set such that every element x of X is the supremum of some subset of $\mathcal{P}$.

We then define $\mathcal{P}_\downarrow(x) := \{p \in \mathcal{P} : p \leq x\}$ (note that $x = \sup \mathcal{P}_\downarrow(x)$). Any element of X can be described by a sequence $\overrightarrow{p}$ of elements of $\mathcal{P}$. Note that the least element $\bot$ need not belong to $\mathcal{P}$: it can be described by the empty set, of which it is the supremum.

Definition 3.1.2. The canonical representation on an enumerative lattice $(X, \leq, \mathcal{P})$ is the induced by the partial surjection $\delta_{\leq}(\overrightarrow{p}) = \sup \overrightarrow{p}$ (where the sequence $\overrightarrow{p}$ may be empty).

From here and beyond, each set X endowed with an enumerative structure $(X, \leq, \mathcal{P})$ will be implicitly represented using the canonical representation. Hence, canonical constructivity notions derive directly from definition 2.2.1. Let us focus on an example: the identity function from X to X is computed by an algorithm outputting exactly what is provided by the oracle. Hence, when the oracle is empty, which describes $\perp$, the algorithm runs forever and outputs nothing, which is a description of $\perp$.

examples:

(1) $(\overline{\mathbb{R}}, \leq, \mathbb{Q})$ with $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$: the constructive elements are the so-called *lower semi-computable* real numbers,
(2) $(2^{\mathbb{N}}, \subseteq, \{\text{finite sets}\})$: the constructive elements are the r.e sets from classical recursion theory,
(3) $(\{\perp, \top\}, \leq, \{\top\})$ with $\perp < \top$.

We recall that a real number x is *computable* if both x and $-x$ are lower semi-computable.

Here is the main interest of enumerative lattices:

Proposition 3.1.1. *Let $(X, \leq, \mathcal{P})$ be an enumerative lattice. There is an enumeration $(x_i)_{i \in \mathbb{N}}$ of all the constructive elements of X such that x_i is constructive uniformly in i.*

Proof. there is an enumeration φ of the r.e subsets of $\mathbb{N}$: for every r.e subset E of $\mathbb{N}$, there is some i such that $E = E_i := \{\varphi(\langle i, n \rangle) : n \in \mathbb{N}\}$. Moreover, we can take φ such that whenever $E_i \neq \emptyset$ the function $\varphi(\langle i, . \rangle) : \mathbb{N} \to \mathbb{N}$ is total (this is a classical construction from recursion theory, see [Rog87]). Then consider the associated algorithm $\mathcal{A}_\varphi = \nu_{\mathcal{P}} \circ \varphi$: for every constructive element x there is some i such that $\mathcal{A}_\varphi(\langle i, . \rangle) : \mathbb{N} \to \mathcal{P}$ enumerates x ($\emptyset$ is an enumeration of $\perp$). $\square$

Remark 3.1.1. Observe that on every enumerative lattice the Scott topology can be defined: a Scott open set O is an upper subset ($x \in O, x \leq y \Rightarrow y \in O$) such that for each sequence $\overrightarrow{p} = (p_{n_i})_{i \in \mathbb{N}}$ such that $\sup \overrightarrow{p} \in O$, there is some k such that $\sup\{p_{n_0}, \ldots, p_{n_k}\} \in O$.

If Y and Z have enumerative lattice structures, a function $f : Y \to Z$ is said to be Scott-continuous if it is monotonic and commutes with suprema of increasing sequences (one can prove that f is Scott-continuous if and only if it is continuous for the Scott topologies on Y and Z) and is easy to see that a Scott-continuous function $f : Y \to Z$ such that all $f(\sup\{p_{n_1}, \ldots, p_{n_k}\})$ are constructive uniformly in $\langle n_1, \ldots, n_k \rangle$, is in fact a constructive function.

3.2. **Functions from a computable metric space to an enumerative lattice.** Given a computable metric space $(X, d, \mathcal{S})$ and an enumerative space $(Y, \leq, \mathcal{P})$, we define the numbered set $\mathcal{F}$ of ***step functions*** from X to Y:

$$f_{\langle i,j \rangle}(x) = \begin{cases} p_j & \text{if } x \in B_i \\ \perp & \text{otherwise} \end{cases}$$

We then define $\mathcal{C}(X, Y)$ as the closure of $\mathcal{F}$ under pointwise suprema, with the pointwise ordering $\sqsubseteq$. We have directly:

Proposition 3.2.1. *$(\mathcal{C}(X,Y),\sqsubseteq,\mathcal{F})$ is an enumerative lattice.*

example: the set $\overline{\mathbb{R}}^+ = [0,+\infty)\cup\{+\infty\}$ has an enumerative lattice structure $(\overline{\mathbb{R}}^+,\leq,\mathbb{Q}^+)$, which induces the enumerative lattice $\mathcal{C}(X,\overline{\mathbb{R}}^+)$ of positive lower semi-continuous functions from X to $\overline{\mathbb{R}}^+$. Its constructive elements are the positive *lower semi-computable* functions.

We now show that the constructive elements of $\mathcal{C}(X,Y)$ are exactly the constructive functions from X to Y.

To each algorithm $\mathcal{A}$ we associate a constructive element of $\mathcal{C}(X,Y)$, enumerating a sequence of step functions: enumerate all $\langle n,i_0,\dots,i_k\rangle$ with $d(s_{i_j},s_{i_{j+1}})<2^{-(j+1)}$ for all $j<k$ (prefix of a super-fast sequence). Keep only those for which the computation of $\mathcal{A}^{[i_0,\dots,i_k,0,0,\dots]}(n)$ halts without trying to read beyond i_k. For each one, the latter computation outputs some element p_l: then output the step function $f_{\langle i,l\rangle}$ where $B_i=B(s_{i_k},2^{-k})$. We denote by $f_{\mathcal{A}}$ the supremum of the enumerated sequence of step functions.

Lemma 3.2.1. *For all x on which $\mathcal{A}$ is extensional, $f_{\mathcal{A}}(x)$ is the element of Y described by $\mathcal{A}^{[x]}$.*

Proof. let y be the element described by $\mathcal{A}^{[x]}$.

For all $\langle n,i_0,\dots,i_k\rangle$ for which some $f_{\langle i,j\rangle}$ is enumerated with $x\in B_i$, there is a fast sequence $\overrightarrow{s}$ converging to x starting with $s_{i_0},\dots,s_{i_k}$, for which $\mathcal{A}^{[\overrightarrow{s}]}(n)=p_j$. Then $y\geq p_j=f_{\langle i,j\rangle}(x)$. Hence $y\geq f_{\mathcal{A}}(x)$.

There is a super-fast sequence $\overrightarrow{s}$ converging to x: for all n, $\mathcal{A}^{[\overrightarrow{s}]}(n)$ stops and outputs some p_{j_n}, so there is some i_n with $x\in B_{i_n}$ such that $f_{\langle i_n,j_n\rangle}$ is enumerated. Hence, $y=\sup_n p_{j_n}=\sup f_{\langle i_n,j_n\rangle}(x)\leq f_{\mathcal{A}}(x)$.

□

Proposition 3.2.2. *The constructive elements of $\mathcal{C}(X,Y)$ are exactly the (total) constructive functions from X to Y.*

Proof. the supremum of a r.e subset E of $\mathcal{F}$ is a total constructive function: semi-decide in dovetail $x\in B_i$ for all $f_{\langle i,j\rangle}\in E$, and enumerate p_j each time a test stops.

Given a total constructive function f, there is an algorithm $\mathcal{A}$ which on each $x\in X$ is extensional and describes $f(x)$, so $f=f_{\mathcal{A}}$.

□

The proof even shows that the equivalence is constructive: the evaluation of any $f:X\to Y$ on any $x\in X$ can be achieved by an algorithm having access to any description of $f\in\mathcal{C}(X,Y)$, and any algorithm evaluating f can be converted into an algorithm describing $f\in C(X,Y)$. More precisely:

Proposition 3.2.3. *Let X,X' be computable metric spaces and Y be an enumerative lattice:*

Evaluation: *The function $Eval:\mathcal{C}(X,Y)\times X\to Y$ is constructive,*

Curryfication: *If a function $f:X'\times X\to Y$ is constructive then the function from X' to $C(X,Y)$ mapping $x'\in X'$ to $f(x',.)$ is constructive.*

Lemma 3.2.1 and proposition 3.2.2 implie:

Corollary 3.2.1. *The x-constructive elements of Y are exactly the images of x by* total *constructive functions from X to Y.*

This is a particular property of the enumerative lattice structure: a partial constructive function from some represented space to another cannot in general be extended to a total constructive one.

3.3. **The Open Subsets of a computable metric space.** Following [BW99], [BP03], we define constructivity notions on the open subsets of a computable metric space. The topology τ induced by the metric has the numbered set $\mathcal{B}$ of ideal balls as a countable basis: any open set can then be described as a countable union of ideal balls. Actually $(\tau, \subseteq, \mathcal{B})$ is an enumerative space (cf section 3), the supremum operator being union. The canonical representation on enumerative lattices (definition 3.1.2) induces constructivity notions on τ, a constructive open set being called a ***recursively enumerable (r.e) open set***.

On the integers, it may be unnatural to show that some subset is recursively enumerable, and the equivalent notion of semi-decidable set is often used. This notion can be extended to subsets of a computable metric space, and it happens to be very useful in the applications. We recall from section 3 that $\{\bot, \top\}$ is an enumerative lattice, which induces canonically the enumerative lattice $\mathcal{C}(X, \{\bot, \top\})$.

Definition 3.3.1. A subset A of X is said to be ***semi-decidable*** if its indicator function $1_A : X \to \{\bot, \top\}$ (mapping $x \in A$ to $\top$ and $x \notin A$ to $\bot$) is constructive.

In other words, A is semi-decidable if there is a recursive function φ such that for all $x \in X$ and all description $\overrightarrow{s}$ of x, $\varphi^{\lceil \overrightarrow{s} \rceil}$ stops if and only if $x \in A$. It is a well-known result (see [BP03]) that the two notions are effectively equivalent:

Proposition 3.3.1. *A subset of X is semi-decidable if and only if it is a r.e open set. Moreover, the enumerative lattices $(\tau, \subseteq, \mathcal{B})$ and $\mathcal{C}(X, \{\bot, \top\})$ are constructively isomorphic.*

The isomorphism is the function $U \mapsto 1_U$ and its inverse $f \mapsto f^{-1}(\top)$. In other words, $f^{-1}(\top)$ is f-r.e uniformly in f and 1_U is U-lower semi-computable uniformly in U. It implies in particular that:

Corollary 3.3.1. *The intersection $(U, V) \mapsto U \cap V$ and union $(U, V) \mapsto U \cup V$ are constructive functions from $\tau \times \tau$ to τ.*

For computable functions between computable metric spaces, we have the following useful characterization:

Proposition 3.3.2. *Let (X, d_X, S_X) and (Y, d_Y, S_Y) be computable metric spaces. A function $f : X \to Y$ is computable on $D \subseteq X$ if and only if the preimages of ideal balls are uniformly r.e open (in D) sets. That is, for all i, $f^{-1}(B_i) = U_i \cap D$ where U_i is a r.e open set uniformly in i.*

We will use the following notion:

Definition 3.3.2. A Π^0_2-set is a set of the form $\bigcap_n U_n$ where $(U_n)_n$ is a sequence of uniformly r.e open sets.

4. Computing with probability measures

4.1. **Measures as points of the computable metric space $\mathcal{M}(X)$.** Here, following [Gác05], we define computable measures in the following way: first the space $\mathcal{M}(X)$ is endowed with a computable metric space structure compatible with the weak topology and then computable measures are defined as the constructive points.

Given a metric space (X, d), the set $\mathcal{M}(X)$ of Borel probability measures over X can be endowed with the *weak topology*, which is the finest topology for which $\mu_n \to \mu$ if and only if $\int f d\mu_n \to \int f d\mu$ for all continuous bounded function $f : X \to \mathbb{R}$. This topology is metrizable and when X is separable and complete, $\mathcal{M}(X)$ is also separable and complete (see [Bil68]). Moreover, a computable metric structure on X induces in a canonical way a computable metric structure on $\mathcal{M}(X)$.

Let $\mathcal{D} \subset \mathcal{M}(X)$ be the set of those probability measures that are concentrated in finitely many points of $\mathcal{S}$ and assign rational values to them. It can be shown that this is a dense subset ([Bil68]). The numberings $\nu_{\mathcal{S}}$ of ideal points of X and $\nu_{\mathbb{Q}}$ of the rationals numbers induce a numbering $\nu_{\mathcal{D}}$ of ideal measures: $\mu_{\langle\langle n_1,\dots,n_k\rangle,\langle m_1,\dots,m_k\rangle\rangle}$ is the measure concentrated over the finite set $\{s_{n_1}, \dots, s_{n_k}\}$ where q_{m_i} is the weight of s_{n_i}.

4.1.1. *The Prokhorov metric.* Let us consider the particular metric on $\mathcal{M}(X)$:

Definition 4.1.1. The ***Prokhorov metric*** ρ on $\mathcal{M}(X)$ is defined by:

$$\rho(\mu, \nu) := \inf\{\epsilon \in \mathbb{R}^+ : \mu(A) \le \nu(A^\epsilon) + \epsilon \text{ for every Borel set } A\}. \tag{1}$$

where $A^\epsilon = \{x : d(x, A) < \epsilon\}$.

It is known that it is indeed a metric, which induces the weak topology on $\mathcal{M}(X)$ (see [Bil68]). Moreover, we have that:

Proposition 4.1.1. *$(\mathcal{M}(X), \mathcal{D}, \rho)$ is a computable metric space.*

Proof. We have to show that the real numbers $\rho(\mu_i, \mu_j)$ are all computable, uniformly in $\langle i, j\rangle$. First observe that if U is a r.e open subset of X, $\mu_i(U)$ is lower semi-computable uniformly in i and U. Indeed, if $(s_{n_1}, q_{m_1}), \dots, (s_{n_k}, q_{m_k})$ are the mass points of μ_i together with their weights (recoverable from i) then $\mu_i(U) = \sum_{s_{n_j} \in U} q_{m_j}$. As the s_{n_j} which belong to U can be enumerated from any description of U, this sum is lower-semi-computable. In particular, $\mu_i(B_{i_1} \cup \dots \cup B_{i_k})$ is lower semi-computable and $\mu_i(\overline{B}_{i_1} \cup \dots \cup \overline{B}_{i_k})$ is upper semi-computable, both of them uniformly in $\langle i, i_1, \dots, i_k\rangle$

Now we prove that $\rho(\mu_i, \mu_j)$ is computable uniformly in $\langle i, j\rangle$.

Observe that if μ_i is an ideal measure concentrated over S_i, then (1) becomes $\rho(\mu_i, \mu_j) = \inf\{\epsilon \in \mathbb{Q} : \forall A \subset S_i,\ \mu_i(A) < \mu_j(A^\epsilon) + \epsilon\}$. Since μ_j is also an ideal measure and A^ϵ is a finite union of open ideal balls, the number $\mu_j(A^\epsilon)$ is lower semi-computable (uniformly) and then $\rho(\mu_i, \mu_j)$ is upper semi-computable, uniformly in $\langle i, j\rangle$. To see that $\rho(\mu_i, \mu_j)$ is lower-semicomputable, uniformly in $\langle i, j\rangle$, observe that $\rho(\mu_i, \mu_j) = \sup\{\epsilon \in \mathbb{Q} : \exists A \subset S_i,\ \mu_i(A) > \mu_j(A^{\overline{\epsilon}}) + \epsilon\}$, where $A^{\overline{\epsilon}} = \{x : d(x, A) \le \epsilon\}$ (a finite union of closed ideal balls when $A \subset S_i$) and use the upper semi-computability of $\mu_j(A^{\overline{\epsilon}})$. □

Definition 4.1.2. A measure μ is ***computable*** if it is a constructive point of $(\mathcal{M}(X), \mathcal{D}, \rho)$.

The effectivization of the space of Borel probability measures $\mathcal{M}(X)$ is of theoretical interest, and opens the question: what kind of information can be (algorithmically) recovered from a description of a measure as a point of the computable metric space $\mathcal{M}(X)$? The two most current uses of a measure are to give weights to measurable sets and means to measurable functions. Can these quantities be computed ?

4.1.2. *The Wasserstein metric.* In the particular case when the metric space X is bounded, an alternative metric can be defined on $\mathcal{M}(X)$. When f is a real-valued function, μf denotes $\int f d\mu$.

Definition 4.1.3. The *Wasserstein metric* on $\mathcal{M}(X)$ is defined by:

$$W(\mu,\nu) = \sup_{f \in 1-Lip(X)} (|\mu f - \nu f|) \tag{2}$$

where $1 - Lip(X)$ is the space of 1-Lipschitz functions from X to $\mathbb{R}$.

We recall (see [LNG05]) that W has the following properties:

Proposition 4.1.2.

(1) *W is a distance and if X is separable and complete then $\mathcal{M}(X)$ with this distance is a separable and complete metric space.*
(2) *The topology induced by W is the weak topology and thus W is equivalent to the Prokhorov metric.*

Moreover, if $(X, \mathcal{S}, d)$ is a computable metric space (and X bounded), then:

Proposition 4.1.3. *$(\mathcal{M}(X), \mathcal{D}, W)$ is a computable metric space.*

Proof. We have to show that the distance $W(\mu_i, \mu_j)$ between ideal measures is uniformly computable. From $\langle i, j\rangle$ we can compute the set $S_{i,j} = supp(\mu_i) \cup supp(\mu_j)$. Let $s_0 \in S_{i,j}$, then we can suppose that the supremum in (2) is taken over $1 - Lip^0_{s_0}(X) := \{f \in 1 - Lip(X) : 1 - Lip^0_{s_0}(X) f(s) = 0\}$. Given some precision ϵ we construct a finite set $\mathcal{N}_\epsilon \subset 1 - Lip^0_{s_0}(X)$ made of uniformly computable functions such that for each $f \in 1 - Lip^0_{s_0}(X)$ there is some $l \in \mathcal{N}_\epsilon$ satisfying $\sup\{|f(x) - l(x)| : x \in S_{i,j}\} < \epsilon$: compute an integer m such that $S_{i,j} \subset B(s, m)$; then $|f| < m$ for every $f \in 1 - Lip^0_s(X)$. Let n be such that $m/n < 2\epsilon$. For each $s \in S_{i,j}$ and $a \in \{\frac{lm}{n}\}^m_{l=-m}$ let us consider the functions defined by $\phi^+_{s,l}(x) := a + d(s, x)$ and $\phi^-_{s,l}(x) := a - d(s, x)$. Then it is not difficult to see that $\mathcal{N}_\epsilon$ defined as the set of all possible combinations of max and min made with the $\phi^{+-}_{s,l}(x)$ satisfy the required condition.

Therefore, since $\sup(|f - g|) < \epsilon$ implies $|\mu(f - g)| < \epsilon$ we have that:

$$W(\mu_i, \mu_j) \in [\sup_{g \in \mathcal{N}_\epsilon} (|\mu_i g - \mu_j g|), \sup_{g \in \mathcal{N}_\epsilon} (|\mu_i g - \mu_j g|) + 2\epsilon]$$

where the $\mu_i g$ are computable, uniformly in i. The result follows.

□

When X is bounded, the effectivisation using the Prokhorov or the Wasserstein metrics turn out to be equivalent.

Theorem 4.1.1. *The Prokhorov and the Wasserstein metrics are computably equivalent. That is, the identity function $id : (\mathcal{M}(X), \mathcal{D}, \rho) \to (\mathcal{M}(X), \mathcal{D}, W)$ is a computable isomorphism, as well as its inverse.*

Proof. Let M be an integer such that $\sup_{x,y\in X} d(x,y) < M$. Suppose $\rho(\mu,\nu) < \epsilon/(M+1)$. Then, by the coupling theorem [Bil68], for every $f \in 1-Lip(X)$ it holds $|\mu f - \nu f| \leq \epsilon$, then $W(\mu,\nu) < \epsilon$. Conversely, suppose $W(\mu,\nu) < \epsilon^2 < 1$. Let A be a Borel set and define $g_\epsilon^A := |1 - d(x,A)/\epsilon|^+$. Then $\epsilon g_\epsilon^A \in 1-Lip(X)$. $W(\mu,\nu) < \epsilon^2$ implies $\mu\epsilon g_\epsilon^A < \nu\epsilon g_\epsilon^A + \epsilon^2$ and since $\mu(A) \leq \mu g_\epsilon^A$ and $\nu g_\epsilon^A \leq \nu(A^\epsilon)$, we conclude $\mu(A) \leq \nu(A^\epsilon) + \epsilon$ and then $\rho(\mu,\nu) < \epsilon$. Therefore, given a fast sequence of ideal measures converging to μ in the Prokhorov metric, we can construct a fast sequence of ideal measures converging to μ in the W metric and vice-versa.

□

This equivalence offers an alternative method to prove computability of measures. It is used for example in [GHR07b] to show the computability of the physical measures for some classes of dynamical systems.

4.2. **Measures as valuations.** We now investigate the first problem: can the measure of sets be computed from the Cauchy description of a measure? Actually, the answer is positive for a very small part of the Borel sigma-field. It is a well-known fact that a Borel (probability) measure μ is characterized by the measure of open sets, which generate the Borel sigma-field. That is, by the valuation $v_\mu : \tau \to [0,1]$ which maps an open set to its μ-measure. The question is then so study this characterization from a computability viewpoint.

The first result is that the measure of open sets can be lower semi-computed, using the Cauchy description of the measure.

Proposition 4.2.1. *The valuation operator $v : \mathcal{M}(X) \times \tau \to [0,1]$ mapping (μ, U) to $\mu(U)$ is lower semi-computable.*

Proof. as $v_\mu = v(\mu,.)$ is Scott-continuous (see remark 3.1.1), it suffices to show that it is uniformly lower semi-computable on finite unions of balls.

We first restrict to ideal measures μ_i: we have already seen (proof of proposition 4.1.1) that all $\mu_i(B_{i_1} \cup \ldots \cup B_{i_k})$ are lower semi-computable real numbers, uniformly in $\langle i, i_1, \ldots, i_k\rangle$.

Now let $(\mu_{k_n})_{n\in\mathbb{N}}$ a description of a measure μ, that is a fast sequence converging to μ for the Prokhorov distance: then $\rho(\mu_{k_n}, \mu) \leq \epsilon_n$ where $\epsilon_n = 2^{-n+1}$. For $n \geq 1$, and $U = B(s_{i_1}, q_{j_1}) \cup \ldots \cup B(s_{i_k}, q_{j_k})$ define:

$$U_n = \bigcup_{m\leq k} B(s_{i_m}, q_{j_m} - \epsilon_n)$$

note that $U_{n-1}^{\epsilon_n} \subseteq U_n$ and $U_n^{\epsilon_n} \subseteq U$. We show that $\mu(U) = \sup_n(\mu_{j_n}(U_n) - \epsilon_n)$:

• $\mu_{j_n}(U_n) \leq \mu(U) + \epsilon_n$ for all n, so $\mu(U) \geq \sup_n(\mu_{j_n}(U_n) - \epsilon_n)$.

• $\mu(U_{n-1}) \leq \mu_{j_n}(U_n) + \epsilon_n$ for all n. As U_{n-1} increases towards U as $n \to \infty$, $\mu(U) = \sup_n(\mu(U_{n-1}) - 2\epsilon_n) \leq \sup_n(\mu_{j_n}(U_n) - \epsilon_n)$.

As the quantity $\mu_{j_n}(U_n) - \epsilon_n$ is lower semi-computable uniformly in n, we are done (observe that everything is uniform in the finite description of U).

□

The second result is stronger: the lower semi-computability of the measure of the r.e open sets even characterizes the computability of the measure.

Theorem 4.2.1. *Given a measure $\mu \in \mathcal{M}(X)$, the following are equivalent:*

(1) *μ is computable,*
(2) *$v_\mu : \tau \to [0,1]$ is lower-semi-computable,*

(3) $\mu(B_{i_1} \cup \ldots \cup B_{i_k})$ *is lower-semi-computable uniformly in* $\langle i_1, \ldots, i_k \rangle$*.*

Proof. [$1 \Rightarrow 2$] Direct from proposition 4.2.1. [$2 \Rightarrow 3$] Trivial. [$3 \Rightarrow 1$] We show that $\rho(\mu_n, \mu)$ is upper semi-computable uniformly in n, and then use proposition 2.4.2. Since $\rho(\mu_n, \mu) < \epsilon$ iff $\mu_n(A) < \mu(A^\epsilon) + \epsilon$ for all $A \subset S_n$ where S_n is the finite support of μ_n, and $\mu(A^\epsilon)$ is lower semi-computable (A^ϵ is a finite union of open ideal balls) $\rho(\mu_n, \mu) < \epsilon$ is semi-decidable, uniformly in n and ϵ. This allows to construct a fast sequence of ideal measures converging to μ.

□

It means that a representation which would be "tailor-made" to make the valuation constructive, describing a measure μ by the set of integers $\langle i_1, \ldots, i_k, j \rangle$ satisfying $\mu(B_{i_1} \cup \ldots \cup B_{i_k}) > q_j$, would be constructively equivalent to the Cauchy representation. This is the approach taken in [Wei99] for the special case $X = [0, 1]$ and in [Sch07] on an arbitrary sequential topological space. In both case, the topology on $\mathcal{M}(X)$ induced by this representation is proved to be equivalent to the weak topology. A domain theoretical approach was also developed in [Eda96] on a compact space, the Scott topology being proved to induce the weak topology.

4.2.1. *The examples of the Cantor space and the unit interval.* On the Cantor space $\Sigma^{\mathbb{N}}$ (where Σ is a finite alphabet) with its natural computable metric space structure, the ideal balls are the cylinders. As a finite union of cylinders can always be expressed as a disjoint (and finite) union of cylinders, and the complement of a cylinder is a finite union of cylinders, we have:

Corollary 4.2.1. *A measure* $\mu \in \mathcal{M}(\Sigma^{\mathbb{N}})$ *is computable iff the measures of the cylinders are uniformly computable.*

On the unit real interval, ideals balls are open rational intervals. Again, a finite union of such intervals can always be expressed as a disjoint (and finite) union of open rational intervals. Then:

Corollary 4.2.2. *A measure* $\mu \in \mathcal{M}([0, 1])$ *is computable iff the measures of the rational open intervals are uniformly lower-semi-computable.*

If μ has no atoms, a rational open interval is the complement of at most two disjoint open rational intervals, up to a null set. In this case, μ is then computable iff the measures of the rational intervals are uniformly *computable.*

4.3. **Measures as integrals.** We now answer the second question: is the integral of functions computable from the description of a measure ?

The computable metric space structure of X and the enumerative lattice structure of $\overline{\mathbb{R}}^+$ induce in a canonical way the enumerative space $\mathcal{C}(X, \overline{\mathbb{R}}^+)$ (see section 3.2), which is actually the set of lower semi-continuous functions from X to $\overline{\mathbb{R}}^+$. We have:

Proposition 4.3.1. *The integral operator* $\int : \mathcal{M}(X) \times \mathcal{C}(X, \overline{\mathbb{R}}^+) \to \overline{\mathbb{R}}^+$ *is lower semi-computable.*

Proof. the integral of a finite supremum of step functions can be expressed by induction on the number of functions: first, $\int f_{\langle i,j\rangle} d\mu = q_j \mu(B_i)$ and

$$\int \sup\{f_{\langle i_1,j_1\rangle},\dots,f_{\langle i_k,j_k\rangle}\} d\mu \quad = \quad q_{j_m}\mu(B_{i_1}\cup\ldots\cup B_{i_k}) + \int \sup\{f_{\langle i_1,j'_1\rangle},\dots,f_{\langle i_k,j'_k\rangle}\} d\mu$$

where q_{j_m} is minimal among $\{q_{j_1},\dots,q_{j_k}\}$ and $q_{j'_1} = q_{j_1} - q_{j_m}, q_{j'_2} = q_{j_2} - q_{j_m}, etc.$ Note that $f_{\langle i_m,j'_m\rangle}$ being the zero function can be removed.

Now, m can be computed and by proposition 4.2.1 the measure of finite unions of ideal balls can be uniformly μ-lower semi-computed, so the integral above can be uniformly μ-lower semi-computed. For any fixed measure μ, the integral operator $\int d\mu : \mathcal{C}(X,\overline{\mathbb{R}}^+) \to \overline{\mathbb{R}}^+$ is Scott-continuous, so it is lower semi-computable.

□

Again, the lower semi-computability of the integral of lower semi-computable functions characterizes the computability of the measure:

Corollary 4.3.1. *Given a measure $\mu \in \mathcal{M}(X)$, the following are equivalent:*
(1) *μ is computable,*
(2) *$\int d\mu : \mathcal{C}(X,\overline{\mathbb{R}}^+) \to \overline{\mathbb{R}}^+$ is lower semi-computable,*
(3) *$\int \sup\{f_{i_1},\dots,f_{i_k}\} d\mu$ is lower-semi-computable uniformly in $\langle i_1,\dots,i_k\rangle$.*

Proof. $[2 \Leftrightarrow 3]$ holds by Scott-continuity of the operator,
$[1 \Rightarrow 2]$ is a direct consequence of proposition 4.3.1,
$[2 \Rightarrow 1]$ is a direct consequence of theorem 4.2.1, composing the integral operator with the function from τ to $\mathcal{C}(X,\overline{\mathbb{R}}^+)$ mapping an open set to its indicator function (which is computable, see proposition 3.3.1).

□

It means that a representation of measures which would be "tailor-made" to make the integration constructive, describing a measure by the set of integers $\langle i_1,\dots,i_k,j\rangle$ satisfying $\int \sup\{f_{i_1},\dots,f_{i_k}\} d\mu > q_j$, would be constructively equivalent to the Cauchy representation.

A corollary of proposition 4.3.1 will be used in the last section: let $(f_i)_i$ be a sequence of uniformly computable functions, i.e. such that the function $(i,x) \mapsto f_i(x)$ is computable. If moreover f_i has a bound M_i computable uniformly in i, then the function $(\mu,i) \to \int f_i d\mu$ is computable. Indeed, $f_i + M_i$ (resp. $M_i - f_i$) is uniformly lower (resp. upper) semi-computable, so $\int f_i d\mu = \int (f_i + M_i) d\mu - M_i = M_i - \int (M_i - f_i) d\mu$ and proposition 4.3.1 allow to conclude.

5. Computable Probability Spaces

Definition 5.0.1. A ***computable probability space*** is a pair $(\mathcal{X},\mu)$ where $\mathcal{X}$ is a computable metric space and μ a computable Borel probability measure on X.

Definition 5.0.2. A ***morphism of computable probability spaces*** $F : (\mathcal{X},\mu) \to (\mathcal{Y},\nu)$, is a computable measure-preserving function $F : D_F \subseteq X \to Y$ where D_F is a (full-measure) Π^0_2-set.

An ***isomorphism*** $(F,G) : (\mathcal{X},\mu) \rightleftarrows (\mathcal{Y},\nu)$ is a pair (F,G) of morphisms such that $G \circ F = id$ on $F^{-1}(D_G)$ and $F \circ G = id$ on $G^{-1}(D_F)$.

We recall that F is measure-preserving if $\nu(A) = \mu(F^{-1}(A))$ for all Borel set A.

5.1. **Generalized binary representations.** The Cantor space 2^ω (2 denotes $\{0, 1\}$) is a privileged place for computability. This can be understood by the fact that it is the countable product (with the product topology) of a finite space (with the discrete topology). A consequence of this is that membership of a basic open set (cylinder) boils down to a pattern-matching and is then *decidable*. As decidable sets must be clopen, this property cannot hold in connected spaces. As a result, a computable metric space is not in general constructively homeomorphic to the Cantor space.

Nevertheless, the real unit interval $[0, 1]$ is not so far away from the Cantor space. The binary numeral system provides a correspondence between real numbers and binary sequences, which is certainly not homeomorphic, unless we remove the small set of dyadic numbers. In particular, the remaining set is totally disconnected, and the dyadic intervals form a basis of clopen sets.

Actually, this correspondence makes the computable probability space $[0, 1]$ with the Lebesgue measure isomorphic to the Cantor space with the uniform measure. This fact has been implicitly used, for instance, to extend algorithmic randomness on the Cantor space with the uniform measure to the unit interval with the Lebesgue measure.

We extend this to any computable probability space defining the notion of *binary representation*, and show that every computable probability space has a binary representation, which implies in particular that every computable probability space is isomorphic to the Cantor space with a computable measure. To carry out this generalization, let us briefly scrutinize the binary numeral system on the unit interval:

$\delta : 2^\omega \to [0, 1]$ is a total surjective morphism. Every non-dyadic real has a unique expansion, and the inverse of δ, defined on the set D of non-dyadic numbers, is computable. Moreover, D is large both in a topological and measure-theoretical sense: it is a residual (a countable intersection of dense open sets) and has measure one. (δ, δ^{-1}) is then an isomorphism.

In our generalization, we do not require every binary sequence to be the expansion of a point, which would force X to be compact.

Definition 5.1.1. A ***binary representation*** of a computable probability space $(\mathcal{X}, \mu)$ is a pair (δ, μ_δ) where μ_δ is a computable probability measure on 2^ω and $\delta : (2^\omega, \mu_\delta) \to (\mathcal{X}, \mu)$ is a surjective morphism such that, calling $\delta^{-1}(x)$ the set of ***expansions*** of $x \in X$:

- there is a dense full-measure Π^0_2-set D of points having a unique expansion,
- $\delta^{-1} : D \to \delta^{-1}(D)$ is computable.

Remark that when the support of the measure (the smallest closed set of full measure) is the whole space X, like the Lebesgue measure on the interval, a full-measure Π^0_2-set is always a residual, but in general it is only dense on the support of the measure: that is the reason why we explicitly require D to be dense. Also remark that a binary representation δ always induces an isomorphism (δ, δ^{-1}) between the Cantor space and the computable probability space.

The sequel of this section is devoted to the proof of the following result:

Theorem 5.1.1. *Every computable probability space $(\mathcal{X}, \mu)$ has a binary representation.*

The space, restricted to the domain D of the isomorphism, is then totally disconnected: the preimages of the cylinders form a basis of clopen and even *decidable* sets. In the whole space, they are not decidable any more. Instead, they are *almost decidable*.

Definition 5.1.2. A set A is said to be ***almost decidable*** if there are two r.e open sets U and V such that:

$$U \subset A, \quad V \subseteq A^{\mathcal{C}}, \quad U \cup V \text{ is dense and has measure one}$$

Definition 5.1.3. A measurable set A is said to be ***μ-continuous*** or a ***μ-continuity set*** if $\mu(\partial A) = 0$ where $\partial A = \overline{A} \cap \overline{X \setminus A}$ is the boundary of A.

Remark that, as for subsets of $\mathbb{N}$, a set is almost decidable if and only if its complement is a.s. decidable. An almost decidable set is always a continuity set. Let $B(s, r)$ be a μ-continuous ball with computable radius: in general it is not an almost decidable set (for instance, isolated points may be at distance exactly r from s). But if there is no ideal point is at distance r from s, then $B(s, r)$ is almost decidable: take $U = B(s, r)$ and $V = X \setminus \overline{B}(s, r)$.

We say that the elements of a sequence $(A_i)_{i\in\mathbb{N}}$ are uniformly a.s. decidable if there are two sequences $(U_i)_{i\in\mathbb{N}}$ and $(V_i)_{i\in\mathbb{N}}$ of uniformly r.e sets satisfying the conditions above.

Lemma 5.1.1. *There is a sequence $(r_n)_{n\in\mathbb{N}}$ of uniformly computable reals such that $(B(s_i, r_n))_{\langle i,n\rangle}$ is a basis of uniformly almost decidable balls.*

Proof. define $U_{\langle i,k\rangle} = \{r \in \mathbb{R}^+ : \mu(\overline{B}(s_i, r)) < \mu(B(s_i, r)) + 1/k\}$: by computability of μ, this is a r.e open subset of $\mathbb{R}^+$, uniformly in $\langle i, k\rangle$. It is furthermore dense in $\mathbb{R}^+$: the spheres $S_r = \overline{B}(s_i, r) \setminus B(s_i, r)$ form a partition of the space when r varies in $\mathbb{R}^+$ and μ is finite, so the set of r for which $\mu(S_r) \geq 1/k$ is finite.

Define $V_{\langle i,j\rangle} = \mathbb{R}^+ \setminus \{d(s_i, s_j)\}$: this is a dense r.e open set, uniformly in $\langle i, j\rangle$.

Then by the computable Baire Category Theorem (see [YMT99], [Bra01]), the dense Π^0_2-set $\bigcap_{\langle i,k\rangle} U_{\langle i,k\rangle} \cap \bigcap_{\langle i,j\rangle} V_{\langle i,j\rangle}$ contains a sequence $(r_n)_{n\in\mathbb{N}}$ of uniformly computable real numbers which is dense in $\mathbb{R}^+$. In other words, all r_n are computable, uniformly in n. By construction, for any s_i and r_n, $B(s_i, r_n)$ is almost decidable.

We recall that from an enumeration $(I_n)_{n\in\mathbb{N}}$ of all the rational compact intervals of $\mathbb{R}^+$, r_n is constructed computing a nested shrinking sequence $(J^n_k)_{k\in\mathbb{N}}$ of rational compact intervals starting from $J^n_0 = I_n$, and such that $J^n_{k+1} \subseteq J^n_k \cap U_k \cap V_k$. Then $\{r_n\} = \bigcap_k J^n_k$.

□

We will denote $B(s_i, r_n)$ by B^μ_k where $k = \langle i, n\rangle$. Note that different algorithmic descriptions of the same μ may yield different sequences $(r_n)_{n\in\mathbb{N}}$, so B^μ_k is an abusive notation. It is understood that some algorithmic description of μ has been chosen and fixed. This can be done only because the measure μ is computable, which is then a crucial hypothesis. We denote $X \setminus \overline{B}(s_i, r_n)$ by C^μ_k and define:

Definition 5.1.4. For $w \in 2^*$, the ***cell*** $\Gamma(w)$ is defined by induction on $|w|$:

$$\Gamma(\epsilon) = X, \quad \Gamma(w0) = \Gamma(w) \cap C^\mu_i \quad \text{and} \quad \Gamma(w1) = \Gamma(w) \cap B^\mu_i$$

where ϵ is the empty word and $i = |w|$.

This an almost decidable set, uniformly in w.

Proof. (of theorem 5.1.1). We construct an encoding function $b : D \to 2^\omega$, a decoding function $\delta : D_\delta \to X$, and show that δ is a binary representation, with $b = \delta^{-1}$.

Encoding.

Let $D = \bigcap_i B_i^\mu \cup C_i^\mu$: this is a dense full-measure Π_2^0-set. Define the computable function $b : D \to 2^\omega$ by:

$$b(x)_i = \begin{cases} 1 & \text{if } x \in B_i^\mu \\ 0 & \text{if } x \in C_i^\mu \end{cases}$$

Let $x \in D$: $\omega = b(x)$ is also characterized by $\{x\} = \bigcap_i \Gamma(\omega_{0..i-1})$. Let μ_δ be the image measure of μ by b: $\mu_\delta = \mu \circ b^{-1}$. b is then a morphism from (X, μ) to $(2^\omega, \mu_\delta)$.

Decoding.

Let D_δ be the set of binary sequences ω such that $\bigcap_i \overline{\Gamma(\omega_{0..i-1})}$ is a singleton. We define the decoding function $\delta : D_\delta \to X$ by:

$$\delta(\omega) = x \text{ if } \bigcap_i \overline{\Gamma(\omega_{0..i-1})} = \{x\}$$

ω is called an expansion of x. Remark that $x \in B_i^\mu \Rightarrow \omega_i = 1$ and $x \in C_i^\mu \Rightarrow \omega_i = 0$, which implies in particular that if $x \in D$, x has a unique expansion, which is $b(x)$. Hence, $b = \delta^{-1} : \delta^{-1}(D) \to D$ and $\mu_\delta(D_\delta) = \mu(D) = 1$.

We now show that $\delta : D_\delta \to X$ is a surjective morphism. For seek of clarity, the center and the radius of the ball B_i^μ will be denoted s_i and r_i respectively. Let us call i an *n-witness* for ω if $r_i < 2^{-(n+1)}, \omega_i = 1$ and $\Gamma(\omega_{0..i}) \neq \emptyset$.

• D_δ is a Π_2^0-set: we show that $D_\delta = \bigcap_n \{\omega \in 2^\omega : \omega \text{ has a } n\text{-witness}\}$.

Let $\omega \in D_\delta$ and $x = \delta(\omega)$. For each n, $x \in B(s_i, r_i)$ for some i with $r_i < 2^{-(n+1)}$. Since $x \in \overline{\Gamma(\omega_{0..i})}$, we have that $\Gamma(\omega_{0..i}) \neq \emptyset$ and $\omega_i = 1$ (otherwise $\overline{\Gamma(\omega_{0..i})}$ is disjoint of B_i^μ). In other words, i is an n-witness for ω.

Conversely, if ω has a n-witness i_n for all n, since $\overline{\Gamma(\omega_{0..i_n})} \subseteq \overline{B_{i_n}^\mu}$ whose radius tends to zero, the nested sequence $(\overline{\Gamma(\omega_{0..i_n})})_n$ of closed cells has, by completeness of the space, a non-empty intersection, which is a singleton.

• $\delta : D_\delta \to X$ is computable. For each n, find some n-witness i_n of ω: the sequence $(s_{i_n})_n$ is a fast sequence converging to $\delta(\omega)$.

• δ is surjective: we show that each point $x \in X$ has at least one expansion. To do this, we construct by induction a sequence $\omega = \omega_0\omega_1 \ldots$ such that for all i, $x \in \overline{\Gamma(\omega_0 \ldots \omega_i)}$. Let $i \geq 0$ and suppose that $\omega_0 \ldots \omega_{i-1}$ (empty when $i = 0$) has been constructed. As $B_i^\mu \cup C_i^\mu$ is open dense and $\Gamma(\omega_{0..i-1})$ is open, $\overline{\Gamma(\omega_{0..i-1})} = \overline{\Gamma(\omega_{0..i-1}) \cap (B_i^\mu \cup C_i^\mu)}$ which equals $\overline{\Gamma(\omega_{0..i-1}0)} \cup \overline{\Gamma(\omega_{0..i-1}1)}$. Hence, one choice for $\omega_i \in \{0, 1\}$ gives $x \in \overline{\Gamma(\omega_{0..i})}$.

By construction, $x \in \bigcap_i \overline{\Gamma(\omega_{0..i-1})}$. As $(B_i^\mu)_i$ is a basis and $\omega_i = 1$ whenever $x \in B_i^\mu$, ω is an expansion of x.

□

5.2. **Another characterization of the computability of measures.** The existence of a basis of almost decidable sets also leads to another characterization of the computability of measures, which is reminiscent of what happens on the Cantor space (see corollary 4.2.1). Let us say that two bases $(U_i)_i$ and $(V_i)_i$ of the topology

τ are constructively equivalent if both $id_\tau : (\tau, \subseteq, \mathcal{U}) \to (\tau, \subseteq, \mathcal{V})$ and its inverse are constructive functions between enumerative lattices.

Corollary 5.2.1. *A measure $\mu \in \mathcal{M}(X)$ is computable if and only if there is a basis $\mathcal{U} = (U_i)_{i\in\mathbb{N}}$ of uniformly almost decidable open sets which is constructively equivalent to $\mathcal{B}$ and such that all $\mu(U_{i_1} \cup \ldots \cup U_{i_k})$ are computable uniformly in $\langle i_1, \ldots, i_k\rangle$.*

Proof. if μ is computable, the a.s. decidable balls $U_{\langle i,n\rangle} = B(s_i, r_n)$ are basis which is constructively equivalent to $\mathcal{B}$: indeed, $B(s_i, r_n) = \bigcup_{q_j<r_n} B(s_i, q_j)$ and $B(s_i, q_j) = \bigcup_{r_n<q_j} B(s_i, r_n)$, and r_n is computable uniformly in n.

For the converse, the valuation function f_μ is lower semi-computable. Indeed, the r.e open sets are uniformly r.e relatively to the basis $\mathcal{U}$, so their measures can be lower-semi-computed, computing the measures of finite unions of elements of $\mathcal{U}$. Hence μ is computable by theorem 4.2.1.

□

6. Algorithmic randomness

On the Cantor space with a computable measure μ, Martin-Löf originally defined the notion of an individual random sequence as a sequence passing all μ-randomness tests. A μ-randomness test *à la Martin-Löf* is a sequence of uniformly r.e open sets $(U_n)_n$ satisfying $\mu(U_n) \leq 2^{-n}$. The set $\bigcap_n U_n$ has null measure, in an effective way: it is then called an effective null set.

Equivalently, a μ-randomness test can be defined as a positive lower semi-computable function $t : 2^\omega \to \mathbb{R}$ satisfying $\int t d\mu \leq 1$ (see [VV93] for instance). The associated effective null set is $\{x : t(x) = +\infty\} = \bigcap_n \{x : t(x) > 2^n\}$. Actually, every effective null set can be put in this form for some t. A point is then called μ-random if it lies in no effective null set.

Following Gács, we will use the second presentation of randomness tests which is more suitable to express uniformity.

6.1. Randomness w.r.t any probability measure.

Definition 6.1.1. Given a measure $\mu \in \mathcal{M}(X)$, a ***μ*-randomness test** is a μ-constructive element t of $\mathcal{C}(X, \overline{\mathbb{R}}^+)$, such that $\int t d\mu \leq 1$. Any subset of $\{x \in X : t(x) = +\infty\}$ is called a ***μ*-effective null set**.

A ***uniform randomness test*** is a constructive function T from $\mathcal{M}(X)$ to $\mathcal{C}(X, \overline{\mathbb{R}}^+)$ such that for all $\mu \in \mathcal{M}(x)$, $\int T^\mu d\mu \leq 1$ where T^μ denotes $T(\mu)$.

Note that T can be also seen as a lower-semi-computable function from $\mathcal{M}(X) \times X$ to $\overline{\mathbb{R}}^+$ (see section 3.2).

A presentation *à la Martin-Löf* can be directly obtained using the functions below:

$$F: \begin{array}{ccc} \mathcal{C}(X, \overline{\mathbb{R}}^+) & \to & \tau^{\mathbb{N}} \\ t & \mapsto & (t^{-1}(2^n, +\infty))_n \end{array} \qquad G: \begin{array}{ccc} \tau^{\mathbb{N}} & \to & \mathcal{C}(X, \overline{\mathbb{R}}^+) \\ (U_n)_n & \mapsto & (x \mapsto \sup\{n : x \in \bigcap_{i\leq n} U_i\}) \end{array}$$

which are constructive, satisfy $F \circ G = id : \tau^{\mathbb{N}} \to \tau^{\mathbb{N}}$ and preserve the corresponding effective null sets.

A uniform randomness test T induces a μ-randomness test T^μ for all μ. We show two important results which hold on any computable metric space:

- the two notions are actually equivalent (theorem 6.1.1),
- there is a *universal* uniform randomness test (theorem 6.1.2).

The second result was already obtained by Gács, but only on spaces which have recognizable Boolean inclusions, which is an additional computability property on the basis of ideal balls.

By proposition 3.2.2, constructive functions from $\mathcal{M}(X)$ to $\mathcal{C}(X, \overline{\mathbb{R}}^+)$ can be identified to constructive elements of the enumerative lattice $\mathcal{C}(\mathcal{M}(X), \mathcal{C}(X, \overline{\mathbb{R}}^+))$. Let $(H_i)_{i\in\mathbb{N}}$ be an enumeration of all its constructive elements (proposition 3.1.1): $H_i = \sup_k f_{\varphi(i,k)}$ where $\varphi : \mathbb{N}^2 \to \mathbb{N}$ is some recursive function and the f_n are step functions.

Lemma 6.1.1. *There is a constructive function $T : \mathbb{N} \times \mathcal{M}(X) \to \mathcal{C}(X, \overline{\mathbb{R}}^+)$ satisfying:*

- *for all i, $T_i = T(i, .)$ is a uniform randomness test,*
- *if $\int H_i(\mu)d\mu < 1$ for some μ, then $T_i(\mu) = H_i(\mu)$.*

Proof. To enumerate only tests, we would like to be able to semi-decide $\int \sup_{k<n} f_{\varphi(i,k)}(\mu)d\mu < 1$. But $\sup_{k<n} f_{\varphi(i,k)}(\mu)$ is only lower semi-computable (from μ). To overcome this problem, we use another class of basic function.

Let $\mathcal{Y}$ be a computable metric space: for an ideal point s of Y and positive rationals q, r, ϵ, define the *hat* function:

$$h_{q,s,r,\epsilon}(y) := q.[1 - [d(y, s) - r]^+/\epsilon]^+$$

where $[a]^+ = \max\{0, a\}$. This is a continuous function whose value is q in $B(s, r)$, 0 outside $B(s, r + \epsilon)$. The numberings of $\mathcal{S}$ and $\mathbb{Q}_{>0}$ induce a numbering $(h_n)_{n\in\mathbb{N}}$ of all the hat functions. They can be taken as an alternative to step functions in the enumerative lattice $\mathcal{C}(Y, \overline{\mathbb{R}}^+)$: they yield the same computable structure. Indeed, step functions can be constructively expressed as suprema of such functions: $f_{\langle i,j\rangle} = \sup\{h_{q_j,s,r-\epsilon,\epsilon} : 0 < \epsilon < r\}$ where $B_i = B(s, r)$, and conversely.

We apply this to $Y = \mathcal{M}(X) \times X$ endowed with the canonical computable metric structure. By Curryfication it provides functions $h_n \in \mathcal{C}(\mathcal{M}(X), \mathcal{C}(X, \overline{\mathbb{R}}^+))$ with which the H_i can be expressed: there is a recursive function $\psi : \mathbb{N}^2 \to \mathbb{N}$ such that for all i, $H_i = \sup_k h_{\psi(i,k)}$.

Furthermore, $h_n(\mu)$ (strictly speaking, $Eval(h_n, \mu)$, see proposition 3.2.3) is bounded by a constant computable from n and independent of μ. Hence, the integration operator $\int : \mathcal{M}(X)\times\mathbb{N} \to [0, 1]$ which maps $(\mu, \langle i_1, \dots, i_k\rangle)$ to $\int \sup\{h_{i_1}(\mu), \dots, h_{i_k}(\mu)\}d\mu$ is computable.

We are now able to define T: $T(i, \mu) = \sup\{H_i^k(\mu) : \int H_i^k(\mu)d\mu < 1\}$ where $H_i^k = \sup_{n<k} h_{\psi(i,n)}$. As $\int H_i^k(\mu)d\mu$ can be computed from i, k and a description of μ, T is a constructive function from $\mathbb{N} \times \mathcal{M}(X)$ to $\mathcal{C}(X, \overline{\mathbb{R}}^+)$.

□

As a consequence, every randomness test for a particular measure can be extended to a uniform test:

Theorem 6.1.1 (Uniformity vs non-uniformity)**.** *Let μ_0 be a measure. For every μ_0-randomness test t there is a uniform randomness test $T : \mathcal{M}(X) \to \mathcal{C}(X, \overline{\mathbb{R}}^+)$ with $T(\mu_0) = \frac{1}{2}t$.*

Proof. let μ_0 be a measure and t a μ_0-randomness test: $\frac{1}{2}t$ is then a μ_0-constructive element of the enumerative lattice $\mathcal{C}(X,\overline{\mathbb{R}}^+)$, so by lemma 3.2.1 there is a constructive element H of $\mathcal{C}(\mathcal{M}(X),\mathcal{C}(X,\overline{\mathbb{R}}^+))$ such that $H(\mu_0)=\frac{1}{2}t$. There is some i such that $H=H_i$: T_i is a uniform randomness test satisfying $T_i(\mu_0)=\frac{1}{2}t$ because $\int H_i(\mu_0)d\mu_0=\frac{1}{2}\int td\mu_0<1$. □

Theorem 6.1.2 (Universal uniform test)**.** *There is a universal uniform randomness test, that is a uniform test T_u such that for every uniform test T there is a constant c_T with $T_u\geq c_TT$.*

Proof. it is defined by $T_u:=\sum_i 2^{-i-1}T_i$: as every T_i is a uniform randomness test, T_u is also a uniform randomness test, and if T is a uniform impossibility test, then in particular $\frac{1}{2}T$ is a constructive element of $\mathcal{C}(\mathcal{M}(X),\mathcal{C}(X,\overline{\mathbb{R}}^+))$, so $\frac{1}{2}T=H_i$ for some i. As $\int H_i(\mu)d\mu=\frac{1}{2}\int T(\mu)d\mu<1$ for all μ, $T_i(\mu)=H_i(\mu)=\frac{1}{2}T(\mu)$ for all μ, that is $T_i=\frac{1}{2}T$. So $T_u\geq 2^{-i-2}T$. □

Definition 6.1.2. Given a measure μ, a point $x\in X$ is called ***μ-random*** if $T_u^\mu(x)<\infty$. Equivalently, x is μ-random if it lies in no μ-effective null set.

The set of μ-random points is denoted by R_μ. This is the complement of the maximal μ-effective null set $\{x\in X: T_u^\mu(x)=+\infty\}$.

6.2. **Randomness on a computable probability space.** We study the particular case of a computable measure. As a morphism of computable probability spaces is compatible with measures and computability structures, it shall be compatible with algorithmic randomness. Indeed:

Proposition 6.2.1. *Morphisms of computable probability spaces are defined on random points and preserve randomness.*

To prove it, we shall use the following lemma:

Lemma 6.2.1. *In a computable probability space $(\mathcal{X},\mu)$, every random point lies in every r.e open set of full measure.*

Proof. let $U=\bigcup_{\langle i,j\rangle\in E}B(s_i,q_j)$ be a r.e open set of measure one, with E a r.e subset of $\mathbb{N}$. Let F be the r.e set $\{\langle i,k\rangle:\exists j,\langle i,j\rangle\in E, q_k<q_j\}$. Define:

$$U_n=\bigcup_{\langle i,k\rangle\in F\cap[0,n]}B(s_i,q_k)\quad\text{and}\quad V_n^{\mathcal{C}}=\bigcup_{\langle i,k\rangle\in F\cap[0,n]}\overline{B}(s_i,q_k)$$

Then U_n and V_n are r.e uniformly in n, $U_n\nearrow U$ and $U^{\mathcal{C}}=\bigcap_n V_n$. As $\mu(U_n)$ is lower semi-computable uniformly in n, a sequence $(n_i)_{i\in\mathbb{N}}$ can be computed such that $\mu(U_{n_i})>1-2^{-i}$. Then $\mu(V_{n_i})<2^{-i}$, and $U^{\mathcal{C}}=\bigcap_i V_{n_i}$ is a μ-Martin-Löf test. Therefore, every μ-random point is in U. □

Proof. (of proposition 6.2.1) let $F:D\subseteq X\to Y$ be a morphism. From lemma 6.2.1, every random point is in D which is an intersection of full-measure r.e open sets.

Let $t:Y\to\overline{\mathbb{R}}^+$ be the universal ν-test. The function $t\circ F:D\to\overline{\mathbb{R}}^+$ is lower semi-computable. Let $\mathcal{A}$ be any algorithm lower semi-computing it: the associated lower semi-computable function $f_{\mathcal{A}} : X \to \overline{\mathbb{R}}^+$ extends $t \circ F$ to the whole space X (see lemma 3.2.1). As $\mu(D) = 1$, $\int t \circ F d\mu$ is well defined and equals $\int f_{\mathcal{A}} d\mu$. As F is measure-preserving, $\int t \circ F d\mu = \int t d\nu \leq 1$. Hence $f_{\mathcal{A}}$ is a μ-test. Let $x \in X$ be a μ-random point: as $x \in D$, $t(F(x)) = f_{\mathcal{A}}(x) < +\infty$, so $F(x)$ is ν-random.

□

Corollary 6.2.1. *Let $(F, G) : (\mathcal{X}, \mu) \rightleftarrows (\mathcal{Y}, \nu)$ be an isomorphism of computable probability spaces. Then $F_{|_{R_\mu}}$ and $G_{|_{R_\nu}}$ are total computable bijections between R_μ and R_ν, and $(F_{|_{R_\mu}})^{-1} = G_{|_{R_\nu}}$.*

In particular:

Corollary 6.2.2. *Let δ be a binary representation on a computable probability space $(\mathcal{X}, \mu)$. Each point having a μ_δ-random expansion is μ-random and each μ-random point has a unique expansion, which is μ_δ-random.*

This proves that algorithmic randomness over a computable probability space could have been defined encoding points into binary sequences using a binary representation: this would have led to the same notion of randomness. Using this principle, a notion of Kolmogorov complexity characterizing Martin-Löf randomness comes for free. For $x \in D$, define:

$$H_n(x) = H(\omega_{0..n-1}) \text{ and } \Gamma_n(x) = \delta([\omega_{0..n-1}])$$

where ω is the expansion of x and H is the prefix Kolmogorov complexity.

Corollary 6.2.3. *Let δ be a binary representation on a computable probability space $(\mathcal{X}, \mu)$. Then x is μ-random if and only if there is c such that for all n:*

$$H_n(x) \geq -\log \mu(\Gamma_n(x)) - c$$

All this allows to treat algorithmic randomness within probability theory over general metric spaces. In [GHR07a] for instance, it is applied to show that in ergodic systems over metric spaces, algorithmically random points are well-behaved: they are *typical* with respect to any computable measure preserving transformation, generalizing what has been proved in [V'y97] for the Cantor space.

Aknowledgments

We would like to thank Stefano Galatolo, Peter Gács and Giuseppe Longo for useful comments and remarks.

References

[AJ94] Samson Abramsky and Achim Jung. Domain theory. In S. Abramsky, D. Gabbay, and T. S. E. Maibaum, editors, *Handbook of Logic in Computer Science Volume 3*, pages 1–168. Oxford University Press, 1994.

[Bil68] Patrick Billingsley. *Convergence of Probability Measures*. John Wiley, New York, 1968.

[BP03] Vasco Brattka and Gero Presser. Computability on subsets of metric spaces. *Theoretical Computer Science*, 305(1-3):43–76, 2003.

[Bra01] Vasco Brattka. Computable versions of baire's category theorem. In *MFCS '01: Proceedings of the 26th International Symposium on Mathematical Foundations of Computer Science*, pages 224–235, London, UK, 2001. Springer-Verlag.

[BW99] Vasco Brattka and Klaus Weihrauch. Computability on subsets of euclidean space i: Closed and compact subsets. *Theoretical Computer Science*, 219(1-2):65–93, 1999.

[Eda96] Abbas Edalat. The Scott topology induces the weak topology. In *LICS '96: Proceedings of the 11th Annual IEEE Symposium on Logic in Computer Science*, page 372, Washington, DC, USA, 1996. IEEE Computer Society.

[Gác05] Peter Gács. Uniform test of algorithmic randomness over a general space. *Theoretical Computer Science*, 341:91–137, 2005.

[GHR07a] Stefano Galatolo, Mathieu Hoyrup, and Cristóbal Rojas. Algorithmically random points in measure preserving systems, statistical behaviour, complexity and entropy. *Submitted to Information and Computation*, 2007. Available on arxiv.

[GHR07b] Stefano Galatolo, Mathieu Hoyrup, and Cristóbal Rojas. An effective Borel-Cantelli lemma. Constructing orbits with required statistical properties. *Submitted to JEMS*, 2007. Available on arxiv.

[Hem02] Armin Hemmerling. Effective metric spaces and representations of the reals. *Theor. Comput. Sci.*, 284(2):347–372, 2002.

[HW98] Peter Hertling and Klaus Weihrauch. Randomness spaces. In Kim G. Larsen, Sven Skyum, and Glynn Winskel, editors, *Automata, Languages and Programming*, volume 1443 of *Lecture Notes in Computer Science*, pages 796–807, Berlin, 1998. Springer. 25th International Colloquium, ICALP'98, Aalborg, Denmark, July 1998.

[HW03] Peter Hertling and Klaus Weihrauch. Random elements in effective topological spaces with measure. *Information and Computation*, 181(1):32–56, 2003.

[KF82] Ker-I Ko and H. Friedman. Computational complexity of real functions. *Theoretical Computer Science*, 20(3):323–352, 1982.

[Ko91] Ker-I Ko. *Complexity Theory of Real Functions*. Birkhauser Boston Inc., Cambridge, MA, USA, 1991.

[LNG05] Ambrosio L, Gigli N, and Savaré G. *Gradient flows: in metric spaces and in the space of probability measures*. Birkhauser, Zürich, 2005.

[ML66] Per Martin-Löf. The definition of random sequences. *Information and Control*, 9(6):602–619, 1966.

[Rog87] Hartley Jr. Rogers. *Theory of Recursive Functions and Effective Computability*. MIT Press, Cambridge, MA, USA, 1987.

[Sch07] Matthias Schröder. Admissible representations of probability measures. *Electronic Notes in Theoretical Computer Science*, 167:61–78, 2007.

[VV93] Vladimir G. Vovk and Vladimir V. Vyugin. On the empirical validity of the bayesian method. *Journal of the Royal Statistical Society*, B 55(1):253–266, 1993.

[V'y97] Vladimir V. V'yugin. Effective convergence in probability and an ergodic theorem for individual random sequences. *SIAM Theory of Probability and Its Applications*, 42(1):39–50, 1997.

[Wei99] Klaus Weihrauch. Computability on the probability measures on the borel sets of the unit interval. *Theoretical Computer Science*, 219:421–437, 1999.

[Wei00] Klaus Weihrauch. *Computable Analysis*. Springer, Berlin, 2000.

[YMT99] Mariko Yasugi, Takakazu Mori, and Yoshiki Tsujii. Effective properties of sets and functions in metric spaces with computability structure. *Theoretical Computer Science*, 219(1-2):467–486, 1999.

LIENS, Ecole Normale Supérieure, Paris. email: hoyrup@di.ens.fr

LIENS, Ecole Normale Supérieure and CREA, Ecole Polytechnique, Paris. email: rojas@di.ens.fr