Hyperbolic surfaces: complexity of Delaunay triangulations via systoles

Matthijs Ebbens

University of Groningen

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Overview

- Background
- Delaunay triangulations on hyperbolic surfaces
- Systole of ‘regular’ surfaces

Recap

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- Systole
- δ_S for point set S

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- Projection $\pi : \mathbb{H}^2 \rightarrow \mathbb{H}^2/\Gamma$

- **Question:** Is $\pi(\mathrm{DT}_{\mathbb{H}}(\Gamma S))$ a triangulation?

Delaunay triangulations on hyperbolic surfaces

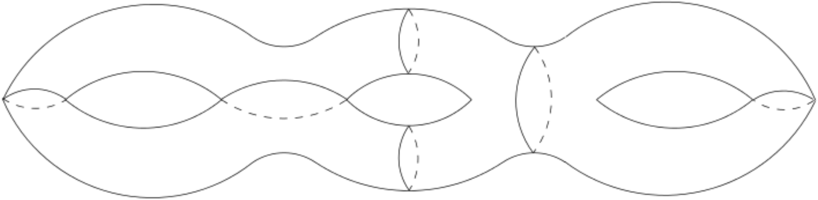
Theorem

If

$$\mathrm{syst}(\mathbb{H}^2/\Gamma) > 2\delta_S,$$

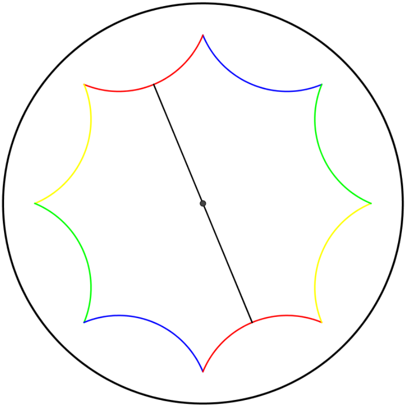
then $\pi(\mathrm{DT}_{\mathbb{H}}(\Gamma S))$ *is a triangulation.*

How to find the systole of $\mathbb{H}^2/\Gamma$?



Relation between geodesics and transformations

$\{\text{closed geodesics of } \mathbb{H}^2/\Gamma\} \leftrightarrow \{\text{conjugacy classes in } \Gamma\}$



Relation between geodesics and transformations

- Geodesic $c \leftrightarrow$ transformation γ
- Length of $c \leftrightarrow$ trace of matrix γ:

$$\cosh(\tfrac{1}{2}\ell(c)) = \tfrac{1}{2}|\operatorname{tr}(\gamma)|$$

Finding the systole

- Optimization problem:

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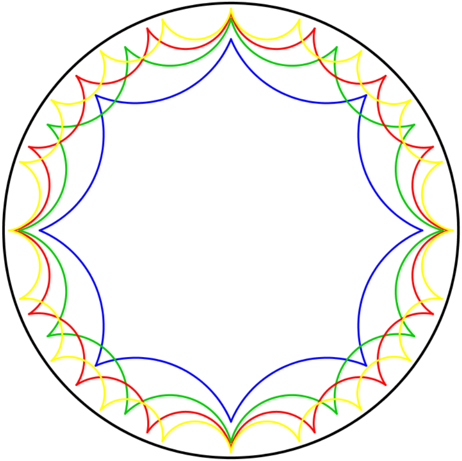
- Sufficient to solve:

$$\min \tfrac{1}{2}|\operatorname{tr}(\gamma)|,$$
$$\text{subject to } \gamma \in \Gamma \setminus \{\mathrm{Id}\}$$

‘Regular’ surfaces



‘Regular’ surfaces

- Hyperbolic surface M_g of genus $g \geq 2$
- Represented by a regular $4g$-gon
- Side pairing transformations pair opposite sides

Systole of ‘regular’ surfaces

Conjecture

$$\cosh\left(\tfrac{1}{2}\,\mathrm{syst}(M_g)\right) = 1 + 2\cos\left(\frac{\pi}{2g}\right)$$

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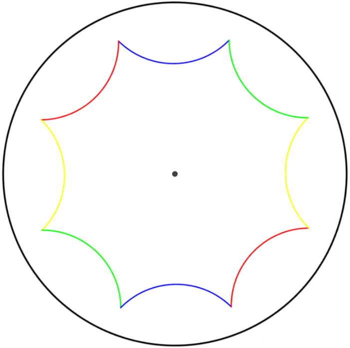
with equality for $g = 2, 3$

Genus $g = 2$

Fuchsian group generated by

$$A_k = \begin{bmatrix} 1+\sqrt{2} & \exp(\frac{ik\pi}{4})\sqrt{2+2\sqrt{2}} \\ \exp(-\frac{ik\pi}{4})\sqrt{2+2\sqrt{2}} & 1+\sqrt{2} \end{bmatrix}$$

for $k = 0, \ldots, 7$



Finding the systole

- Optimization problem

$$\begin{aligned} \min \tfrac{1}{2}|\operatorname{tr}(\gamma)|, \\ \text{subject to } \gamma \in \Gamma \setminus \{\mathrm{Id}\} \end{aligned}$$

- Look at products of the A_k's

Arbitrary products of the A_k

- Of the form

$$\begin{bmatrix} \alpha & \beta\sqrt{2+2\sqrt{2}} \\ \bar{\beta}\sqrt{2+2\sqrt{2}} & \bar{\alpha} \end{bmatrix}$$

- $\alpha - 1 \in 2\mathbb{Z}[\exp(\frac{\pi i}{4})]$

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- See [Aurich, Bogomolny, Steiner 1990]

Genus $g = 3$

- Products of generators of the form

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Fuchsian group for arbitrary genus

Group generated by

$$A_k = \begin{bmatrix} \cot(\frac{\pi}{4g}) & \exp(\frac{ik\pi}{2g})\sqrt{\cot^2(\frac{\pi}{4g}) - 1} \\ \exp(-\frac{ik\pi}{2g})\sqrt{\cot^2(\frac{\pi}{4g}) - 1} & \cot(\frac{\pi}{4g}) \end{bmatrix}$$

for $k = 0, \ldots, 4g - 1$

Upper bound

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- $\frac{1}{2}\,\mathrm{tr}(A_k A_{k+2g-1}) = 1 + 2\cos(\frac{\pi}{2g})$
- $\Rightarrow \cosh(\frac{1}{2}\,\mathrm{syst}(M_g)) \leq 1 + 2\cos(\frac{\pi}{2g})$

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$$\begin{bmatrix} \alpha & \beta\sqrt{\cot^2(\frac{\pi}{4g})-1} \\ \bar{\beta}\sqrt{\cot^2(\frac{\pi}{4g})-1} & \bar{\alpha} \end{bmatrix}$$

- $\alpha \in \mathbb{Z}[\zeta_{4g}]$
- $\alpha - 1 \in 2\mathbb{Z}[\zeta_{4g}]$ or $\alpha - \cot(\frac{\pi}{4g}) \in 2\mathbb{Z}[\zeta_{4g}]$

Automorphisms

$$\begin{bmatrix} \alpha & \beta\sqrt{\cot^2(\frac{\pi}{4g}) - 1} \\ \bar{\beta}\sqrt{\cot^2(\frac{\pi}{4g}) - 1} & \bar{\alpha} \end{bmatrix}$$

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- $\Rightarrow |\psi_k(\alpha)| < 1$

Intuition for the automorphisms

- Consider the optimization problem for $g = 2$ without the automorphism constraint:

$$\begin{aligned} &\min |m + n\sqrt{2}|, \\ \text{subject to } &m, n \in \mathbb{Z}, \\ &(m, n) \neq (0, 0), (1, 0) \end{aligned}$$

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- Consider $1 + (\sqrt{2} - 1)^n$ for $n \to \infty$

Resulting optimization problem

$$\begin{aligned} &\min |\operatorname{Re}(\alpha)|, \\ \text{subject to } &\alpha \in 1 + 2\mathbb{Z}[\zeta_{4g}] \cup \cot(\tfrac{\pi}{4g}) + 2\mathbb{Z}[\zeta_{4g}], \\ &|\psi_k(\alpha)| < 1 \text{ for } g < k < 3g, \gcd(k, 4g) = 1 \end{aligned}$$

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- Conjectured that this does not affect the minimum

Example: $g = 2$

$$\begin{aligned} &\min |\operatorname{Re}(\alpha)|, \\ \text{subject to } &\alpha \in 1 + 2\mathbb{Z}[\zeta_8] \cup 1 + \sqrt{2} + 2\mathbb{Z}[\zeta_8], \\ &|\psi_k(\alpha)| < 1 \text{ for } 2 < k < 6, \gcd(k, 8) = 1 \end{aligned}$$

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