Design of the CGAL Spherical Kernel

Pedro M. M. de Castro, Frédéric Cazals, **Sébastien Loriot**, Monique Teillaud

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The Computational Geometry Algorithms Library

Open Source project

`www.cgal.org`

$>$ 400.000 lines of C++ code
$>$ 3.000 pages manual
$\sim$ 12.000 downloads per year
$\sim$ 850 users on public mailing list
$\sim$ 50 developers
licenses LGPL or QPL
start-up GeometryFactory
interfaces: Python, Scilab

Robustness and efficiency

Quality:

- Editorial board
  (3 members in Geometrica / 12 members)
- Test-suites each night

...

CGAL kernels

A ***kernel*** *consists of constant-size non-modifiable geometric primitive objects and operations on these objects*.

[CGAL manual]

Predicates are basic units of geometric algorithms $\Longleftrightarrow$ decisions;
Value returned belong to an enum

Constructions generate objects that are neither of type bool nor enum types

CGAL Kernels

- Primitives : elementary geometric objects
  (points, segments, lines, . . .)
- Predicates and constructions : Elementary operations on them
  (intersection tests, intersection computations,. . .)

For example `CGAL::Polyhedron` is not in a kernel

Design of a CGAL kernel

A kernel concept defines requirements for a kernel in order to be able to construct **generic** *geometric algorithms based only on requirements (usable with any kernel model of the concept)*.
$\longrightarrow$ Each kernel in CGAL is a **model** of a kernel Concept

Kernel concept design guidelines

– Code reuse: ability to reuse the CGAL kernel for points, circles, number types,. . .
– Flexibility: possibility to use other implementations for points, circles, number types,. . .
possibility to use several algebraic implementations

Up to release 3.1 (Dec’04):
essentially linear objects
Release 3.2 (May ’06):
2D circular kernel **[Pion-Teillaud]**

2D circular kernel concept

[Emiris-Kakargias-Pion-Teillaud-Tsigaridas socg'04]

```
template < LinearKernel, AlgebraicKernel >
class CircularKernel
```

Types :

- Must be defined by `LinearKernel`
  basic number types, points, lines,. . .
- Must be defined by `AlgebraicKernel`
  algebraic numbers, polynomials
- Defined by `CircularKernel`
  `Circular_arc_2, Line_arc_2,`
  `Circular_arc_point_2`

2D circular kernel concept

[Emiris-Kakargias-Pion-Teillaud-Tsigaridas socg'04]

```
template < LinearKernel, AlgebraicKernel >
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Types :

- Must be defined by `LinearKernel`
  basic number types, points, lines,...
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  algebraic numbers, polynomials
- Defined by `CircularKernel`
  `Circular_arc_2, Line_arc_2, Circular_arc_point_2`

Predicates : intersection tests, comparisons of intersection points,...
Constructions : computation of intersection points

3D Spherical Kernel : Concept

Following the same design, we define a new **geometric** kernel : the 3D spherical kernel.

```
template < LinearKernel, AlgebraicKernel >
class SphericalKernel
```

which must define the following types :

`Circle_3`
`Circular_arc_3`
`Line_arc_3`
`Circular_arc_point_3`

3D Spherical Kernel : Concept

Access functions: Define the interface with kernel objects

- **`Circle_3`**
  - `center()`
  - `squared_radius()`
  - `supporting_plane()`
  - `diametrial_sphere()`
- **`CircularArcPoint_3`**
  - `x()`, `y()`, `z()`
- **`LineArc_3`**
  - `source(),target()`
  - `supporting_line()`
- **`CircularArc_3`**
  - `source(),target()`
  - `supporting_circle()`

3D Spherical Kernel Objects : Default implementation

Coordinate system chosen: **Cartesian Coordinates**.

- **`Circle_3`**
  represented by a plane and a sphere.
- **`Circular_arc_3`**
  represented by a circle and two endpoints (`Circular_arc_point_3`)
- **`Line_arc_3`**
  represented by a kernel line with two endpoints (`Circular_arc_point_3`)
- **`Circular_arc_point_3`**
  represented by an algebraic number per each cartesian coordinates

User frontend to 3D Spherical Kernel : Geometric functions

Predicates :

- `Has_on_3`, `Do_overlap_3`
- `Compare_x_3`, `Compare_y_3`, `Compare_z_3`
  Compare cartesian coordinates of `Circular_arc_point_3`
- `Side_of_3`
  position of a `Circular_arc_point_3` wrt a plane or a sphere

Constructions :

- `Intersect_3`
  (from 2 or 3 objects among planes, circle arcs, line and spheres)

Requirements to Algebraic Kernel

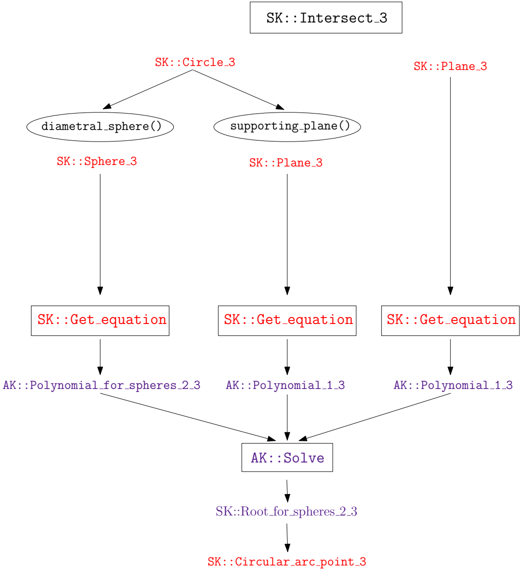
Type :

- FT
- Polynomial_1_3
- Polynomial_for_spheres_2_3
- Polynomial_for_lines_3
- Polynomial_for_circles_3
- Root_of_2
- Root_for_spheres_2_3

Constructions and predicates :

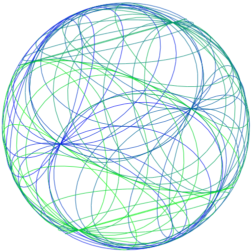
- Constructors for algebraic types from geometric objects
- Solve
- Sign_at

Example : Using algebra for geometric constructions



Application : specialization

Spherical Bentley-Ottmann **[Cazals,Loriot06]**.



- Input
  - A central sphere
  - Set of spheres intersecting the central one (or set of planes)
- Output
  - HDS containing faces of the arrangement of intersection circles
  - For each face, a list of sphere which ball covering it

Specialization on a given sphere

Natural extension to handle objects on a commun sphere using cylindrical coordinates

Primitives :

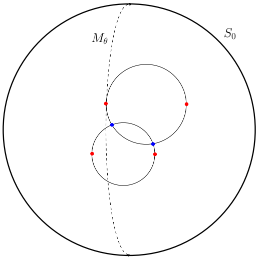
– Circle_on_reference_sphere_3
– Circular_arc_point_on_reference_sphere_3
– Circular_arc_on_reference_sphere_3
– Theta_rep

Predicates:

– Compare_theta_3
– Compare_z_at_theta_3
– Compare_z_to_left_3

Constructions :

– Intersect_3
– Make_theta_monotonic_3
– Theta_extremal_point_3

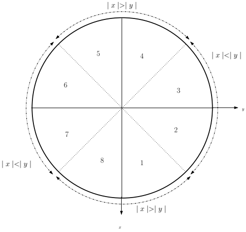


Example : `Compare_theta_3`

Bentley-Ottmann on a sphere $\Longrightarrow$ sort event points (critical and intersection points) according to (θ, z) value.

Our implementation :

– Need `Theta_extremal_point_3` for critical points
– Need new `Intersect_3` for intersection points



– Each event point is in a hquadrant
– We obtain $\tan\theta$ and $\cot\theta$ as AN of degree 2
$\Longrightarrow$ comparison of θ coordinates :
  – compare hquadrant indices
  – compare AN of degree 2

Needs of the Spherical BO

- Predicates
  - Initialize vertical ordering (`Compare_z_to_left_3`,`Compare_z_at_theta_3`)
  - Sort event points using cylindrical coordinates (`Compare_theta_z_3`)
  - Detect/create intersection points (`Do_intersect_3`,`Intersect_3`)
  - Insert circle starting (`Compare_z_at_theta_3`)
- DS : Face, Vertex and halfedge types for HDS encoding arrangement of circles
- Gauss-Bonnet formula to compute approximate area of a spherical face.

Illustration

Video

Improvements

Several level of filtering

- Arithmetic Filtering
  - Computation on intervals : failure implies exact computation
- Filtered constructions
  - `CGAL::Lazy_kernel` : a construction create one node in the dag (vs set of operations) **[Fabri,Pion06]**
- Geometric Filtering
  - Using predicates on Bbox
- Static Filters **[Melquiond,Pion05]**
  - Design bounds for arithmetic operations in order to guarantee **`double`** computations (specific to each predicates).

We take advantage of these strategies to design filtered version of kernels.

Conclusion and Future work

The SphericalKernel provides

- a generic framework for algorithms involving spheres
- Robust and efficient primitives and predicates
- extension for manipulating circle arcs on a common sphere

Future work

- Traits class for `Arrangement_2` for arrangement of circle on a sphere using the Spherical Kernel.
- Efficient DS to encode arrangement of spheres.

An example

```
typedef CGAL::Quotient< CGAL::MP_Float>                    NT;
typedef CGAL::Cartesian<NT>                                Linear_k;
typedef CGAL::Algebraic_kernel_for_spheres_2_3<NT>         Algebraic_k;
typedef CGAL::Spherical_kernel_3<Linear_k,Algebraic_k>     SK;

int main(){
  //construction of 3 spheres from their centers and squared radii
  SK::Sphere_3 s1(SK::Point_3(0,0,0),2);
  SK::Sphere_3 s2(SK::Point_3(0,1,0),1);
  SK::Sphere_3 s3(SK::Point_3(1,0,0),3);

  SK::Intersect_3 inter;
  SK::Compare_xyz_3 cmp;
  std::vector< CGAL::Object > intersections;
  inter(s1,s2,s3,std::back_inserter(intersections));

  std::pair<SK::Circular_arc_point_3,unsigned> p1,p2;
  //unsigned integer indicates multiplicity of intersection point
  if (intersections.size() >1){
    //as intersection can return several types (points with multiplicity, circle,...),
    //CGAL::Object and CGAL::assign are used to recover the expected type
    if (CGAL::assign(p1,intersections[0]) && CGAL::assign(p2,intersections[1]))
      std::cout << "Two different intersection points" << std::endl;
    else
      std::cout << "Error" << std::endl;
  }
  //intersection points are sorted lexicographically
  CGAL_assertion(cmp(p1.first,p2.first)==CGAL::SMALLER);
  return 0;
}
```