BRITISH AIRCRAFT CORPORATION LIMITED
COMMERCIAL AIRCRAFT DIVISION
WEYBRIDGE.

17 September 1974
retypeset and footnotes added July 1999

A Class of Surfaces Closed under Five Important Geometric Operations

A set is closed under an operation if, whenever the operation is applied to members of the set, the result is also a member. It is clearly an advantage in a computer based numerical geometry system if the representations used are closed under all the operations required.

Operations

In the context of sculptured surfaces we can identify five such operations, under which we would like closure:

1. Scaling
2. Translation

The requirement for these two has been so obvious that it has never needed spelling out.

3. Rotation

The 'conic lofting' schemes developed during the 1940's (7) did not have this property except by the somewhat artificial association of a rotation matrix with each surface. When vector-valued parametric bipolynomials were introduced the fact that they had this property was quickly recognised as an advantage (2)(5). This has also been claimed for the general curve-based system (3).

4. Offsets

The result of displacing each point of a surface a fixed distance along the local surface normal is obviously important when dealing with surfaces of constant thickness, or when machining a surface with a ball end cutter. This operation we term offsetting. If the displacement vector is a more complex function of the point position and derivatives the operation is **generalised offsetting**, which includes, for example, the calculation of the tool centre surface for a toroidal cutter. Closure under this operation has been approached in two ways. Flutter (1) has substituted approximate offsetting for the precise operation, using the good approximation properties of the bicubic to minimize the error. This works well for offsetting, particularly for small offsets, but runs into difficulties in regions like that due to the singularity at the end of a toroidal cutter. Sabin (4) uses the fact that general biparametric surfaces form a class closed under generalised offsetting, a solution which accepts somewhat increased mill[1] times.

[1] 'mill' = CPU

5. Convolution

This recently identified operation (6) can be regarded as that of offsetting a cutter whose shape is another sculptured surface. Closure under convolution implies closure under general offsetting if the cutter form is included in the class of surfaces represented.

Surface equations

Because the vector valued bi-polynomial form does not express at all transparently the conditions for closure under offsetting and convolution, we seek forms in which these relationships are clearer.

Such an one is the envelope equation in which the tangent plane is expressed as a function of surface normal direction.

Now the tangent plane T can be represented by the quadruple $(l\ m\ n\ h)$ such that $lx + my + nz + h = 0$ if the point $(x\ y\ z)$ lies on the plane.

The triple $(l\ m\ n)$ is itself the normal N, and so the envelope form can be expressed entirely in the variation of h with N. Although the general surface equation in this form is $f(h, N) = 0$, we shall consider for the moment only the explicit form

$$h = h(N)$$

Each closure condition on the surface class corresponds to a closure condition[2] on the form of $h(N)$.

1. Scaling

   $h(N)$ must include $\alpha h(N)$

2. Translation

   $h(N)$ must include $h(N) + A.N$ where A is the displacement vector

3. Rotation

   $h(N)$ must include $h(AN)$ where A is an orthornormal[3] rotation matrix

4. Offset

   $h(N)$ must include $h(N) + r$

5. Convolution

   $h(N)$ must include $h_1(N) + h_2(N)$

---

[2] The notation here is somewhat elliptic. A better expression of the scaling condition, for example, might be *If S is the set of permitted functions $h(N)$, then*

$$h(N) \in S \rightarrow \alpha h(N) \in S$$

[3] original misspelling

The simplest form satisfying all these conditions is

$$h = r + A.N$$

This represents the sphere, or if applied piecewise, the convex hull of a configuration of spheres. The non-spherical surfaces (6) correspond to lines or points on the unit sphere, rather than to regions.

To cater for more general shapes this can be generalised to the full **outer polynomial**

$$h = \sum_{i=0}^{n} a_i N^i$$

where powers of N are generated by the outer product, so that a_i is a tensor of order i.

This form appears to have a very large number of coefficients as n increases $(3^{n+1} - 1)/2$, but for two reasons the number is much lower, $(n+1)^2$. The first is the full symmetry of the tensors; the second is that because $N.N = 1$ identically all other terms can be subsumed into the highest order two terms by multiplying them by $N.N$ to the appropriate power.

Properties

Point coordinates can be determined from an envelope equation by the dual of the method for determining tangent planes from the point equation. In the latter case we find the plane through the point and two points independently infinitesimally perturbed from it. Here we calculate the intersection point of the tangent plane and the two planes corresponding to independent infinitesimal changes of N. This gives three equations which may be solved for the three components of P.

$$\begin{aligned} P.N + h &= 0 \\ P.\delta_1 N + \delta_1 h &= 0 \\ P.\delta_2 N + \delta_2 h &= 0 \end{aligned}$$

which may be written in homogeneous coordinate notation as the triple product[4]

$$\begin{bmatrix} wP \\ w \end{bmatrix} = \begin{bmatrix} N, & \delta_1 N, & \delta_2 N \\ h & \delta_1 h & \delta_2 h \end{bmatrix}$$

[4] what is implied here is the 1×4 matrix obtained by taking the minors of the right hand side

The setting up of independent perturbations of N can be simplified by mapping the unit sphere on to a parameter plane. Stereographic projection is computationally simplest.

$$N = \begin{bmatrix} l \\ m \\ n \end{bmatrix} = \begin{bmatrix} 2u/(1+u^2+v^2) \\ 2v/(1+u^2+v^2) \\ (1-u^2-v^2)/(1+u^2+v^2) \end{bmatrix}$$

The combination of the parametrization of N, the surface equation , and the calculation of point coordinates indicates that these surfaces are in fact parametric rational bi-polynomials.

The geometric properties, however, are much more restrictive than those of the general bipolynomials. We can only represent surfaces which are single valued with respect to surface normal. This implies that we are concerned only with topologically spherical surfaces. Further, we can only represent compact surfaces because in the equation for a surface point we can always choose N, $\delta_1 N$ and $\delta_2 N$ to be orthonormal. P is then given by an expression with unit denominator and cannot go to infinity.

We can map regions of negative Gaussian curvature, but these join adjacent regions of positive curvature in cusps, vertex singularities or astigmatic self-intersections rather than in the more acceptable parabolic lines.

Positive and negative regions can, of course, abut at patch boundaries, but only with discontinuities of curvature.

Fitting Surface to Data

Two conditions are of interest. The first is the condition that the surface should touch a particular plane. This demands a particular value for h at a particular value of N, and so imposes one scalar condition on the coefficients of the surface. Because

$$h = a_i N^i$$

is linear in the coefficients the condition is a linear one.

Thus the sphere

$$h = r + A.N$$

with four degrees of freedom can be chosen to touch four planes by the simultaneous solution of four linear equations. Note that the planes are orientated, the orientation selecting which of the four possible spheres is solved for.

The second condition is that the surface should touch a particular plane at a particular point. Here there are three scalar conditions, corresponding to the three equations calculating the point coordinates. Again, because h is linear in the coefficients so will $\partial h/\partial N.\delta_1 N$ and $\partial h/\partial N.\delta_1 N$ be, and so all three equations are linear.

Thus the sphere can be chosen to touch one plane at a given point and another plane somewhere. Again, the quadratic surface can be fitted to touch three planes, each at its own specific point, by the solution of nine simultaneous linear equations.

This configuration appears attractive from the point of view of convenience of use and therefore warrants further attention.

Quadratic surface

Because of the $N.N = 1$ property noted above, the the[5] canonical form of this surface is

$$h = A_i N^i + A_{ij} N^i N^j$$

Now by displacement of the origin by the vector A_i, we reduce this to

$$h = A_{ij} N^i N^j$$

If in this equation we replace N by $-N$, h is left unchanged. We therefore conclude that the surface is symmetric about the centre A_i.

Further, A_{ij} is itself a symmetric matrix which can be pre and post multiplied by an appropriate orthonormal rotation matrix to give the diagonal form

$$\begin{aligned} h &= A_{ii} N^i N^i \\ &= a_1 l^2 + a_2 m^2 + a_3 n^2 \end{aligned}$$

Within this form negation of any component of N leaves h invariant, thus the surface in general has three orthogonal planes of symmetry. The values a_1 a_2 and a_3 are the principal radii.

The shape is reminiscent of an ellipsoid and, like the ellipsoids, contains the sphere as a special case. The singularities, however, appear in the form of cusps rather than asymptotes, so there is no equivalent of the hyperboloids.

Difficulties and Outstanding Questions

The first difficulty is that of the likelihood of cusps appearing. This, however, appears to be inherent in all methods offering closure under offsetting, and can at least be detected numerically.

---

[5] original typo

The second is a very real practical difficulty. Toroidal shapes appear as edges[6] where cylinders intersect planes and also occur frequently as cutter shapes. The equation of a toroid has to be quadratic because there are two tangent planes at any given orientation. It can be expressed either as

$$h^2 + lh + q = 0$$

where l is linear in N and q quadratic. or as

$$h = a + A.N + r\sqrt{1 - (B.N)^2}$$

Neither of these forms is closed under convolution. This reflects an underlying geometric difficulty. The convolution surface of a set of torus shapes does have as many singularities as there are toroids, and these singularities do not combine. This means that any numerical representation closed under convolution which includes the torus must allow for an indefinitely long list of singularity terms like the $\sqrt{1 - (B.N)^2}$ above.

Both these difficulties are inherently geometric and are not, therefore, going to be solved by variation of representation. The outer polynomial is as good a representation as any which offers all five closure properties.

There is one outstanding question which may determine the usability of the above theory. This is the question of continuity.

All envelope surfaces are continuous in slope if the partitioning is of the entire unit sphere. Continuity of position can be constructed if the boundary between two pieces lies on the locus

$$h_1(N) = h_2(N) \text{ in } N\text{-space}$$

A developable will then fit between corresponding points on the edges of the two patches.

For example, the spheres $(r_1\ A_1)$, $(r_2\ A_2)$ can be joined by a cone if the boundary on the unit sphere is

$$r_1 - r_2 + (A_1 - A_2).N = 0$$

It is not yet clear, however, what the conditions are for direct position continuity between, for example, two adjacent quadratic triangles.

M.A.Sabin

[6] as fillets when the edge is rounded

References

1. Flutter,A.G. The Polysurf System
proceedings of 2nd Prolamat Conference, Budapest, 1973
2. Forrest,A.R. Curves and Surfaces for computer aided design.
Cambridge University Computer Lab.
3. Mehlum,E. Private Communications 1970
4. Sabin,M.A. Interrogation Techniques for Parametric Surfaces.
proceedings of CG70 Conference, Brunel University, 1970
5. Sabin,M.A. An existing system in the aircraft industry.
Proc.Roy.Soc.Lond.A 321,197-205 (1971).
6. Sabin,M.A. The convolution of two regions VTO/MS/206.
British Aircraft Corporation, Weybridge.
7. Shelley,J.H. The development of curved surfaces for aero-design.
Gloster Aircraft Company Limited.