4 Robustness issues: numerical issues, degenerate cases.

4.1 `double` arithmetic

4.1.1 Small questions

Let us use `double` arithmetic. For each of the following statements, answer whether it is true or false, and justify in less than one line.

$$a > b \quad \Leftrightarrow \quad a - b > 0 \tag{1}$$
$$(a * b) * c \quad = \quad a * (b * c) \tag{2}$$
$$a + b \quad = \quad b + a \tag{3}$$
$$a * (b + c) \quad = \quad a * b + a * c \tag{4}$$
$$x > y \quad \Rightarrow \quad sqrt(x) > sqrt(y) \tag{5}$$
$$(\text{for } x, y \geq 0) \;\; x * x \geq y * y \quad \Rightarrow \quad x \geq y \tag{6}$$
$$a, b, c \text{ integers in } [-2^{20}, 2^{20}] \quad \Rightarrow \quad (a - b) * (a - c) = a * a + a * (c - b) - b * c \tag{7}$$

4.1.2 A function

What does the following function return when called on a `double` in the open interval $]-2^{50}, 2^{50}[$?

```
double WhoAmI{double x}
  {
    double a = 3377699720527872.0;       // 2^50 + 2^51
    double s = x+a;
    double r = s-a;
    return r;
  }
```

4.1 Correction:

4.1.1 Small questions

(1) true. That's what denormalized numbers are for.
(2) false. Computer multiplication is not associative
(3) true. Rounding is the same in both cases.
(4) false. No distributivity.
(5) false. It is possible that $sqrt(x) = sqrt(y)$ even for $x \neq y$.
(6) true. Rounding preserves order (in the broad sense).
(7) true. All values are integers $< 2^{54}$ so they are exact with 53 significant digits. (Of course there was a mistake in the question, it should read $(a - b) * (a - c) = a * a - a * (b + c) + b * c$.)

4.1.2 A function

Answer: Rounding to closest half-integer.

Proof: $2^{51} = 2^{51} + 2^{50} - 2^{50} < x + a < 2^{51} + 2^{50} + 2^{50} = 2^{52}$. So, the value of first significant bit of `s` is $2^{52-1} = 2^{51}$, and the value of the 53^{rd} significant bit of `s` is $2^{52-53} = 0.5$. Since the rounding mode is to closest, `s` becomes the half-integer closest to `x+a`. Finally, `r` is the half-integer that is closest to `x`.

4.2 Circle intersection

Let C_1 and C_2 be two circles of respective centers (x_1, y_1) and (x_2, y_2) and respective radii r_1 and r_2,

4.2.1 Predicate

Write the predicate testing if C_1 and C_2 intersect as the sign of a polynomial in $x_1, y_1, r_1, x_2, y_2, r_2$.

4.2.2 Precision

Assume that the input data $x_1, y_1, r_1, x_2, y_2, r_2$ are integers in $[-2^b, 2^b]$, and that the computations are performed with `double`. For which values of b is the predicate guaranteed to give the correct result?

(Recall that according to IEEE754 norm, `double` are stored with 53 significant bits.)

4.2 Correction:

4.2.1 Predicate

p_i denotes the center of $C_i, i = 1, 2$.
If $\|p_1p_2\| > r_1 + r_2$ then disjoint
If $\|p_1p_2\| = r_1 + r_2$ then tangent
If $\|p_1p_2\| < \|r_1 - r_2\|$ then disjoint (one circle is contained in the other)
If $\|p_1p_2\| = \|r_1 - r_2\|$ then tangent
otherwise intersecting.

To put this in a single polynomial:
$s = \text{sign}\left(((x_1 - x_2)^2 + (y_1 - y_2)^2 - (r_1 + r_2)^2)((x_1 - x_2)^2 + (y_1 - y_2)^2 - (r_1 - r_2)^2)\right)$
If $s < 0$ then intersection; If $s = 0$ then tangent; If $s > 0$ then disjoint.

4.2.2 Precision

The sign of a product is the product of the signs of the factors.
$x_1 - x_2$, $y_1 - y_2$, and $r_1 + r_2$ are integers in $[-2^{b+1}, 2^{b+1}]$
$(x_1 - x_2)^2$, $(y_1 - y_2)^2$, and $(r_1 + r_2)^2$ are integers in $[0, 2^{2b+2}]$
$(x_1 - x_2)^2 + (y_1 - y_2)^2 \in [0, 2^{2b+3}]$
$(x_1 - x_2)^2 + (y_1 - y_2)^2 - (r_1 + r_2)^2 \in [-2^{2b+2}, 2^{2b+3}]$
For the computations to be exact it is enough that $2b + 3 \leq 53$, i.e. $b \leq 25$. Same for $(x_1 - x_2)^2 + (y_1 - y_2)^2 - (r_1 - r_2)^2$. Remark: there is no risk of overflow.